ANALOG OF "ABC" CONJECTURE FOR P-ADIC HOLOMORPHIC FUNCTIONS

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Abstract. By using the Wronskian technique we proved the theorem, which is an analog of "abc" conjecture for p-adic holomorphic functions.
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1. Introduction

Let $f(z)$ be a polynomial with coefficients in an algebraically closed field F of characteristic 0 and let $\overline{n}(1/f)$ be the number of distinct zeros of f. Let us start by recalling Marson's theorem :

Marson's theorem. *(see [4]) Let $a(z), b(z), c(z)$ be relatively prime polynomials in F and not all constants such that $a + b = c$. Then*

$$\max\left\{deg(a), deg(b), deg(c)\right\} \leq \overline{n}\left(\frac{1}{abc}\right) - 1.$$

Influenced by Marson's theorem, and considerations of Szpiro and Frey, Masser and Oestrelé formulated the " abc " conjecture for integer as follows:

"abc" conjecture. *Let q be a non-zero integer. Define the radical of q to be*

$$N\left(\frac{1}{q}\right) = \prod_{p/q} p,$$

i.e. the product of the distinct primes dividing q. Given $\epsilon > 0$, there exists a number $C(\epsilon)$ having the following property. For any non-zero relatively prime integers a, b, c such that $a + b = c$, we have

$$\max\left\{|a|, |b|, |c|\right\} \leq C(\epsilon)\overline{N}\left(\frac{1}{abc}\right)^{1+\epsilon}.$$

Julie Tzu-Yuen Wang in [8] also gived a generalization of function fields version of abc-conjecture due to Mason, Voloch, Browanawell and Masser.

In [2], Hu and Yang shows that the analogue "abc"conjecture for one variable non-Archimedean holomorphic functions is true. In [7], An and Manh proved a similar result for non-Archimedean holomorphic function in several variables. In this paper by using the Wronskian technique we give a generalization of the Hu and Yang's theorem in p-adic case.

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2. The main results

Let p be a prime number, Q_p the field of p-adic number, and C_p the p-adic completion of the algebraic closure of Q_p. The absolute value in Q_p is normalized so that $|p| = p^{-1}$. We further use the notion $v(z) = -\log|z|$ for the additive valuation on C_p which extends ord_p. We first recall some standard notations and definitions in p-adic Nevanlinna theory. Let $\varphi = \frac{f}{g}$ be a meromorphic function on C_p. The *counting function* of φ is defined as following $N(\varphi, t) = N(f, t) - N(g, t)$, where $N(f, t)$ is counting function of holomorphic function f (see [1]).

The following results are well-known in p-adic Nevanlinna theory.

Lemma 2.1. *Let φ, ψ be meromorphic functions on C_p. Then*

$$N(\varphi + \psi, t) \le \max\{N(f, t), N(g, t)\} + O(1),$$
$$N(\varphi\psi, t) = N(\varphi, t) + N(\psi, t),$$

where $O(1)$ is bounded when $t \to -\infty$.

Lemma 2.2. *Let φ be a meromorphic function on C_p and let k be positive integral number such that $\varphi^{(k)} \not\equiv 0$. Then*

$$N\left(\frac{\varphi^{(k)}}{\varphi}, t\right) \le kt + O(1).$$

Note that we use some notations and definitions of p-adic Nevanlinna theory as in [1]. Recall that for every entire function f we use the notation $N_k(f, t)$ to denote the k-truncated counting function. The necessary properties of counting functions can be found in [1]. Here we will use the symbol $\overline{N}(f, t)$ to denote $N_1(f, t)$.

Theorem 2.3. *Let $f_1, f_2, ..., f_n$ $(n \ge 3)$ be entire functions on C_p without common zeros on C_p. Assume that $f_1, f_2, ..., f_{n-1}$ are linearly independent and $f_1 + f_2 + \cdots + f_n = 0$. Then*

$$\max_{1 \le j \le n} N(f_j, t) \le \left(n - \frac{k+3}{2}\right) \sum_{i=1}^{n} \overline{N}(f_i, t) + \frac{(n-2)(n-1)}{2} t + O(1),$$

where $k := \min_{\alpha \text{ is a zero of some } f_i} \#\{f_j : f_j(\alpha) = 0\}$ and $O(1)$ is bounded as $t \to -\infty$.

