PROBABILITY MEASURES WITH FINITE SUPPORTS ON TOPOLOGICAL SPACES

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Abstract. Let X be a topological Hausdorff space. For each $k \in N$, by $P_k(X)$ we denote the set of all probability measures on X, whose supports of no more than k points. Then probability measure functors P_k preserve some topological properties: compactness, regularity, contractiveness...

I. Introduction

In [2] Fedorchuk introduced the concept of probability measure functors with finite supports and proved that these functors preserve the ANR-property of compact metric spaces. Therefore by Torunczyk's Theorem [4] they preserved the topology of Q-manifolds.

In this paper we study the action of probability measure functors on topological spaces. Our results show that numbers of topological properties are preserved under action of probability measure functors.

II. Probability measure with supports

Let X be a topological Hausdorff space. A probability measure with finite supports on X is a function $\mu : X \to [0,1]$ satisfying the conditions:

$$\text{Supp}\mu = \{x \in X : \mu(x) > 0\} \text{ is finite,} \tag{1}$$

$$\sum_{x \in \text{Supp}\mu} \mu(x) = 1 \tag{2}$$

for each $k \in N$. Let $P_k(X)$ denote the set of all probability measures on X, whose supports of no more than k points. Then every $\mu \in P_k(X)$ can be written in the form

$$\mu = \sum_{i=1}^{q} m_i \delta_{x_i},\ q \leq k,$$

where δ_x is Dirac function, that is

$$\delta_x(y) = \begin{cases} 0 & \text{if } y \neq x \\ 1 & \text{if } y = x \end{cases}$$

and $m_i = \mu(x_i) > 0$, $\sum_{i=1}^{q} m_i = 1$. m_i called the *mass* of μ at x_i.

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Fedorchuk [2] introduced a topology on $P_k(X)$ as follows:

Each point $\mu_0 = \sum_{i=1}^{q} m_i^0 \delta_{x_i^0} \in P_k(X)$ has a neighborhood of the form $O\langle \mu_0, U_1, U_2, \ldots, U_q, \varepsilon \rangle$, where $\varepsilon > 0$, $U_1, U_2, \ldots, U_q$ are disjoint neighborhoods of $x_1^0, x_2^0, \ldots, x_q^0$ respectively (note that U_i can be taken from a fixed basic of topology of X). We have

$$O\langle \mu_0, U_1, U_2, \ldots, U_q, \varepsilon \rangle = \Big\{ \mu \in P_k(X) : \ \mu = \sum_{i=1}^{q+1} \mu_i, \ \mathrm{Supp}\mu \subset U_i, \\ |m_i^0 - \|\mu_i\|| < \varepsilon, \ i = 1, 2, \ldots, q+1; \\ U_{q+1} = X \setminus \bigcup_{i=1}^{q} U_i, \ m_{q+1}^0 = 0 \Big\}.$$

Observe that μ_i is not necessarily a probability measure, because in general $\|\mu_i\| \neq 1$.

It is easy to see that the family $\{O\langle \mu, U_1, U_2, \ldots, U_q, \varepsilon \rangle\}$ forms a basis of topology of $P_k(X)$. This topology is called *Fedorchuk topology.*

III. The results

Our aim is to indicate topological invariants preserved under the action of P_k. The following example shows that not every topological invariant is preserved under the action of probability measure functors.

3.1. Example. Let X be discrete space having more than one point, then $P_k(X)$ is not a discrete space for every $k \geq 2$.

Proof In fact let $x_1, x_2 \in X$ and define $G(x_1, x_2)$ by the formula

$$G(x_1, x_2) = \{\mu = m_1\delta_{x_1} + m_2\delta_{x_2} : m_1 + m_2 = 1\}.$$

Then it is easy to see that set

$$G(x_1, x_2) = \left\{\mu \in G_1(x_1, x_2) : \mu = m_1\delta_{x_1} + m_2\delta_{x_2} : m_1 < \frac{1}{2}\right\}$$

is closed but not open in $P_k(X)$ for every $k \geq 2$. Therefore $P_k(X)$ is not discrete space.

Thus the discrete property is a topological invariant which is not preserved under the action of P_k. However, we shall see that the functors P_k preserve plenty of topological invariant.

Namely, our results are the following.

