ON THE S - QUASI REGULAR ALGEBRAS

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Abstract. In this paper we introduce the definition of S-regular algebras and establish some diagrams to define concret radical classes.

I. Introduction

In this note we shall work in the variety W of algebras over an associative and commutative ring K with unity element. For details of radical theory we refer to [11]. We now come to the definition of radical in the sense of Kurosh and Amitsur:

The class R is called a radical class in W if R satisfies the following three conditions

i) R is homomorphically closed;

ii) Every algebra A has an $R-$ ideal $R(A)$;

iii) The factor algebra $A/R(A)$ of A with respect to $R(A)$ is $R-$ semisimple i.e. $R(A/R(A)) = 0$.

In ring theory many regularities and quasi-regularities have been defined and studied by many authors. In [6] we have introduced the notion of $S-$ regular algebras and showed the radical characteristic of regularities in this sense. In this note we are going to introduce the definition of $S-$ quasi regular algebras and to establish some diagrams to define concret radical classes. It is a generalization of the $f-$ regularies in [5].

We recall the following definition:

Let there be assigned to each an algebra A belonging to W a mapping S_A which maps the dicscrete direct sum $A^\infty = \oplus_{i=1}^{\infty} A_i$ into the algebra A, where $A_i = A$, for $i = 1, 2, \ldots$. The class S consisting of all mappings S_A is called a regularity in W if the following condition is satisfied: For every $K-$ homomorphism $f : A \to B$ we have the commutative diagram below

$$\begin{array}{ccc} A^\infty & \xrightarrow{S_A} & A \\ {\scriptstyle f^\infty}\downarrow & & {\scriptstyle f}\downarrow \\ B^\infty & \xrightarrow{S_B} & B \end{array} \tag{1}$$

where $f^\infty = (f, f, f\ldots)$.

An algebra A is said to be $S-$ regular if $S_A(A^\infty) = A$.

2. $S-$ quasi regular algebras

Definition 1. *Let $S = \{S_A : A^\infty \to A\}_{A\in W}$ be the $S-$ regularity. An element a of the algebra A is called $S-$ quasi regular if there exists an element x of A^∞ such that $pr_1(x) = a$ and $S_A(x) = 0$, where pr_1 is the projection of A^∞ onto 1^{th} component.*

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An algebra A is said to be an $S-$ quasi regular algebra if every element of A is $S-$ quasi regular

Theorem 1. *If the class $S = \{S_A : A^\infty \to A\}_{A \in W}$ is a regularity then the class R of all $S-$ quasi regular algebras is a radical class in W if and only if the following condition is satified:*

If I is an $S-$ quasi regular ideal of algebra A and for every element a of A there exists an element x of A^∞ such that $pr_1(x) = a$ and $S_A(x) = 0 \mod I$, then the algebra A is $S-$ quasi regular.

Proof. Assume that the class R of all $S-$ quasi regular algebras is a radical class. Suppose that I is an S-quasi regular ideal of the algebra A and for every element a of A there exists an element x of A^∞ such that $pr_1(x) = a$ and $S_A(x) = 0 \mod I$. We have to show that the algebra A is S-quasi regular. Let us consider the factor algebra A/I. Take an element $\overline{a}$ of A/I. By hypothesis there exists an element x of A^∞ such that $pr_1(x) = a$ and $S_A(x) = 0 \mod I$. For the natural homomorphism $p : A \longrightarrow A/I$ we have the commutative diagram following

$$\begin{array}{ccc} A^\infty & \xrightarrow{S_A} & A \\ {\scriptstyle p^\infty}\downarrow & & {\scriptstyle p}\downarrow \\ (A/I)^\infty & \xrightarrow{S_{A/I}} & A/I. \end{array} \qquad (2)$$

Take $\overline{x} = p^\infty(x)$. It is clear that $pr_1(\overline{x}) = \overline{a}$.

$$S_{A/I}(\overline{x}) = S_{A/I}(p^\infty(x)) = p(S_A(x)) = \overline{S_A(x)} = \overline{0}.$$

Therefore the element $\overline{a}$ is S-quasi regular. This implies the S- quasi regularity of the algebra A/I. Since radical classes are closed under extension (see [11], p. 31; theorem 4.13) the algebra A is S- quasi regular.