Proof. We set

$$P(z) = \frac{\|f_1, f_2, ..., f_{n-1}\|}{f_1 f_2 \cdots f_{n-1}},$$

and

$$Q(z) = \frac{f_1 f_2 \cdots f_n}{\|f_1, f_2, ..., f_{n-1}\|},$$

where

$$\|f_1, f_2, \ldots, f_{n-1}\| = \begin{vmatrix} f_1 & f_2 & \cdots & f_{n-1} \\ f_1' & f_2' & \cdots & f_{n-1}' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-2)} & f_2^{(n-2)} & \cdots & f_{n-1}^{(n-2)} \end{vmatrix}$$

is the Wronskian of function $f_1, f_2, \dots, f_{n-1}$.

By the hypothesis $f_1, f_2, \dots, f_{n-1}$ are linearly independent, we have $\|f_1, f_2, \dots, f_{n-1}\| \neq 0$. On the other hand, from the equation

$$f_1 + f_2 + \cdots + f_n = 0,$$

we obtain

$$\|f_{\alpha_1}, f_{\alpha_2}, \dots, f_{\alpha_{n-1}}\| = \delta\|f_1, f_2, ..., f_{n-1}\|, \ (\delta = \pm 1),$$

where $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ are the distinct numbers of the set $I = \{1, 2, \dots, n\}$.

We first prove that

$$N(Q,t) \leq \left(n - \frac{k+3}{2}\right) \sum_{j=1}^{n} \overline{N}(f_j, t).$$

Indeed, let α be a zero of $Q(z)$. Then α is a zero of some f_i. By the hypothesis the functions $f_1, f_2, ..., f_n$ have no common zeros, there exists a number $i_0, (1 \leq i_0 \leq n)$ such that $f_{i_0}(\alpha) \neq 0$. Let $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ be $(n-1)$ distinct numbers of the set $I \setminus \{i_0\}$, we have

$$Q(z) = \frac{f_1 f_2 ... f_n}{\|f_1, f_2, ..., f_{n-1}\|} = \delta \frac{f_{\alpha_1} f_{\alpha_2} \cdots f_{\alpha_{n-1}}}{\|f_{\alpha_1} f_{\alpha_2} \cdots f_{\alpha_{n-1}}\|} f_{i_0}.$$

Let

$$R(z) = \frac{\|f_{\alpha_1}, f_{\alpha_2}, \dots, f_{\alpha_{n-1}}\|}{f_{\alpha_1} f_{\alpha_2} \cdots f_{\alpha_{n-1}}}.$$

Now we have

$$R(z) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \frac{f'_{\alpha_1}}{f_{\alpha_1}} & \frac{f'_{\alpha_2}}{f_{\alpha_2}} & \dots & \frac{f'_{\alpha_{n-1}}}{f_{\alpha_{n-1}}} \\ \vdots & \vdots & & \vdots \\ \frac{f^{(n-2)}_{\alpha_1}}{f_{\alpha_1}} & \frac{f^{(n-2)}_{\alpha_2}}{f_{\alpha_2}} & \dots & \frac{f^{(n-2)}_{\alpha_{n-1}}}{f_{\alpha_{n-1}}} \end{vmatrix}.$$

This determinant is a summa of following terms

$$\gamma \frac{f'_{i_1} \cdots f^{(n-2)}_{i_{n-2}}}{f_{i_1} \cdots f_{i_{n-2}}}, \tag{1}$$

where $1 \leq i_1, \dots, i_{n-2} \leq n$ and $\gamma = \pm 1$.

We set $q := \#\{f_j : f_j(\alpha) = 0\}$. Then $q \geq k$. Assume that for every term in (1), there are l functions f_j such that $f_j(\alpha) = 0$. Then we have the following inequalities

$$\begin{aligned} ord_\alpha \left(\frac{f'_{i_1} \cdots f^{(n-2)}_{i_{n-2}}}{f_{i_1} \cdots f_{i_{n-2}}} \right) &= ord_\alpha \left(\frac{f'_{i_1}}{f_{i_1}} \right) + \cdots + ord_\alpha \left(\frac{f^{(n-2)}_{i_{n-2}}}{f_{i_{n-2}}} \right) \\ &\geq -(n-2) - (n-3) - \cdots - (n-1-l) \\ &\geq -(n-2) - (n-3) - \cdots - (n-1-q) \\ &= -\left(n - \frac{q+3}{2}\right) q \geq -\left(n - \frac{k+3}{2}\right) q, \end{aligned}$$

where $ord_\alpha(f)$ is the order of f at α.