3.2. Theorem. *Let X be a topological Hausdorff space. Let $P_k(X)$ denote the space of all probability measures whose supports consist of no more than k points equipped with the Fedorchuk topology. Then*

(1) $P_k(X)$ is Hausdorff space ;

(2) If X is completely regular then so is $P_k(X)$;

(3) If X is separable then so is $P_k(X)$;

(4) *If* X *is compact then so is* $P_k(X)$;

(5) *If* X *is contractible then so is* $P_k(X)$;

(6) *If* X *homeomorphic to* Y *then* $P_k(X)$ *is homeomorphic to* $P_k(Y)$;

(7) *If* X *path connected then so is* $P_k(X)$;

(8) *If* X *satisfies the first axiom of countability then so does* $P_k(X)$;

(9) *If* X *satisfies the second axiom of countability then so does* $P_k(X)$.

4. Proof of the results

Proof of (1). Let $\mu, \mu' \in P_k(X)$ with $\mu \neq \mu'$. Write μ, μ' in the forms

$$\mu = \sum_{i=1}^{p} m_i \delta_{x_i},\ \mu' = \sum_{j=1}^{q} m'_j \delta_{x_i},\ p, q \leq k.$$

We consider two cases:

Case 1

$$\text{Supp}\mu \neq \text{Supp}\mu'.$$

Without loss of generality we may assume that $x_1 \neq x'_1$. Since X is Hausdorff space we can choose a neighborhoods U_1 of x_1, and U'_1 of x'_1 such that $U_1 \cap U'_1 = \emptyset$ and $U_i \cap U_j = \emptyset$ if $i \neq j$; $i, j = 1, ..., p$ and $U'_i \cap U'_j = \emptyset$ if $i \neq j$; $i, j = 1, ..., q$.

Let $\varepsilon < \frac{1}{4} \min\{m_1, m_2, ..., m_p, m'_1, ..., m'_q\}$. Let us

$$O = O\langle \mu, U_1, U_2, \ldots, U_p, \varepsilon \rangle,$$

$$O' = O\langle \mu', U'_1, U'_2, \ldots, U'_q, \varepsilon \rangle.$$

We shall show that $O \cap O' = \emptyset$.

In fact, if it is not the case then there exists

$$\mu^* \in O \cap O'.$$

Since $\mu^* \in O\langle \mu, U_1, U_2, \ldots, U_p, \varepsilon \rangle$, we have

$$\mu^* = \sum_{i=1}^{p+1} \mu_i,\ \text{Supp}\mu_i \subset U_i,\ |\|\mu_i\| - m_i| < \varepsilon,\ i = 1, 2, ..., p+1. \tag{1}$$

On the other hand, since $\mu^* \in O\langle \mu', U'_1, U'_2, \ldots, U'_q, \varepsilon \rangle$, μ^* can be written in the form

$$\mu^* = \sum_{j=1}^{q+1} \mu'_j,\ \text{Supp}\mu'_j \subset U'_j,\ |\|\mu'_j\| - m'_j| < \varepsilon,\ j = 1, 2, ..., q+1. \tag{2}$$

Since $U_1 \cap U'_j = \emptyset$ for $j = 1, 2, ..., q$, it follows that

$$\text{Supp}\mu_1 \subset \text{Supp}\mu'_{q+1}.$$

Therefore
$$\|\mu_1\| \leq \|\mu'_{q+1}\| < \varepsilon. \tag{3}$$
On the other hand from (1) we have
$$\|\mu_1\| > m_1 - \varepsilon. \tag{4}$$
From (3) and (4) we get
$$m_1 - \varepsilon < \varepsilon \text{ or } m_1 < 2\varepsilon.$$
Since $\varepsilon < \frac{1}{4}m_1$, we have $m_1 < \frac{1}{2}m_1$, a contradiction.

Case 2
$$\text{Supp}\mu = \text{Supp}\mu'.$$
Since $\mu \neq \mu'$, there exists at least an index i such that $m_i \neq m'_i$. Without loss of generality we may assume that $m_1 \neq m'_1$.

