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Oscilation and Convergence for a Neutral Difference Equation

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Abstract. The oscillation and convergence of the solutions of neutral difference equation

$$\Delta(x_n + \delta x_{n-\tau}) + \sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}) = 0, \quad n = 0, 1, \cdots$$

are investigated, where $m_i \in \mathbb{N}_0, \forall i = \overline{1, r}$ and F is a function mapping $\mathbb{R}$ to $\mathbb{R}$.

Keyworks: Neutral difference equation, oscillation, nonoscillation, convergence.

1. Introduction

It is well-known that difference equation

$$\Delta(x_n + \delta x_{n-\tau}) + \alpha(n) x_{n-\sigma} = 0, \tag{1}$$

where $n \in \mathbb{N}$, the operator Δ is defined as $\Delta x_n = x_{n+1} - x_n$, the function $\alpha(n)$ is defined on $\mathbb{N}$, δ is a constant, τ is a positive integer and σ is a nonnegative integer, was first considered by Brayton and Willoughby from the numerical point of view (see [1]). In recent years, the asymptotic behavior of solutions of this equation has been studied extensively (see [2-7]). In [4, 6, 7], the oscillation of solutions of the difference equation (1) was discussed.

Motivated by the work above, in this paper, we aim to study the oscillation and convergence of solutions of neutral difference equation

$$\Delta(x_n + \delta x_{n-\tau}) + \sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}) = 0, \tag{2}$$

for $n \in \mathbb{N}, n \geqslant a$ for some $a \in \mathbb{N}$, where $r, m_1, m_2, \cdots, m_r$ are fixed positive integers, the functions $\alpha_i(n)$ are defined on $\mathbb{N}$ and the function F is defined on $\mathbb{R}$.

Put $A = \max\{\tau, m_1, \cdots, m_r\}$. Then, by a solution of (2) we mean a function which is defined for $n \geqslant -A$ and sastisfies the equation (2) for $n \in \mathbb{N}$. Clearly, if

$$x_n = a_n, \quad n = -A, -A+1, \cdots, -1, 0$$

are given, then (2) has a unique solution, and it can be constructed recursively.

A nontrivial solution $\{x_n\}_{n \geqslant a}$ of (2) is called *oscillatory* if for any $n_1 \geqslant a$ there exists $n_2 \geqslant n_1$ such that $x_{n_2} x_{n_2+1} \leqslant 0$. The difference equation (2) is called oscillatory if all its solutions are oscillatory. Otherwise, it is called nonoscillatory.

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2. Main results

2.1. The Oscillation

Consider neutral difference equation

$$\Delta(x_n + \delta x_{n-\tau}) + \sum_{i=1}^{r} \alpha_i(n) x_{n-m_i} = 0, \tag{3}$$

for $n \in \mathbb{N}, n \geqslant a$ for some $a \in \mathbb{N}$, where $r, m_1, m_2, \cdots, m_r$ are fixed positive integers and the functions $\alpha_i(n)$ are defined on $\mathbb{N}$. It is clear that equation (3) is a particular case of (2). We shall establish some sufficient criterias for the oscillation of solutions of the difference equation (3). First of all we have

Theorem 1. *Assume that*

$$\frac{(\tilde{m}+1)^{\tilde{m}+1}}{\tilde{m}^{\tilde{m}}} \sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n) > 1, \tag{4}$$

where $\delta = 0$, $\alpha_i(n) \geqslant 0, n \in \mathbb{N}, 1 \leqslant i \leqslant r$ *and* $\tilde{m} = \min_{1\leqslant i\leqslant r} m_i$. *Then, (3) is oscillatory.*

Proof. We first prove that the inequality

$$\Delta x_n + \sum_{i=1}^{r} \alpha_i(n) x_{n-m_i} \leqslant 0, \quad n \in \mathbb{N} \tag{5}$$

has no eventually positive solution. Assume, for the sake of contradiction, that (5) has a solution $\{x_n\}$ with $x_n > 0$ for all $n \geqslant n_1, n_1 \in \mathbb{N}$. Setting $v_n = \frac{x_n}{x_{n+1}}$ and dividing this inequality by x_n, we obtain

$$\frac{1}{v_n} \leqslant 1 - \sum_{i=1}^{r} \alpha_i(n) \prod_{\ell=1}^{m_i} v_{n-\ell}, \tag{6}$$

where $n \geqslant n_1 + m, \quad m = \max_{1\leqslant i\leqslant r} m_i$.