It follows that

$$ord_\alpha\left(\frac{f'_{i_1}\cdots f^{(n-2)}_{i_{n-2}}}{f_{i_1}\cdots f_{i_{n-2}}}\right)\geq -\left(n-\frac{k+3}{2}\right)q.$$

Hence

$$ord_\alpha\left(R(z)\right)\geq -\left(n-\frac{k+3}{2}\right)\sum_{\substack{1\leq j\leq n\\ f_j(\alpha)=0}}1.$$

Therefore

$$ord_\alpha\left(Q(z)\right)=-ord_\alpha\left(R(z)\right)\leq\left(n-\frac{k+3}{2}\right)\sum_{\substack{1\leq j\leq n\\ f_j(\alpha)=0}}1.$$

Consequently from the definition of the counting function, we have

$$N(Q,t)\leq\left(n-\frac{k+3}{2}\right)\sum_{j=1}^{n}\overline{N}(f_j,t).$$

Now, we will show that

$$N(P,t)\leq\frac{(n-2)(n-1)t}{2}+O(1). \quad (\text{:}$$

Indeed, we have

$$P(z)=\begin{vmatrix}1 & 1 & \cdots & 1\\ \dfrac{f'_1}{f_1} & \dfrac{f'_2}{f_2} & \cdots & \dfrac{f'_{n-1}}{f_{n-1}}\\ \vdots & \vdots & & \vdots\\ \dfrac{f^{(n-1)}_1}{f_1} & \dfrac{f^{(n-1)}_2}{f_2} & \cdots & \dfrac{f^{(n-1)}_{n-1}}{f_{n-1}}\end{vmatrix}.$$

This determinant is the summa of the following terms

$$\beta\frac{f'_{j_1}\cdots f^{(n-2)}_{j_{n-2}}}{f_{j_1}\cdots f_{j_{n-2}}}\qquad(\beta=\pm1).$$

For every term, from lemma 2.2, we obtain

$$\begin{aligned}N\left(\frac{f'_{j_1}\cdots f^{(n-2)}_{j_{n-2}}}{f_{j_1}\cdots f_{j_{n-2}}},t\right)&=N\left(\frac{f'_{j_1}}{f_{j_1}},t\right)+\cdots+N\left(\frac{f^{(n-2)}_{j_{n-2}}}{f_{j_{n-2}}},t\right)\\&\leq t+2t+\cdots+(n-2)t+O(1)=\\&=\frac{(n-2)(n-1)}{2}t+O(1).\end{aligned}$$

By the lemma 2.1, we still have

$$N(P,t)\leq\frac{(n-2)(n-1)}{2}t+O(1). \quad (3)$$

Therefore inequalities (2) and (3) imply that

$$N(P,t) + N(Q,t) \le \left(n - \frac{k+3}{2}\right) \sum_{i=1}^{n} \overline{N}(f_i, t) + \frac{(n-2)(n-1)}{2} t + O(1).$$

Thus, we have

$$\begin{aligned} N(f_n, t) &= N(PQ, t) = N(P,t) + N(Q,t) \\ &\le \left(n - \frac{k+3}{2}\right) \sum_{i=1}^{n} \overline{N}(f_i, t) + \frac{(n-2)(n-1)}{2} t + O(1). \end{aligned}$$

Therefore, similar to $f_1, f_2, \dots, f_{n-1}$ we conclude that

$$\max_{1 \le j \le n} \{N(f_j, t)\} \le \left(n - \frac{k+3}{2}\right) \sum_{i=1}^{n} \overline{N}(f_i, t) + \frac{(n-2)(n-1)}{2} t + O(1).$$

This completes the proof of the theorem.

By theorem 2.3 we have the following corollary, which is another statement of Hu-Yang's theorem ([2]).

