SOME THEOREMS ON THE STRONG LAW OF LARGE NUMBERS IN VON NEUMANN ALGEBRA

Phan Duc Thanh, Nguyen Van Quang

Department of Mathematics, Vinh University

Abstract. The aim of this paper is to give some theorems on the strong law of large numbers in von Neumann algebras, related to the functions satisfying the condition (A).

I. Introduction and notations

Throughout this paper, let $\mathcal{A}$ denote a von Neumann algebra with faithfull norman tracial state τ, $\tilde{\mathcal{A}}$ the algebra of measurable operators in the Segal-Nelson's sence (see [3]). On $\tilde{\mathcal{A}}$ one can define the notions of sequence of successively independent operators; martingale; martinggale differences....and can consider various versions of convergence: almost uniformly (a.u) convergence, bilaterally almost uniformly (b.a.u) convergence... (see [3]). For further infomation of von Neumann algebras we refer to [3], [5] and [7].

We say that a restricly increasing, differentiable funtion φ $R^+ \to R^+$ satisfy the condition (A) if: $\varphi(x) \to \infty$ as $x \to \infty$ and both $x/\varphi(x)$ and $\varphi(x)/x^2$ nonincreasing.

Recently, various results concerning the strong limit theorems, in patticular, the strong laws of large numbers have been extended to the non-commutative context. In [1] the Kolmogorov's law has been proved. The non-commutative extension of the Marcinkiewicz-Zygmund law was shown in [4]. Some theorems on the convergence of weighted sums can be found in [6]. The aim of this paper is to give some theorems on the strong law of large numbers in von Neumann algebras , which are related to the functions satisfying the condition (A). Our results here get some results in [6] as consequences in the special case: $\varphi(x) = x^2$.

In classical case, the results here are analogue to some theorems in [2] and [8].

II. Results

In the sequel we'll need the following lemmas

Lemma 2.1. *([4], lemma 3.3). Let (x_n), (y_n) be sequences of measuable operators and (c_n) asequence of positive numbers. If*

$$\sum_{n=1}^{\infty} \tau(e_{[c_n,\infty)}(|x_n|)) < \infty,$$

then the series $\sum_{n=1}^{\infty}(x_n + y_n)$ converges almost uniformly if and only if the series $\sum_{n=1}^{\infty}(x_n^{(c_n)} + y_n)$ converges almost uniformly, where $x_n^{(c_n)} = x_n e_{[0,c_n)}(|x_n|)$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

For $x > 0$, we define $N(x)$ as the number of n such that $b_n = \frac{A_n}{a_n} \leq x$

It is easy to see

$$N(x) = \sum_{n=1}^{\infty} I_{\{x:\ \frac{A_n}{a_n} \leq x\}}.$$

Lemma 2.2. *Let $\varphi(x)$ satisfy the condition (A) and $\frac{N(u)}{\varphi(u)} \to 0$ as $u \to \infty$. Then we have*

$$\sum_{n:b_n \geq x} \frac{1}{b_n^2} \leq \frac{\varphi(x)}{x^2} \int_x^{\infty} \frac{dN(y)}{\varphi(y)} < \frac{\varphi(x)}{x^2} \int_x^{\infty} \frac{N(y)\varphi'(y)}{\varphi^2(y)} dy \qquad \text{(i)}$$

$$\sum_{n:b_n < x} \frac{1}{b_n} \leq \frac{\varphi(x)}{x} \int_1^{x} \frac{dN(y)}{\varphi(y)} < \frac{\varphi(x)}{x} \int_1^{x} \frac{N(y)\varphi'(y)}{\varphi^2(y)} dy. \qquad \text{(ii)}$$

Proof. (i) Using the nonincreasing of $\varphi(x)/x^2$ we have

$$\frac{x^2}{\varphi(x)} \leq \frac{b_n^2}{\varphi(b_n)} \quad \forall x < b_n.$$

This implies

$$\frac{1}{b_n^2} \leq \frac{\varphi(x)}{x^2} \frac{1}{\varphi(b_n)}.$$

Thus

$$\sum_{n:b_n \geq x} \frac{1}{b_n^2} \leq \frac{\varphi(x)}{x^2} \sum_{n:b_n \geq x} \frac{1}{\varphi(b_n)}.$$

On the other hand

$$\sum_{n:b_n \geq x} \frac{1}{\varphi(b_n)} = \lim_{u \to \infty} \sum_{n:x \leq b_n \leq u} \frac{1}{\varphi(b_n)} = \lim_{u \to \infty} \int_x^u \frac{dN(y)}{\varphi(y)} =$$

$$= \lim_{u \to \infty} \left[\frac{N(u)}{\varphi(u)} - \frac{N(x)}{\varphi(x)} + \int_x^u \frac{N(y)\varphi'(y)}{\varphi^2(y)} dy \right] < \int_x^u \frac{N(y)\varphi'(y)}{\varphi^2(y)} dy,$$

because $\frac{N(u)}{\varphi(u)} \to 0$ as $u \to \infty$.