Let $\varepsilon < \frac{1}{4}|m_1 - m'_1| > 0$, and choose disjoint neighborhoods U_i of x_i, $i = 1, 2, ..., p$ and put
$$O = O\langle\mu, U_1, U_2, \ldots, U_p, \varepsilon\rangle,$$
$$O' = O\langle\mu', U'_1, U'_2, \ldots, U'_p, \varepsilon\rangle.$$
We shall show that $O \cap O' = \emptyset$.

In fact, assume on the contrary that $O \cap O' \neq \emptyset$. Then there exists $\mu^* \in O \cap O'$. Since $\mu^* \in O$, we have
$$\mu^* = \sum_{i=1}^{p+1} \mu_i,\ \text{Supp}\mu_i \subset U_i,\ |\|\mu_i\| - m_i| < \varepsilon,\ i = 1, 2, ..., p+1. \tag{5}$$
On the other hand since $\mu^* \in O'$, we infer that
$$\mu^* = \sum_{i=1}^{p+1} \mu_i,\ \text{Supp}\mu_i \subset U_i,\ |\|\mu_i\| - m'_i| < \varepsilon,\ i = 1, 2, ..., p+1. \tag{6}$$
From (5) and (6) we get
$$|m_1 - m'_1| = |m_1 - \|\mu_1\| + \|\mu_1\| - m'_1| \leq |m_i - \|\mu_1\|| + |\|\mu_1\| - m'_1|$$
$$= 2 \cdot \frac{1}{4}|m_1 - m'_1| = \frac{1}{2}|m_1 - m'_1|.$$
This contradiction shows that
$$O \cap O' = \emptyset.$$
Therefore (1) is proved.

Proof of (2) Assume that X is completely regular and let
$$\mu_0 = \sum_{i=1}^{p} m_i^0 \delta_{x_i^0} \in P_k(X),$$

and let

$$O = O\langle \mu_0, U_1, U_2, \dots, U_p, \varepsilon\rangle$$

be a neighborhood of μ_0, where $\varepsilon > 0$. We shall show that there exists a map $F : P_k(X) \to [0,1]$ such that

$$F(\mu_0) = 1 \text{ and } F(P_k(X) \setminus O) = O. \tag{7}$$

We may assume that $\varepsilon < \frac{1}{4}\min\{m_1^0, m_2^0, ..., m_p^0\}$. Since X is completely regular, for every $i = 1, 2, ..., p$ there exists a map $f_i : X \to [0,1]$ such that

$$f_i(x_i^0) = 1 \text{ and } f_i(X \setminus U_i) = 0,\ i = 1, 2, ..., p. \tag{8}$$

Define $\widehat{f}_i : P_k(X) \to [0,1]\, i = 1, 2, ..., p$ by the formula

$$\begin{aligned} &\widehat{f}_i(\mu) = \sum_{j=1}^{p} m_j f_i(x_j) \text{ for } \mu = \sum_{j=1}^{p} m_j \delta_{x_j} \in P_k(X) \quad \text{and} \\ &\widehat{f_{p+1}}(\mu) = 1 \text{ for every } \mu \in P_k(X). \end{aligned} \tag{9}$$

Let $V_i = (m_i^0 - \varepsilon, m_i^0 + \varepsilon),\ i = 1, 2, ..., p$ and let $\varphi_i : [0,1] \to [0,1]$ be Urynsohn function satisfying the conditions

$$\begin{aligned} &\varphi_i(m_i^0) = 1 \text{ and } \varphi_i([0,1] \setminus V_i) = 0 \text{ for every } i = 1, 2, ..., p \text{ and} \\ &\varphi_{p+1}(O) = 1,\ \varphi_{p+1}([\varepsilon, 1]) = 0. \end{aligned} \tag{10}$$

Observe that for every $\mu \in P_k(X)$ we have

$$\mu = \sum_{i=1}^{p+1} \mu_i,$$

where $\mu_i = \mu|_{U_i},\ i = 1, 2, ..., p+1,\ U_{p+1} = X \setminus \bigcup_{i=1}^{p} U_i$.

Define $F : P_k(X) \to [0,1]$ by the formula

$$F(\mu) = \frac{1}{m} \prod_{i=1}^{p+1} \varphi_i(\|\mu_i\|) \widehat{f}_i(\mu), \tag{11}$$

where $m = m_1^0 \cdots m_p^0$.