Clearly, $\{x_n\}$ is nonincreasing with $n \geqslant n_1 + m$, and so $v_n \geqslant 1$ for all $n \geqslant n_1 + m$. From (4) and (6) we see that $\{v_n\}$ is a above bounded sequence. Putting $\liminf_{n\to\infty} v_n = \beta$, we get

$$\limsup_{n\to\infty} \frac{1}{v_n} = \frac{1}{\beta} \leqslant 1 - \liminf_{n\to\infty} \sum_{i=1}^{r} \alpha_i(n) \prod_{\ell=1}^{m_i} v_{n-\ell},$$

or

$$\frac{1}{\beta} \leqslant 1 - \sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n) \cdot \beta^{m_i}. \tag{7}$$

Since

$$\beta^{m_i} \geqslant \beta^{\tilde{m}}, \quad \forall i = \overline{1, r},$$

we have

$$\liminf_{n\to\infty} \alpha_i(n)\beta^{m_i} \geqslant \liminf_{n\to\infty} \alpha_i(n)\beta^{\tilde{m}}, \quad \forall i = \overline{1, r}$$

and

$$1 - \sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n)\beta^{m_i} \leqslant 1 - \sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n)\beta^{\tilde{m}}.$$

From (7) we have

$$\liminf_{n\to\infty}\sum_{i=1}^{r}\alpha_i(n)\leqslant\frac{\beta-1}{\beta^{\tilde{m}+1}}.$$

But

$$\frac{\beta-1}{\beta^{\tilde{m}+1}}\leqslant\frac{\tilde{m}^{\tilde{m}}}{(\tilde{m}+1)^{\tilde{m}+1}},$$

so

$$\frac{(\tilde{m}+1)^{\tilde{m}+1}}{\tilde{m}^{\tilde{m}}}\sum_{i=1}^{r}\liminf_{n\to\infty}\alpha_i(n)\leqslant 1,$$

which contradicts condition (4). Hence, (5) has no eventually positive solution.

Similarly, we can prove that the inequality

$$\Delta x_n+\sum_{i=1}^{r}\alpha_i(n)x_{n-m_i}\geqslant 0,\quad n\in\mathbb{N}$$

has no eventually negative solution. So, the proof is complete.

Corollary. *Assume that*

$$r\Big[\prod_{i=1}^{r}\liminf_{n\to\infty}\alpha_i(n)\Big]^{\frac{1}{r}}>\frac{\hat{m}^{\hat{m}}}{(\hat{m}+1)^{\hat{m}+1}},\tag{8}$$

where $\delta=0$, $\alpha_i(n)\geqslant 0, n\in\mathbb{N}, 1\leqslant i\leqslant r$ *and* $\hat{m}=\frac{1}{r}\sum_{i=1}^{r}m_i$. *Then, (3) is oscillatory.*

Proof. We will prove that the inequality (5) has no eventually positive solution. Assume, for the sake of contradiction, that (5) has a solution $\{x_n\}$ with $x_n>0$ for all $n\geqslant n_1, n_1\in\mathbb{N}$. Using arithmetic and geometric mean inequality, we obtain

$$\sum_{i=1}^{r}\liminf_{n\to\infty}\alpha_i(n)\cdot\beta^{m_i}\geqslant r\left(\prod_{i=1}^{r}\liminf_{n\to\infty}\alpha_i(n)\beta^{m_i}\right)^{\frac{1}{r}},$$

which is the same as

$$\sum_{i=1}^{r}\liminf_{n\to\infty}\alpha_i(n)\cdot\beta^{m_i}\geqslant r\left(\prod_{i=1}^{r}\liminf_{n\to\infty}\alpha_i(n)\right)^{\frac{1}{r}}\beta^{\hat{m}}.$$

This yields

$$1-\sum_{i=1}^{r}\liminf_{n\to\infty}\alpha_i(n)\cdot\beta^{m_i}\leqslant 1-r\left(\prod_{i=1}^{r}\liminf_{n\to\infty}\alpha_i(n)\right)^{\frac{1}{r}}\beta^{\hat{m}}.$$

By using the inequality (7) we have

$$r\Big[\prod_{i=1}^{r}(\liminf_{n\to\infty}\alpha_i(n))\Big]^{\frac{1}{r}}\leqslant\frac{\hat{m}^{\hat{m}}}{(\hat{m}+1)^{\hat{m}+1}},$$

which contradicts condition (8). Hence, (5) has no eventually positive solution.