Conversely, assume that the S- regularity satisfying the condition of the theorem. We shall show that the class R of all S - quasi regular algebras is a radical class. Since the zero algebra is S-quasi regular, the class R is not empty.

Let B be an image of S- quasi regular algebra A under the homomophism f. Now let b be an arbitrary element of the algebra B. Then there exists an element a of A such that $b = f(a)$. Since A is an S- quasi regular algebra there is an element x of A^∞ such that $pr_1(x) = a$ and $S_A(x) = 0$. Take $y = f^\infty(x)$. It is clear that $pr_1(y) = b$. By the commutative diagram (1) we have

$$S_B(y) = S_B(f^\infty(x)) = f.S_A(x) = f(S_A(x)) = f(0) = 0.$$

Therefore the algebra B is S-quasi regular. This implies the class R of all S-quasi regular algebras is homomorphically closed. Let P denote the set of all R-ideals of an algebra A. Since zero ideal is an R-ideal, this set is not empty. Now consider a chain $\{B_\alpha, \alpha \in I\}$ in P. The set $B = \bigcup B_\alpha$ is an ideal of A. Take an arbitrary element b of B. Then there is $\alpha \in I$ such that $b \in B_\alpha$. By hypothesis there exists an element x of B_α^∞ such that $pr_1(x) = b$ and $S_{B_\alpha}(x) = 0$. For the embedding $i_\alpha : B_\alpha \longrightarrow B$, we have the following commutative diagram

$$\begin{array}{ccc} B_\alpha^\infty & \xrightarrow{S_{B_\alpha}} & B_\alpha \\ {\scriptstyle i^\infty}\downarrow & \cdot & \downarrow{\scriptstyle i} \\ B^\infty & \xrightarrow{S_B} & B \end{array} \tag{3}$$

Take $y = i^\infty(x)$. It is clear that $pr_1(y) = i_\alpha(b) = b$. By the commutative diagram (3) we have

$$S_B(y) = S_B(i^\infty(x)) = i.S_B(x) = i(0) = 0.$$

Therefore the element b is S-quasi regular. This implies that B is an R-ideal of A. The ideal B is an upper bound of the chain $\{B_\alpha, \alpha \in I\}$ in the set P. By Zorn's lemma the set P has a maximal R-ideal, say $R(A)$.

We have to show that $R(A/R(A)) = \{0\}$. Assume that $R(A/R(A)) = B/R(A)$, where B is an ideal of A and $R(A) \subseteq B$. Since the algebra $B/R(A)$ is S- quasi regular, every element b of B there exists an element α of $(B/R(A))^\infty$ such that $pr_1(\alpha) = \bar{b}$ and $S_{B/R(A)}(\alpha) = \bar{0}$. Clearly, there exists an element x of B^∞ such that $pr_1(x) = b$ and $p^\infty(x) = \alpha$, where p is the natural homomorphism of B onto $B/R(A)$. We have the following commutative diagram

$$\begin{array}{ccc} B^\infty & \xrightarrow{S_B} & B \\ {\scriptstyle p^\infty}\downarrow & & \downarrow{\scriptstyle p} \\ (B/R(A))^\infty & \xrightarrow{S_{B/R(A)}} & B/R(A). \end{array} \tag{4}$$

We have

$$p(S_B(x)) = S_{B/R(a)}.p^\infty(x) = S_{B/R(A)}(\alpha) = \bar{0}.$$

This implies $S_B(x) = 0 \mod R(A)$. By the condition of the theorem the ideal B is S-quasi regular. Therefore B is an R-ideal containing R-ideal $R(A)$. By the maximal property of $R(A)$ we have $B = R(A)$. Hence $R(A/R(A)) = \{0\}$. The proof of the theorem is therefore finish.