Since $\varphi_i, \widehat{f}_i,\ i = 1, 2, ..., p+1$ are continuous functions, we infer that F is continuous. It is easy to see that

$$F(\mu) \in [0,1] \text{ for every } \mu \in P_k(X).$$

Observe that

$$\begin{aligned} F(\mu_0) &= \frac{1}{m} \prod_{i=1}^{p+1} \varphi_i(m_i^0) \widehat{f}_i(\mu_0) = \frac{1}{m} \prod_{i=1}^{p} m_i^0 \\ &= \frac{m_1^0 \cdots m_p^0}{m_1^0 \cdots m_p^0} = 1. \end{aligned}$$

On the other hand if $\mu = \sum_{i=1}^{p+1} \mu_i \notin O\langle \mu_0, U_1, \ldots, U_p, \varepsilon \rangle$, then there exists an index $i \leq p+1$ such that $|m_i^0 - \|\mu_i\|| \geq \varepsilon$. If $i \leq p$ then by (10) we have $\varphi_i(\|\mu_i\|) = 0$.

From (11) we infer that $F(\mu) = 0$.

If $i = p+1$ then $\|\mu_{p+1}\| \geq \varepsilon$. Therefore from (10) we infer that $\varphi_{p+1}(\|\mu_{p+1}\|) = 0$. Consequently from (11) it follows that $F(\mu) = 0$. Therefore (2) is proved.

Proof of (3). Now we shall prove the separability of $P_k(X)$.

Let $A = \{x_n\}_{n=1}^{\infty}$ be a countable dense subset in X.

Put

$$Q = \left\{ \mu \in P_k(X) : \mu = \sum_{i=1}^{p} m_i \delta_{x_i},\ p \leq k,\ x_i \in A \text{ and } m_i \text{ rational numbers with} \sum_{i=1}^{p} m_i = 1 \right\}.$$

It is easy to see that Q is countable. It remains to prove Q is dense in $P_k(X)$.

Let $\mu_0 = \sum_{i=1}^{p} m_i^0 \delta_{x_i^0} \in P_k(X)$ and let $O\langle \mu_0, U_1, \ldots, U_p, \varepsilon \rangle$ be an arbitrary neighborhood of μ_0. Choose $x_i \in U_i \cap A$ for $i = 1, 2, \ldots, p$ and rational numbers m_i such that

$$0 \leq m_i^0 - m_i < \frac{\varepsilon}{2(k+1)}, i = 1, 2, \ldots, p+1$$

and

$$m_p = 1 - \sum_{i=1}^{p-1} m_i.$$

We shall show that

$$\mu = \sum_{i=1}^{p} m_i \, \delta_{x_i} \ \in \ O\langle \mu_0,\ U_1, \ldots,\ U_p, \varepsilon \ \rangle \cap Q.$$

Obviously $\mu \in Q$. Observe that $x_i \in U_i$, $i = 1, 2, \ldots, p$ and

$$0 \leq m_i^0 - m_i < \frac{\varepsilon}{2(k+1)} < \varepsilon,\ i = 1, 2, \ldots, p-1.$$

For $i = p$ we have

$$|m_p - m_p^0| = \left| 1 - \sum_{i=1}^{p-1} m_i - m_p^0 \right| = \left| \sum_{i=1}^{p} m_i^0 - \sum_{i=1}^{p-1} m_i - m_p^0 \right| =$$

$$= \left| \sum_{i=1}^{p-1} (m_i^0 - m_i) \right| \leq \sum_{i=1}^{p-1} (m_i^0 - m_i) < \frac{\varepsilon}{2(k+1)} < \varepsilon$$

Morever

$$\sum_{i=1}^{p} m_i = \sum_{i=1}^{p-1} m_i + \left(1 - \sum_{i=1}^{p-1} m_i \right) = 1.$$

It follows that $\mu \in O\langle \mu_0, U_1, ..., U_p, \varepsilon \rangle$. Therefore Q is dense in $P_k(X)$ and (3) is proved.
Proof of (4). Let $\{\mu_\alpha\}_{\alpha \in I}$, where I is a directed set, be an arbitrary net in $P_k(X)$. Hence

$$\mu_\alpha = \sum_{i=1}^{p} m_i^\alpha \delta_{x_i^\alpha}.$$

Since X is compact, we may assume that for every $i = 1, 2, ..., p$ there exists a subnet $x_i^{\alpha_p} \to x_i$ for $i = 1, 2, ..., p$.