Next, we consider the equation (3) in case $\delta\neq 0$. We have the following Lemma.

Lemma 1. *Let* $\alpha_i(n)\geqslant 0$ *for all* $n\in\mathbb{N}$ *and let* $\{x_n\}$ *be an eventually positive solution of (3). Put* $z_n=x_n+\delta x_{n-\tau}$, *we have*

- *(a) If $-1 < \delta < 0$, then $z_n > 0$ and $\Delta z_n < 0$ eventually.*
- *(b) If $\delta < -1$ and $\sum_{\ell=1}^{\infty}[\sum_{i=1}^{r} \alpha_i(\ell)] = \infty$, then $z_n < 0$ and $\Delta z_n \leqslant 0$ eventually.*

Proof. (a) Since $\alpha_i(n) \not\equiv 0$, we have

$$\Delta z_n = -\sum_{i=1}^{r} \alpha_i(n) x_{n-m_i} < 0$$

eventually, so z_n cannot be eventually identically zero. If $z_n < 0$ eventually, then

$$z_n \leqslant z_N < 0, \quad \forall n \geqslant N \in \mathbb{N}.$$

Since $-1 < \delta < 0$, we get

$$z_n = x_n + \delta x_{n-\tau} > x_n - x_{n-\tau},$$

which implies that

$$x_n < z_n + x_{n-\tau} \leqslant z_N + x_{n-\tau}.$$

Therefore,

$$x_{N+\tau n} < z_N + x_{N+\tau n-\tau} = z_N + x_{N+\tau(n-1)} < \cdots < n z_N + x_N.$$

Taking $n \to \infty$ in the above inequality, we have $x_{N+\tau n} < 0$, which is a contradiction to $x_n > 0$.

(b) We have

$$\Delta z_n = -\sum_{i=1}^{r} \alpha_i(n) x_{n-m_i} < 0,$$

for n sufficient large. We shall prove that $z_n < 0$, eventually. Assume, for the sake of a contradiction, that

$$z_n = x_n + \delta x_{n-\tau} \geqslant 0, \quad n \geqslant N,$$

i.e.

$$x_n \geqslant -\delta x_{n-\tau}, \quad n \geqslant N,$$

which implies that

$$0 < x_{N-\tau} \leqslant \left(-\frac{1}{\delta}\right) x_N \leqslant \cdots \leqslant \left(-\frac{1}{\delta}\right)^j x_{N+(j-1)\tau}, \quad j = 1, 2, \cdots.$$

On letting $j \to \infty$ in the above inequality, we get $x_n \to \infty$ as $n \to \infty$. But

$$\Delta z_n = -\sum_{i=1}^{r} \alpha_i(n) x_{n-m_i} \leqslant -M \sum_{i=1}^{r} \alpha_i(n), \tag{9}$$

for n sufficient large, where $M > 0$. Summing (9) from N to n, we obtain

$$z_{n+1} - z_N \leqslant -M \sum_{\ell=N}^{n} [\sum_{i=1}^{r} \alpha_i(\ell)],$$

which implies that $z_n \to -\infty$ as $n \to \infty$. This contradicts the hypothesis that $z_n \geqslant 0, \quad n \geqslant N$.

Theorem 2. *Suppose that*

$$\frac{1}{1+\delta} \frac{(\tilde{m}+1)^{\tilde{m}+1}}{\tilde{m}^{\tilde{m}}} \sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n) > 1, \tag{10}$$

where $-1 < \delta < 0$, $\tilde{m} = \min_{1\leqslant i\leqslant r} m_i$ *and* $\alpha_i(n) \geqslant 0$, $\alpha_i(n) > \alpha_i(n-\tau)$, *for* n *sufficient large,* $1 \leqslant i \leqslant r$. *Then, (3) is oscillatory.*