Proposition 1. *The class R of all S-quasi regular algebras is a radical class if the following condition is satisfied:*

For every element a of the algebra A if there exists an element x of A^∞ such that $pr_1(x) = a$ and the element $S_A(x)$ is S-quasi regular then the element a is also S-quasi regular.

Proof Assume that I is an S-quasi regular ideal of an algebra A and for every element a of A there exists an element x of A such that $pr_1(x) = a$ and $S_A(x) = 0 \mod I$. Therefore the element $S_A(x)$ belongs to the ideal I. Since I is an S-quasi regular ideal, the element $S_A(x)$ is S-quasi regular. By hypothesis the element a is S-quasi regular. Thus the algebra A is S-quasi regular. The condition of Theorem 1 is satisfied. This completes the proof of Proposition 2.

3. Some expressions definiting radical classes

We consider a infinite set of indeterminants $\{t_1, t_2, ...\}$. $\overline{K}[t_1, ..., t_n]$ denotes the K-algebra of polynomials in non-commutative indeterminants $t_1, ..., t_n$.

Definition 2. *The formal serie* $f = \sum_{n=1}^{\infty} f_n$ *is said to be admissible if the following conditions are satisfied for* $n = 1, 2, ...$

i) $f_n \in \overline{K}[t_1, ..., t_n]$; $\deg f_n > 0$

ii) $f(t_1, ..., t_n, 0, 0, ...) \in \overline{K}[t_1, ..., t_n]$

It is clear to see that each admissible formal serie f defines an $S(f)$- regularity

$$S(f) = \{S(f)_A : A^{\infty} \longrightarrow A\}_{A \in W},$$

where $S(f)_A(a_1, a_2, ...) = f(a_1, a_2, ...)$.

For the sake of brevity we shall call an admissible formal serie f a *radical expression* if the class $R(f)$ of all $S(f)$ - quasi regular algebras is a radical class.

Proposition 2. *The following formal series are radical expressions.*

1) $f(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = t_1 + \alpha_1 t_2 + \alpha_2 t_1 t_2 + \sum_{i=1}^{\infty} \alpha_3 t_{2i+1} t_1 t_{2(i+1)} + \alpha_4 t_{2i+1} t_{2(i+1)}$ *where* $\alpha_i \in K, i = 1, 2, 3, 4$ *satisfy the condition* $\alpha_1\alpha_3 = \alpha_2\alpha_4$.

2) $g(\beta_1, \beta_2, \beta_3) = t_1 + \beta_1 t_2 t_1 + \beta_2 t_1 t_3 + \beta_3 \sum_{i=2}^{\infty} t_{2i} t_1 t_{2i+1}$ *where* $\beta_i \in K, i = 1, 2, 3$ *satisfy the condition* $\beta_1\beta_2 = 0$ *or* $\beta_1\beta_2 = \beta_3$.

3) $\varphi(m, \alpha) = t_1 + \alpha \sum_{i=1}^{\infty} t_{i+1+m(i-1)} t_1 t_{i+2+m(i-1)} t_1 ... t_1 t_{(m+1)i+1}$ *where* $\alpha \in K, m \in \mathbb{N}$.

4) $\psi(m, p(t), q(t)) = t_1 + \sum_{i=1}^{\infty} p(t_1) t_{2i+(i-1)m} p(t_1) ... p(t_1) t_{i(m+2)-1} q(t_1)$ *where* $p(t) \in K[t], q(t) \in K[t], m \in \mathbb{N}$.

Proof In order to prore that $f(\alpha_1, \alpha_2, \alpha_3), g(\beta_1, \beta_2, \beta_3), \varphi(m, \alpha)$ and $\psi(m, p(t), q(t))$ are radical expressions, we shall show that each of them defines an S- regularity satisfying the condition of Proposition 2.