By the compactness of $[0,1]$, we may assume that $m_i^{\alpha_p} \to m_i$ for $i = 1, 2, ..., p$. Therefore

$$\mu_{\alpha_\beta} \to \mu = \sum_{i=1}^{p} m_i \delta_{x_i}.$$

Observe that

$$\sum_{i=1}^{p} m_i = \sum_{i=1}^{p} \lim_\beta m_i^{\alpha_\beta} = \lim_\beta \sum_{i=1}^{p} m_i^{\alpha_\beta} = 1.$$

Consequently $\mu \in P_k(X)$ and hence $P_k(X)$ is compact.
Proof of (5). Now we assume that X is contractible. Then there exists a map $\varphi : X \times [0,1] \to X$ such that

(i) $\varphi(x, 0) = x$ for every $x \in X$;

(ii) $\varphi(x, 1) = a$ for every $x \in X, a \in X$ is a fixed point.

We define a map $\Phi : P_k(X) \times [0,1] \to P_k(X)$ by the formula

$$\Phi(\mu, t) = \Phi\left(\sum_{i=1}^{p} m_i \delta_{x_i}, t\right) = \sum_{i=1}^{p} m_i \delta_{\varphi(x_i, t)}$$

for every $\mu = \sum_{i=1}^{p} m_i \delta_{x_i} \in P_k(X)$.

Then we have

$$\Phi(\mu, t) \in P_k(X),$$

for every $\mu \in P_k(X)$,

$$\Phi(\mu, 0) = \sum_{i=1}^{p} m_i \delta_{\varphi(x_i, 0)} = \sum_{i=1}^{p} m_i \delta_{x_i} = \mu,$$

and

$$\Phi(\mu, 1) = \sum_{i=1}^{p} m_i \delta_{\varphi(x_i, 1)} = \sum_{i=1}^{p} m_i \delta_a = 1 \cdot \delta_a \in P_k(X).$$

It is easy to see that Φ is continuous and therefore $P_k(X)$ is contractible.
Proof of (6). Assume that X is locally contractible.

Let $\mu_0 = \sum_{i=1}^{p} m_i^0 \delta_{x_i^0} \in P_k(X)$ and let $O\langle \mu_0, U_1, ..., U_p, \varepsilon \rangle$ be a neighborhood of μ_0. By the local contractibility of X, for each $i = 1, 2, ..., p$ there exists a neighborhood $U_i^0 \subset U_i$ and map

$$\varphi_i : U_i^0 \times [0,1] \to U_i$$

such that

$$\varphi_i(x,0) = x \quad \text{for every} \quad x \in U_i^0,$$
$$\varphi_i(x,1) = x_i^* \quad \text{for every} \quad x \in U_i^0, \text{ where } x_i^* \in U_i^0.$$

Denote $\varphi_{p+1}(x,t) = x$ for every $x \in X$ and $t \in [0,1]$. Put

$$\mu_0 = \sum_{i=1}^{p} m_i^0 \delta_{x_i^0} \quad \text{and} \quad O^0 = O\left\langle \mu_0, U_1^0, ..., U_p^0, \frac{\varepsilon}{2(k+1)} \right\rangle.$$

Obviously $O^0 \subset O$.

Define $F : O^0 \times [0,1] \to O$ by the formula

$$F(\mu,t) = \sum_{i=1}^{p+1} \sum_{x_j \in U_i^0} m_j \delta_{\varphi_i}(x_j,t)$$

for every $\mu = \sum_{j=1}^{q} m_j \delta_{x_j} \in O^0$, where $U_{p+1}^0 = X \setminus \bigcup_{i=1}^{p} U_i^0$. It is easy to see that

$$F(\mu,t) \in O\langle \mu_0, U_1, ..., U_p, \varepsilon \rangle.$$

Observe that

$$F(\mu,0) = \sum_{i=1}^{p+1} \sum_{x_j \in U_i^0} m_j \delta_{\varphi_i}(x_j,0) = \sum_{i=1}^{p+1} \sum_{x_j \in U_i^0} m_j \delta_{x_j} = \sum_{i=1}^{p} \mu_i = \mu.$$

$$F(\mu,1) = \sum_{i=1}^{p+1} \sum_{x_j \in U_i^0} m_j \delta_{\varphi_i}(x_j,1) = \sum_{i=1}^{p+1} \left(\sum_{x_j \in U_i^0} m_j \right) \delta_{x_j}.$$

Therefore, denoting $m_i^* = \sum_{x_j \in U_i^0} m_j$ and $\mu^* = \sum_{i=1}^{p+1} m_i^* \delta_{x_i^*}$ we obtain

$$F(\mu,1) = \mu^* \in O^0.$$

Therefore $P_k(X)$ is locally contractible. This proved (6).