Proof. Assume the contrary and let $\{x_n\}$ be an eventually positive solution of (3). Let $z_n = x_n + \delta x_{n-\tau}$ and $w_n = z_n + \delta z_{n-\tau}$. Then, by the case (a) of Lemma 1, $z_n > 0, \Delta z_n < 0$ and $w_n > 0$. We have

$$\begin{aligned}
\Delta w_n &= \Delta z_n + \delta\Delta z_{n-\tau}\\
&= -\sum_{i=1}^{r}\alpha_i(n)x_{n-m_i} - \delta\sum_{i=1}^{r}\alpha_i(n-\tau)x_{n-\tau-m_i},\\
&\leqslant -\sum_{i=1}^{r}\alpha_i(n)x_{n-m_i} - \delta\sum_{i=1}^{r}\alpha_i(n)x_{n-\tau-m_i},\\
\Delta w_n &\leqslant -\sum_{i=1}^{r}\alpha_i(n)(x_{n-m_i} + \delta x_{n-\tau-m_i}),\\
\Delta w_n &\leqslant -\sum_{i=1}^{r}\alpha_i(n)z_{n-m_i} \leqslant 0.
\end{aligned}$$

Putting $\lim_{n\to\infty} z_n = \beta$, we have $\beta \geqslant 0$ and

$$\lim_{n\to\infty} w_n = \beta + \delta\beta = (1+\delta)\beta \geqslant 0.$$

Therefore, $w_n > 0$ for n sufficient large. On the other hand,

$$w_n = z_n + \delta z_{n-\tau} \leqslant (1+\delta)z_n,$$

which implies that

$$z_{n-m_i} \geqslant \frac{w_{n-m_i}}{1+\delta}.$$

Hence, we obtain

$$\Delta w_n \leqslant -\sum_{i=1}^{r}\alpha_i(n)z_{n-m_i} \leqslant -\frac{1}{1+\delta}\sum_{i=1}^{r}\alpha_i(n)w_{n-m_i},$$

or

$$\Delta w_n + \frac{1}{1+\delta}\sum_{i=1}^{r}\alpha_i(n)w_{n-m_i} \leqslant 0. \tag{11}$$

By Theorem 1 and in view of condition (10), the inequality (11) has no eventually positive solution, which is a contradiction.

Lemma 2. *Assume that* $-1 < \delta < 0$ *and* $\tau > \tilde{m} + 1$, *where* $\tilde{m} = \min_{1\leqslant i\leqslant r} m_i$. *Then, the maximum value of* $f(\beta) = \frac{\beta-1}{\beta^{\tilde{m}+2}}(1+\delta\beta^\tau)$ *on* $[1,\infty)$ *is* $f(\beta^*)$, *in which* $\beta^* \in (1, (-\delta)^{-1/\tau})$ *is a unique real solution of the equation*

$$1 + \delta\beta^\tau + (\beta-1)[\delta\tau\beta^\tau - (\tilde{m}+1)(1+\delta\beta^\tau)] = 0.$$

Proof. The equation $f'(\beta) = 0$ is equivalent to

$$1 + \delta\beta^\tau + (\beta-1)[\delta\tau\beta^\tau - (\tilde{m}+1)(1+\delta\beta^\tau)] = 0. \tag{12}$$

Put

$$\varphi(\beta) = 1 + \delta\beta^{\tau} + (\beta - 1)[\delta\tau\beta^{\tau} - (\tilde{m} + 1)(1 + \delta\beta^{\tau})].$$

It is easy to check that

$$\varphi'(\beta) = \delta\tau\beta^{\tau-1} + \delta\beta^{\tau}[\tau - (\tilde{m} + 1)] - (\tilde{m} + 1) + (\beta - 1)\delta\tau\beta^{\tau-1}[\tau - (\tilde{m} + 1)].$$

Since $\tau > \tilde{m} + 1$, we get $\varphi'(\beta) < 0$. On the other hand, we have $\varphi(1) = 1 + \delta > 0$ and

$$\lim_{\beta\to+\infty} \varphi(\beta) = \lim_{\beta\to+\infty} \{1 + \delta\beta^{\tau} + (\beta - 1)[\delta\beta^{\tau}[\tau - (\tilde{m} + 1)] - (\tilde{m} + 1)]\} = -\infty.$$

It implies that, φ is a decreasing function, starting from a positive value at $\beta = 1$, and hence (12) has a unique real solution $\beta^* \in [1, \infty)$. Further, it is easy to see that $\beta^* \in (1, (-\delta)^{-1/\tau})$ and $f(\beta) \geqslant 0, \quad \forall \beta \in (1, (-\delta)^{-1/\tau})$, which implies that $f(\beta^*)$ is the maximum value of $f(\beta)$ on $[1, \infty)$. The proof is complete