Analoguesly, using the nonincreasing of $x/\varphi(x)$ as $b_n < x$ we can prove (i.i).

Let $(X_n) \subset \tilde{A}$, $X \in \tilde{A}$. If there exists a constant $C > 0$ such that for all $\lambda > 0$ and all $n \in N$

$$\tau(e_{[\lambda,\infty)}(|X_n|)) \leq C\tau(e_{[\lambda,\infty}(|X|))$$

then we write $(X_n) \prec X$

Theorem 2.1. *Let $a_n > 0$, $A_n = \sum_{i=0}^{n} a_i \uparrow \infty$, $b_n = A_n/a_n \to \infty$ as $n \to \infty$, x_n- sequence of succesively independent measurable operators, $\tau(x_n) = 0, (x_n) \prec x, \tau(x) < \infty$ $\tau(N(|x|)) < \infty$ and $\varphi(x)$ satisfy the condition (A) and $\frac{N(u)}{\varphi(u)} \to 0$ as $u \to \infty$. If*

$$\int_0^{\infty} \frac{\varphi(\lambda)}{\lambda} \tau\left(e_{[\lambda,\infty)}(|x|)\right) \int_{y \geq \lambda} \frac{N(y)\varphi'(y)}{\varphi^2(y)} dy d\lambda < \infty, \qquad (2.1)$$

then

$$A_n^{-1}\sum_{k=1}^{n} a_k x_k \to 0\ (a.u) \quad \text{as } n \to \infty. \tag{2.2}$$

Proof. Put

$$y_k = x_k e_{[0,b_k)}(|x_k|).$$

From the polar decomposition

$$x_k - y_k \;=\; u_k|x_k - y_k|$$

we have

$$|\tau(y_k)| = |\tau(x_k - y_k)| \leq \tau(|x_k - y_k|) = \tau(|x_k|e_{[b_k,\infty)}(|x_k|)) \to 0 \text{ as} k \to \infty,$$

because $\tau(|x_k|) \leq C\tau(|x|) < \infty$.

This implies

$$0 \leq \left|A_n^{-1}\sum_{k=1}^{n} a_k \tau(y_k)\right| \leq A_n^{-1}\sum_{k=1}^{n} a_k|\tau(y_k)| \to 0 \text{ as } n \to \infty. \tag{2.3}$$

On the other hand

$$\sum_{k=1}^{n}\tau\Big(|\frac{y_k - \tau(y_k)}{b_n}|^2\Big) \leq \sum_{k=1}^{\infty}\tau\Big(|\frac{y_k}{b_k}|^2\Big) = \sum_{k=1}^{\infty}\frac{1}{b_n^2}\int_{\lambda<b_k}\lambda^2\tau\big(e_{d\lambda}(|x_k|\big) \leq$$

$$\leq 2C\sum_{k=1}^{\infty}\frac{1}{b_k^2}\int_{\lambda<b_k}\lambda\tau\big(e_{[\lambda,\infty)}(|x|)\big)d\lambda = 2C\int_0^{\infty}\lambda\tau\big(e_{[\lambda,\infty)}(|x|)\big)\sum_{k:\, b_k>\lambda}\frac{1}{b_k^2}d\lambda <$$

$$< 2C\int_0^{\infty}\lambda\tau\big(e_{[\lambda,\infty)}(|x|)\big)\frac{\varphi(\lambda)}{\lambda^2}\int_{y\geq\lambda}\frac{N(y)\varphi'(y)}{\varphi^2(y)}dyd\lambda \;< \infty, \tag{2.4}$$

which follows that the series

$$\sum_{k=1}^{\infty}\frac{y_k - \tau(y_k)}{b_k} \tag{2.5}$$

converges almost unifomly.