Corollary 2.4. *Let $a(z), b(z), c(z)$ be entire functions on C_p and without common zeros and not all constants such that*

$$a(z) + b(z) = c(z).$$

Then

$$\max\{N(a,t), N(b,t), N(c,t)\} \le \overline{N}(abc, t) + t + O(1).$$

Proof. Indeed, it suffices to take in theorem 2.3, $f_1 = a(z)$, $f_2 = b(z)$, $f_3 = c(z)$, $k = 1$ and $n = 3$, we obtain

$$\max\{N(a,t), N(b,t), N(c,t)\} \le \overline{N}(a,t) + \overline{N}(b,t) + \overline{N}(c,t) + t + O(1).$$

On the other hand, we have

$$\overline{N}(a,t) + \overline{N}(b,t) + \overline{N}(c,t) = \overline{N}(abc, t).$$

This completes the proof of our corollary.

By using theorem 2.3, we give a generalization of Borel's lemma in the p-adic case (see [6]).

Corollary 2.5. *Let $f_1, f_2, \dots, f_n$, $(n \ge 3)$ be holomorphic functions without common zeros on C_p such that*

$$f_1 + f_2 + \dots + f_n = 0.$$

Then the functions $f_1, f_2, \dots, f_{n-1}$ are linearly dependent over C_p if, for $j = 1, \dots, n$, every zero of f_j is of multiplicity at least d_j and the following condition holds

$$\sum_{j=1}^{n} \frac{1}{d_j} \le \frac{1}{n - \frac{k+3}{2}},$$

where

$$k := \min_{\alpha \text{ is a zero of some } f_i} \#\{f_j : f_j(\alpha) = 0\}.$$

Proof. We have $N(f_j, t) \geq d_j \overline{N}(f_j, t)$. Therefore

$$\frac{1}{d_j} \geq \frac{\overline{N}(f_j, t)}{N(f_j, t)}, \ (j = 1, \ldots, n),$$

whence

$$\sum_{j=1}^{n} \frac{1}{d_j} \geq \frac{\sum_{j=1}^{n} \overline{N}(f_j, t)}{\max_{1 \leq j \leq n} \{N(f_j, t)\}}.$$

Then by theorem 2.3, we have

$$\max_{1 \leq j \leq n} \{N(f_j, t)\} \leq \left(n - \frac{k+3}{2}\right) \sum_{j=1}^{n} \overline{N}(f_j, t) + \frac{(n-2)(n-1)}{2} t + O(1)$$

$$\leq \left(n - \frac{k+3}{2}\right) \left(\sum_{j=1}^{n} \frac{1}{d_j} \max_{1 \leq j \leq n} \{N(f_j, t)\}\right) + \frac{(n-2)(n-1)}{2} t + O(1).$$

Therefore

$$\left(\frac{1}{n - \frac{k+3}{2}} - \sum_{j=1}^{n} \frac{1}{d_j}\right) \max_{1 \leq j \leq n} \{N(f_j, t)\} \leq \frac{(n-2)(n-1)}{2} t + O(1).$$

By hypothesis $\sum_{j=1}^{n} \frac{1}{d_j} \leq \frac{1}{n - \frac{k+3}{2}}$, we obtain a contradiction as $t \to -\infty$.

Definition 2.6. Let $M_j = z_1^{\alpha_{j1}} \cdots z_{n+1}^{\alpha_{jn+1}}$, $1 \leq j \leq s$ be distinct monomial of degree d with non-negative exponents. Let X be a hypersurface of degree d of $P^n(C_p)$ defined by

$$X : c_1 M_1 + \cdots + c_s M_s = 0,$$

where $c_j \in C_p^*$ are non-zero constants. We call X a *perturbation of the Fermat hypersurface* of degree d if $s \geq n+1$ and $M_j = z_j^d$, $j = 1, \ldots, n+1$.

Theorem 2.7. *Let X be a perturbation of the Fermat hypersurface of degree d in $P^n(C_p)$ such that*

$$d \geq \left(s - \frac{k+3}{2}\right) \left(n + 1 + \sum_{j=n+2}^{s} k_j\right),$$

where $k := \min_{1 \leq m \leq n+1} \#\{\alpha_{jm} > 0 : j = 1, \ldots, s\}$,
and $k_j := \#\{\alpha_{jm} > 0 : m = 1, \ldots, n+1\}$, $j = n+2, \ldots, s$.

Then every holomorphic curve in X is degenerate.