Theorem 3. *Assume that* $-1 < \delta < 0$; $\tau > \tilde{m} + 1$; $\alpha_i(n) \geqslant 0$, $\alpha_i(n) > \alpha_i(n - \tau)$, *for* n *sufficient large,* $1 \leqslant i \leqslant r$, $\tilde{m} = \min_{1\leqslant i\leqslant r} m_i$ *and*

$$\sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n) > \frac{\beta^* - 1}{\beta^{*\tilde{m}+2}}(1 + \delta\beta^{*\tau-1}), \tag{13}$$

where $\beta^* \in [1, \infty)$ *is defined as in Lemma 2. Then, (3) is oscillatory.*

Proof. Suppose to the contrary, and let $\{x_n\}$ be an eventually positive solution of (3). By the case (a) of Lemma 1, we get $z_n > 0, \Delta z_n < 0$ eventually. On the other hand,

$$\Delta w_n = \Delta(z_n + \delta z_{n-\tau}) \leqslant -\sum_{i=1}^{r} \alpha_i(n) z_{n-m_i} \leqslant 0. \tag{14}$$

Putting $\gamma_n = \frac{z_{n-1}}{z_n}$, we have $\gamma_n \geqslant 1$ for n sufficient large. Dividing (14) by z_n, we get

$$\frac{1}{\gamma_{n+1}} \leqslant 1 + \delta\Big[\frac{z_{n-\tau}}{z_n} - \frac{z_{n-\tau+1}}{z_n}\Big] - \sum_{i=1}^{r} \alpha_i(n) \frac{z_{n-m_i}}{z_n},$$

or

$$\frac{1}{\gamma_{n+1}} \leqslant 1 + \delta\Big[\gamma_{n-\tau+1} \cdots \gamma_n - \gamma_{n-\tau+2} \cdots \gamma_n\Big] - \sum_{i=1}^{r} \alpha_i(n) \prod_{\ell=0}^{m_i} \gamma_{n-\ell}. \tag{15}$$

Setting $\liminf_{n\to\infty} \gamma_n = \beta$, we get $\beta \geqslant 1$. It is clear that β is finite. From (15) we have

$$\limsup_{n\to\infty} \frac{1}{\gamma_{n+1}} = \frac{1}{\beta} \leqslant 1 + \delta\beta^{\tau-1}(\beta - 1) - \sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n) \cdot \beta^{m_i},$$

$$\sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n) \cdot \beta^{m_i+1} \leqslant 1 + \delta\beta^{\tau-1}(\beta - 1) - \frac{1}{\beta} = (\beta - 1)[\frac{1}{\beta} + \delta\beta^{\tau-1}],$$

$$\sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n) \leqslant \frac{\beta - 1}{\beta^{\tilde{m}+2}}(1 + \delta\beta^{\tau}) = f(\beta).$$

By Lemma 2, we have

$$\sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n) \leqslant f(\beta^*) = \frac{\beta^* - 1}{\beta^{*\tilde{m}+2}}(1 + \delta\beta^{*\tau}),$$

which contradicts condition (13). Hence, (3) has no eventually positive solution.

Theorem 4. *Suppose that*

$$-\frac{1}{\delta+1}\frac{(\tau - m_*)^{\tau - m_*}}{(\tau - m_* - 1)^{\tau - m_* - 1}} \sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n) > 1, \tag{16}$$

where $\alpha_i(n) \leqslant \alpha_i(n-\tau)$ *for* n *sufficient large;* $\delta < -1$, $m_* = \max_{1\leqslant i\leqslant r} m_i$, $\tau > m_* + 1$ *and* $\sum_{\ell=1}^{\infty}[\sum_{i=1}^{r} \alpha_i(\ell)] = \infty$. *Then, (3) is oscillatory.*