We also have

$$\sum_{k=1}^{\infty}\tau(e_{[1,\infty)}(|\frac{x_k}{b_k}|)) = \sum_{k=1}^{\infty}\tau(e_{[b_k,\infty)}(|x_k|))$$

$$\leq C\sum_{k=1}^{\infty}\tau(e_{[b_k,\infty)}(|x|)) = C\tau(N(|x|)) < \infty. \tag{2.6}$$

Applying lemma 2.1 to $(\frac{x_k}{b_k})$ and $(\frac{y_k}{b_k})$ we see that

$$\sum_{k=1}^{\infty}\frac{x_k - \tau(y_k)}{b_k} = \sum_{k=1}^{\infty}\frac{a_k(x_k - \tau(y_k))}{A_k} \tag{2.7}$$

converges almost unifomly.

This together with (2.3) and Kronecker's lemma complete the proof.

In the more general case, when the assumption $A_n = \sum_{i=0}^{n} a_i$ is omited, we need more condition on $N(x)$.

Theorem 2.2.. Let $a_n > 0$, $0 < A_n \uparrow \infty$, $b_n = A_n/a_n \to \infty$ as $n \to \infty$, x_n - *sequence of succesively independent measurable operators,* $\tau(x_n) = 0, (x_n) \prec x, \tau(|x|) < \infty$, $\tau(N(|x|)) < \infty$ *and* $\varphi(x)$ *setisfy the condition (A) and* $\frac{N(u)}{\varphi(u)} \to 0$ *as* $u \to \infty$. *If* $N(x)$ *satisfies the condition (2.1) and*

$$\int_0^{\infty} \frac{\varphi(\lambda)}{\lambda} \tau\left(e_{[\lambda,\infty)}(|x|)\right) \int_{y \le \lambda} \frac{N(y)\varphi'(y)}{\varphi^2(y)} dy d\lambda < \infty \tag{2.8}$$

then

$$A_n^{-1} \sum_{k=1}^{n} a_k x_k \to 0 \ (a.u) \quad \text{as } n \to \infty \tag{2.9}$$

Proof. By the same way as the proof of the Theorem 2.1, we have (2.7) if $N(x)$ satisfy (2.1). Now we shall show that

$$\left| A_n^{-1} \sum_{k=1}^{n} a_k \tau(y_k) \right| \to 0 \quad \text{as } n \to \infty$$

if N(x) satisfies (2.8).

To do this, it suffices to show that

$$\sum_{k=1}^{\infty} \frac{|\tau(y_k)|}{b_k} < \infty.$$

We have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{|\tau(y_k)|}{b_k} &< C \sum_{k=1}^{\infty} \left[\int_{\lambda > b_k} \tau(e_{[\lambda,\infty)}(|x|)) d\lambda + \tau(e_{[b_n,\infty)}(|x|)) \right] \\ &= C \int_0^{\infty} \tau(e_{[\lambda,\infty)}(|x|)) \sum_{(k:\, 0 < b_k \le \lambda)} \frac{1}{b_k} d\lambda + C\tau(N(|x|)) \\ &< C \int_0^{\infty} \frac{\varphi(\lambda)}{\lambda} \tau(e_{[\lambda,\infty)}(|x|)) \int_{y \le \lambda} \frac{N(y)\varphi'(y)}{\varphi^2(y)} dy d\lambda + C\tau(N(|x|)) < \infty. \end{aligned}$$

This completes the proof of the theorem.

References

1 C. J. K. Batty, *The strong laws of large numbers for states and for traces of a W^* algebra*, Z. W. 48 (1979), 177-191.

2 K. L. Chung, *Note on some strong laws of large numbers*, Amer. J. Math. 69 (1947) 189-192.

3 R. Jajte, *Strong limits theorems in non-commutative probability*, Lect. Notes Math. 1110. Springer, Berlin-Heidelberg-New York-Tokyo 1985.

4 A. Luczak, *Laws of large numbers in von-Neumann algebras and related results*, Studia Math. 81 (1985) 231-243.

5 P. Mayer, *Quantum Probability for Probabilists*, Springer-Verlag 1993.

6 Nguyen Van Quang, *Convergence of weighted sums of independent measurable operators in von Neumann algebras*, Vietnam Journal of Math. 22 N. 1+2 (1994) 94-103.

7 K.R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus*, Birkhäuser, Basel 1992.

8 J. Szulga, *Three series theorems for martingales in Banach spaces.* Bull. Acad, Polon. Sci. Ser. sci. Math. Astr. Phys. 25. 2 (1977) 175-180.