Proof. Let $f = (f_1, f_2, \ldots, f_{n+1}) : C_p \to P^n(C_p)$ be a holomorphic curve in X. Then

$$M_1 \circ f + \cdots + M_{n+1} \circ f + M_{n+2} \circ f + \cdots + M_s \circ f \equiv 0.$$

We first claim that $M_j \circ f,\ 1 \le j \le s-1$ are linearly dependent over C_p. Assume that they are linearly independent over C_p. Then by using theorem 2.3, we have

$$\max_{1\le j\le s} N(M_j \circ f, t) \le \left(s - \frac{k+3}{2}\right) \sum_{j=1}^{s} \overline{N}(M_j \circ f, t) + \frac{(s-2)(s-1)}{2} t + O(1).$$

On the other hand, we have

$$N(M_j \circ f, t) = N(f_j^d, t) = dN(f_j, t), (j = 1, \ldots, n+1).$$

For $j = n+2, \ldots, s$ we still have

$$\begin{aligned} N(M_j \circ f, t) &= N\left(f_1^{\alpha_{j_1}} \ldots f_{n+1}^{\alpha_{j_{n+1}}}, t\right) = \sum_{m=1}^{n+1} N\left(f_m^{\alpha_{j_m}}, t\right) \\ &\le (\alpha_{j_1} + \ldots + \alpha_{j_{n+1}}) \max_{1\le m\le n+1} (N(f_m, t)) = \\ &= d \max_{1\le m\le n+1} N(f_m, t). \end{aligned}$$

Therefore

$$\max_{1\le j\le s} N(M_j \circ f, t) = d \max_{1\le m\le n+1} N(f_m, t).$$

Moreover, we have

$$\overline{N}(f_j^d, t) = \overline{N}(f_j, t), \quad (j = 1, \ldots, n+1),$$

and

$$\begin{aligned} \overline{N}\left(f_1^{\alpha_{j_1}} \ldots f_{n+1}^{\alpha_{j_{n+1}}}, t\right) &\le \sum_{\substack{1\le j\le n+1 \\ \alpha_{j_m}>0}} \overline{N}(f_m, t) \\ &\le k_j \max_{1\le m\le n+1} \overline{N}(f_m, t), \ (j = n+2, ..., s). \end{aligned}$$

From the above it follows that

$$\begin{aligned} \sum_{j=1}^{s} \overline{N}(M_j \circ f, t) &\le \left(n + 1 + \sum_{j=n+2}^{s} k_j\right) \max_{1\le m\le n+1} \overline{N}(f_m, t) \le \\ &\le \left(n + 1 + \sum_{j=n+2}^{s} k_j\right) \max_{1\le m\le n+1} N(f_m, t). \end{aligned}$$

Hence

$$\begin{aligned} d \max_{1\le m\le n+1} N(f_m, t) &\le \left(s - \frac{k+3}{2}\right)\left(n + 1 + \sum_{j=n+2}^{s} k_j\right) \max_{1\le m\le n+1} N(f_m, t) + \\ &+ \frac{(s-2)(s-1)}{2} t + O(1). \end{aligned}$$

Therefore

$$\left(d-\left(s-\frac{k+3}{2}\right)\left(n+1+\sum_{j=n+2}^{s}k_j\right)\right)\max_{1\le m\le n+1}N(f_m,t)\le\frac{(s-2)(s-1)}{2}t+O(1).$$

By the hypothesis $d \geq \left(s-\frac{k+3}{2}\right)\left(n+1+\sum_{j=1}^{s}k_j\right)$, we have a contradiction as $t \to -\infty$.

2.9 Example. Here we give a hyperbolic hypersurface in $P^3(C_p)$.

$$X : z_1^{4d}+z_2^{4d}+z_3^{4d}+z_4^{4d}+(z_1z_2z_3z_4)^d=0,\ d>4\ (degX=4d\geq 20),\ t\in C_p^*.$$

In the complex case, Masuda and Noguchi proved that if $d > 6$ then X hyperbolic ([5]). Put $k = 2$, $s = 5$, $k_5 = 4$. Then X satisfies the hypothesis of theorem 2.8 and every holomorphic curve in X is degenerate. Therefore in p-adic case, by using Masuda and Noguchi's method we prove that if $d > 4$ then X is hyperbolic.

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