Proof. Assume the contrary. Without loss of generality, let $\{x_n\}$ be an eventually positive solution of (3). By the case (b) of Lemma 1, we have $z_n < 0$ and $\Delta z_n \leqslant 0$. Putting

$$w_n = z_n + \delta z_{n-\tau},$$

we have

$$w_n = z_n + \delta z_{n-\tau} \leqslant (1+\delta) z_{n-\tau},$$

which is the same as

$$z_{n-\tau} \leqslant \frac{1}{\delta+1} w_n.$$

Therefore, it follows that

$$\begin{aligned}
\Delta w_n &= \Delta z_n + \delta \Delta z_{n-\tau} \\
&= -\sum_{i=1}^{r} \alpha_i(n) x_{n-m_i} - \delta \sum_{i=1}^{r} \alpha_i(n-\tau) x_{n-\tau-m_i}, \\
&\geqslant -\sum_{i=1}^{r} \alpha_i(n) x_{n-m_i} - \delta \sum_{i=1}^{r} \alpha_i(n) x_{n-\tau-m_i}, \\
\Delta w_n &\geqslant -\sum_{i=1}^{r} \alpha_i(n)(x_{n-m_i} + \delta x_{n-\tau-m_i}), \\
\Delta w_n &\geqslant -\sum_{i=1}^{r} \alpha_i(n) z_{n-m_i} \geqslant 0,
\end{aligned}$$

so we get

$$0 \leqslant \Delta w_n + \sum_{i=1}^{r} \alpha_i(n) z_{n-m_i} \leqslant \Delta w_n + \frac{1}{\delta+1} \sum_{i=1}^{r} \alpha_i(n) w_{n-m_i+\tau}.$$

Setting $\gamma_n = \frac{w_{n+1}}{w_n}$, we obtain

$$\gamma_n \geqslant 1 - \frac{1}{\delta+1} \sum_{i=1}^{r} \alpha_i(n) \prod_{\ell=1}^{\tau-m_i} \gamma_{n-m_i+\tau-\ell}. \tag{17}$$

Putting $\beta = \liminf_{n\to\infty} \gamma_n$, we have $\beta \geqslant 1$. Taking lower limit on both sides of (17), we obtain

$$\beta \geqslant 1 - \frac{1}{\delta+1}\sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n) \cdot \beta^{\tau-m_i},$$

or

$$\beta - 1 \geqslant -\frac{1}{\delta+1}\sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n) \cdot \beta^{\tau-m_i}. \tag{18}$$

Since

$$\beta^{\tau-m_i} \geqslant \beta^{\tau-m_*}, \quad \forall i = \overline{1,r},$$

$$-\frac{1}{\delta+1}\liminf_{n\to\infty} \alpha_i(n)\beta^{\tau-m_i} \geqslant -\frac{1}{\delta+1}\liminf_{n\to\infty} \alpha_i(n)\beta^{\tau-m_*}, \quad \forall i = \overline{1,r}.$$

From (18) we get

$$-\frac{1}{\delta+1}\sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n) \leqslant \frac{\beta-1}{\beta^{\tau-m_*}}.$$

But

$$\frac{\beta-1}{\beta^{\tau-m_*}} \leqslant \frac{(\tau-m_*-1)^{\tau-m_*-1}}{(\tau-m_*)^{\tau-m_*}},$$

so

$$-\frac{1}{\delta+1}\frac{(\tau-m_*)^{\tau-m_*}}{(\tau-m_*-1)^{\tau-m_*-1}}\sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n) \leqslant 1,$$

which contradicts condition (16). Hence, (3) has no eventually positive solution.

Theorem 5. *Suppose that*

$$-\frac{1}{\delta}\frac{(\tau-m_*)^{\tau-m_*}}{(\tau-m_*-1)^{\tau-m_*-1}}\sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n) > 1, \tag{19}$$

where $\delta < -1$, $m_* = \max_{1\leqslant i\leqslant r} m_i$, $\tau > m_* + 1$ *and* $\sum_{\ell=1}^{\infty}[\sum_{i=1}^{r} \alpha_i(\ell)] = \infty$. *Then, (3) is oscillatory.*

Proof. Suppose to the contrary, and let $\{x_n\}$ be an eventually positive solution of (3). Put $z_n = x_n + \delta x_{n-\tau}$. By the case (b) of Lemma 1, we obtain $z_n < 0$ and $\Delta z_n \leqslant 0$. On the other hand, we have $z_n > \delta x_{n-\tau}$ or $x_{n-\tau} > \frac{1}{\delta} z_n$, which implies that $x_{n-m_i} > \frac{1}{\delta} z_{n+\tau-m_i}$. Hence,

$$\Delta z_n \leqslant -\frac{1}{\delta}\sum_{i=1}^{r} \alpha_i(n) z_{n+\tau-m_i}. \tag{20}$$