Proof of (7). Since X is homeomorphic to Y there exists a homeomorphism $f : X \to Y$. For every $\mu = \sum_{i=1}^{p} m_i \delta_{x_i}$, we define $F(\mu)$ by the formula

$$F(\mu) = \sum_{i=1}^{p} m_i \delta_{f(x_i)}.$$

It is easy to show that F is a homeomorphism and (7) is proved.

Proof of (8). Assume that X is path connected. Let

$$\mu_1 = \sum_{i=1}^{q} m_i \delta_{x_i} \in P_k(X),$$

$$\mu_2 = \sum_{i=1}^{q} n_i \delta_{y_i} \in P_k(X).$$

Since X is path connected, for each $i = 1, 2, ..., q$ there exists a map

$$g_i(0) = x_i \quad , \quad g_i(1) = y_i.$$

For every $i = 1, 2, ..., q$, let $f_i : [0,1] \to [0,1]$ be a map

$$f_i(0) = m_i \quad , \quad f_i(1) = n_i.$$

Let

$$m_i(t) = \frac{f_i(t)}{\sum_{i=1}^{q} f_i(t)}$$

for $i = 1, 2, ..., q$.

We define $F : [0,1] \to P_k(X)$ by the formula

$$F(t) = \sum_{i=1}^{q} m_i(t) \delta_{g_i(t)}.$$

It is easy to see that F is continuous and $\sum_{i=1}^{q} m_i(t) = 1$ for every $t \in [0,1]$. Therefore $F(t) \in P_k(X)$.

Observe that

$$F(0) = \sum_{i=1}^{q} m_i(0) \delta_{g_i(0)} = \sum_{i=1}^{q} m_i \delta_{x_i} = \mu_1,$$

$$F(1) = \sum_{i=1}^{q} m_i(1) \delta_{g_i(1)} = \sum_{i=1}^{q} n_i \delta_{y_i} = \mu_2.$$

Therefore $P_k(X)$ is path connected.

Proof of (9). Let $\mu_0 = \sum_{i=1}^{p} m_i^0 \delta_{x_i^0} \in P_k(X)$. We have to show that μ_0 has a countable basis of neighborhoods. Since X satisfies the first axiom of countability for every $x_i^0 \in X$, there exists a countable basis of neighborhoods of x_i^0, denoted by $\{U_i^n\}$, $i = 1, 2, ..., p$.

We put

$$O^n = O\left\langle \mu_0, U_1^n, ..., U_p^n, \frac{1}{n} \right\rangle, n \in N, U_i^n \in \{U_i^n\}.$$

It is easy to see that $\{O^n\}$ is countable basis of neighborhoods of μ_0 and (9) is proved. *Proof of (10).* Finally we assume that X satisfies the second axiom, infer X is separable (see [1]). Thus by (5) $P_k(X)$ is separable.

Let $\beta = \{O_m\}_{n=1}^{\infty}$ be a countable basis of X and $\{\mu_m\}_{m=1}^{\infty}$ be a dense set of $P_k(X)$. Denote

$$O_m^n = O\left\langle \mu_m, U_1^n, ..., U_p^n, \frac{1}{n} \right\rangle, n, m \in N.$$

It is easy to see that $\{O_m^n\}_{m,n=1}^{\infty}$ is countable basis of topology of $P_k(X)$ and the theorem is proved.

4.1. Corollary. *If X compact metric space then $P_k(X)$ is compact metrizable.*

Proof. By (4) $P_k(X)$ is compact. Being a compact metric space, X is separable. Therefore by (3) $P_k(X)$ is separable.

Therefore $P_k(X)$ is compact metrizable space.

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