Setting $v_n = \frac{z_{n+1}}{z_n}$ and dividing (20) by z_n, we obtain

$$v_n \geqslant 1 - \frac{1}{\delta}\sum_{i=1}^{r} \alpha_i(n) \frac{z_{n+\tau-m_i}}{z_n},$$

or

$$v_n \geqslant 1 - \frac{1}{\delta}\sum_{i=1}^{r} \alpha_i(n) \prod_{\ell=0}^{\tau-m_i-1} \frac{z_{n+\tau-m_i-\ell}}{z_{n+\tau-m_i-\ell-1}}. \tag{21}$$

Taking lower limit on both sides of (21) and putting $\beta = \liminf_{n\to\infty} v_n$, we have $\beta \geqslant 1$ and

$$\beta - 1 \geqslant -\frac{1}{\delta}\sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n)\cdot \beta^{\tau - m_i}.$$

We can prove

$$-\frac{1}{\delta}\frac{(\tau - m_*)^{\tau - m_*}}{(\tau - m_* - 1)^{\tau - m_* - 1}}\sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n) \leqslant 1$$

similarly as the proof of Theorem 4, which contradicts condition (19). Hence, (3) has no eventually positive solution.

2.2. *The Convergence*

We give conditions implying that every nonoscillatory solution is convergent. To begin with, we have

Lemma 3. *Let $\{x_n\}$ be a nonoscillatory solution of (2). Put $z_n = x_n + \delta x_{n-\tau}$.*

- *(a) If $\{x_n\}$ is eventually positive (negative), then $\{z_n\}$ is eventually nonincreasing (nondecreasing).*
- *(b) If $\{x_n\}$ is eventually positive (negative) and there exists a constant γ such that*

$$-1 < \gamma \leqslant \delta, \tag{22}$$

then eventually $z_n > 0$ ($z_n < 0$).

Proof. Let $\{x_n\}$ be an eventually positive solution of (2). The case $\{x_n\}$ is an eventually negative solution of (2) can be considered similarly.

(a) We have $\Delta z_n = -\sum_{i=1}^{r} \alpha_i(n)F(x_{n-m_i}) \leqslant 0$ for all large n. Thus, $\{z_n\}$ is eventually nonincreasing.

(b) Suppose the conclusion does not hold, then since by (a) $\{z_n\}$ is nonincreasing, it follows that eventually either $z_n \equiv 0$ or $z_n < 0$. Now $z_n \equiv 0$ implies that $\Delta z_n = -\sum_{i=1}^{r} \alpha_i(n)F(x_{n-m_i}) \equiv 0$, but this contradicts the fact that $\alpha_i(n) \not\equiv 0$ for infinitely many n. If $z_n < 0$, then $x_n < -\delta x_{n-\tau}$ so $\delta < 0$. From (22) it follows that $-1 < \gamma < 0$ and $x_n < -\gamma x_{n-\tau}$. Thus, by induction, we obtain $x_{n+j\tau} \leqslant (-\gamma)^j x_n$ for all positive integers j. Hence, $x_n \to 0$ as $n \to \infty$. It implies that $\{z_n\}$ decreases to zero as $n \to \infty$. This contradicts the fact that $z_n < 0$.

Theorem 6. *Assume that*

$$\sum_{\ell=1}^{\infty}\sum_{i=1}^{r} \alpha_i(\ell) = \infty, \tag{23}$$

and there exists a constant η such that

$$-1 < \eta \leqslant \delta \leqslant 0. \tag{24}$$

Suppose further that, if $|x| \geqslant c$ then $|F(x)| \geqslant c_1$ where c and c_1 are positive constants. Then, every nonoscillatory solution of (2) tends to 0 as $n \to \infty$.

Proof. Let $\{x_n\}$ be an eventually positive solution of (2), say $x_n > 0, x_{n-\tau} > 0$ and $x_{n-m_i} > 0$ for $n \geqslant n_0 \in \mathbb{N}$. Put $z_n = x_n + \delta x_{n-\tau}$. We first prove that $z_n \to 0$ as $n \to \infty$. Note that (24) implies (22) with γ replace by η. By Lemma 3 we have $\{z_n\}$ is eventually positive and nonincreasing. Therefore, there exists $\lim_{n\to\infty} z_n$. Put $\lim_{n\to\infty} z_n = \beta$. Now, suppose that $\beta > 0$. By (24), we have $z_n \leqslant x_n$. Thus, there exists an integer $n_1 \geqslant n_0 \in \mathbb{N}$ such that

$$\beta \leqslant z_{n-m_i} \leqslant x_{n-m_i}, \quad \forall n \geqslant n_1, i = 1, \cdots, r.$$

Hence,

$$\Delta z_n = -\sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}) \leqslant -M \sum_{i=1}^{r} \alpha_i(n), \quad \forall n \geqslant n_1$$

for some positive constant M. Summing the last inequality, we obtain

$$z_n \leqslant z_{n_1} - M \sum_{\ell=n_1}^{n-1} \sum_{i=1}^{r} \alpha_i(\ell),$$

which as $n \to \infty$, in view of (23), implies that $z_n \to -\infty$. This is a contradiction.

Since $\lim_{n\to\infty} z_n = 0$, there exists a positive constant A such that $0 < z_n \leqslant A$ and so, by (24) we have

$$x_n \leqslant -\eta x_{n-\tau} + A. \tag{25}$$

Assume that $\{x_n\}$ is not bounded. Then, there exists a subsequence $\{n_k\}$ of $\mathbb{N}$, so that $\lim_{k\to\infty} x_{n_k} = \infty$ and $x_{n_k} = \max_{n_0 \leqslant j \leqslant n_k} x_j, \quad k = 1, 2, \cdots$. From (25), for k sufficiently large, we get

$$x_{n_k} \leqslant -\eta x_{n_k} + A$$

and so

$$(1+\eta) x_{n_k} \leqslant A,$$

which as $k \to \infty$ leads to a contradiction.

Now suppose that $\limsup_{n\to\infty} x_n = \alpha > 0$. Then, there exists a subsequence $\{n_k\}$ of $\mathbb{N}$, with n_1 large enough so that $x_n > 0$ for $n > n_1 - \tau$ and $x_{n_k} \to \alpha$ as $k \to \infty$. Then, from (24), we have

$$z_{n_k} \geqslant x_{n_k} + \eta x_{n_k - \tau}$$

and so

$$x_{n_k-\tau} \geqslant -\frac{1}{\eta}(x_{n_k} - z_{n_k}).$$

As $k \to \infty$, we obtain

$$\alpha \geqslant \lim_{k\to\infty} x_{n_k-\tau} \geqslant -\frac{\alpha}{\eta}.$$

Since $-\eta \in (0,1)$, it follows that $\alpha = 0$, i.e. $x_n \to 0$ as $n \to \infty$. The arguments when $\{x_n\}$ is an eventually negative solution of (2) is similar.

Theorem 7. *Suppose there exists positive constants M, $\alpha_i, i = 1, 2, \cdots, r$ such that*

$$\alpha_i(n) \geqslant \alpha_i, \quad i = 1, 2, \cdots, r, \quad \forall n \in \mathbb{N}, \tag{26}$$

$$|F(x)| \geqslant M|x|, \quad \forall x \in \mathbb{R}, \tag{27}$$

$$\delta \geqslant 0. \tag{28}$$

Then, every nonoscillatory solution of (2) tends to 0 *as* $n \to \infty$.

Proof. Let $\{x_n\}$ be an eventually positive solution of (2), say $x_n > 0, x_{n-\tau} > 0$ and $x_{n-m_i} > 0$ for $n \geqslant n_0 \in \mathbb{N}$. By Lemma 3, $\{z_n\}$ is eventually positive and nonincreasing, so there exists $\lim_{n\to\infty} z_n$. Put $\lim_{n\to\infty} z_n = \beta$. Summing the equation (2) from n to ∞ for $n \geqslant n_0$, we obtain

$$z_n = \beta + \sum_{\ell=n}^{\infty}\sum_{i=1}^{r} \alpha_i(\ell)F(x_{\ell-m_i}), \quad n \geqslant n_0.$$

Now by (26) and (27), we get

$$\alpha M \sum_{\ell=n}^{\infty}\sum_{i=1}^{r} \alpha_i(\ell)x_{\ell-m_i} \leqslant \sum_{\ell=n}^{\infty}\sum_{i=1}^{r} \alpha_i(\ell)F(x_{\ell-m_i}) \leqslant z_n - \beta < \infty,$$

which implies that $x_n \to 0$ as $n \to \infty$. The proof is similar when $\{x_n\}$ is eventually negative.

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