First we prove that the $S(f(\alpha_1, \alpha_2, \alpha_3, \alpha_4))$ - quasi regularity satisfies the condition of Proposition 2. Assume that for the element a of the algebra A there exists an element $x = (a_1, a_2, ...)$ of A^{∞} such that $a = a_1$ and the element $b = S(f(\alpha_1, \alpha_2, \alpha_3, \alpha_4))_A(x)$ is $S(f(\alpha_1, \alpha_2, \alpha_3, \alpha_4))$ - quasi regular, we have

$$b = a_1 + \alpha_1 a_2 + \alpha_2 a_1 a_2 + \sum_{i=1}^{N_1} \alpha_3 a_{2i+1} a_1 a_2(i+1) + \alpha_4 a_{2i+1} a_{2(i+1)},$$

where $N_1 = \min\{\max\{i : a_{2i+1} \neq 0\}; \max\{i : a_{2(i+1)} \neq 0\}\}$. By Definition 1 there exists an element $y = (b_1, b_2, ...)$ of A^{∞} such that $b_1 = b$ and $S(f(\alpha_1, \alpha_2, \alpha_3, \alpha_4))_A(y) = 0$. Therefore we have $a_1 + \alpha_1 a_2 + \alpha_2 a_1 a_2 + \sum_{i=1}^{N_1} \alpha_3 a_{2i+1} a_1 a_{2(i+1)} + \alpha_4 a_{2i+1} a_{2(i+1)} + \alpha_1 b_2 + \alpha_2(a_1 + \alpha_1 a_2 + \alpha_2 a_1 a_2 + \sum_{i=1}^{N_1} \alpha_3 a_{2i+1} a_1 a_{2(i+1)} + \alpha_4 a_{2i+1} a_{2(i+1)}) b_2 + \sum_{j=1}^{N_2} b_{2j+1}(a_1 + \alpha_1 a_2 + \alpha_2 a_1 a_2 + \sum_{i=1}^{N_1} \alpha_3 a_{2i+1} a_1 a_{2(i+1)} + \alpha_4 a_{2i+1} a_{2(i+1)}) b_{2(j+1)} = 0$, where $N_2 = \min\{\max\{j : b_{2j+1} \neq 0\}; \max\{j : b_{2(j+1)} \neq 0\}\}$.

A straighforward caculation shows that

$$c_1+\alpha_1c_2+\alpha_2c_1c_2+\sum_{k=1}^{N}\alpha_3c_{2k+1}c_1c_{2(k+1)}+\alpha_4c_{2k+1}c_{2(k+1)}=0,$$

where

$$N=2(N_1+N_2)+N_1N_2$$
$$c_1=a_1=a$$
$$c_2=a_2+b_2+\alpha_2a_2b_2$$

$$c_{2k+1}=\begin{cases} a_{2k+1} & \text{if } 0<k\le N_1\\ a_{2i+1} & \text{if } N_1<k=N_1+i\le 2N_1\\ b_{2j+1} & \text{if } 2N_1<k=2N_1+j\le 2N_1+N_2\\ \alpha_2b_{2j+1} & \text{if } 2N_1+N_2<k=2N_1+N_2+j\le 2(N_1+N_2)\\ \alpha_3b_{2j+1}a_{2i+1} & \text{if } 2(N_1+N_2)+(j-1)N_1<k=2(N_1+N_2)+(j-1)N_1\\ & +i\le 2(N_1+N_2)+jN_1,\text{ for } j=1,...,N_2\end{cases}$$

and

$$c_{2(k+1)}=\begin{cases} a_{2(k+1)} & \text{if } 0<k\le N_1\\ a_{2(i+1)}b_2 & \text{if } N_1<k=N_1+i\le 2N_1\\ b_{2(j+1)} & \text{if } 2N_1<k=2N_1+j\le 2N_1+N_2\\ a_2b_{2(j+1)} & \text{if } 2N_1+N_2<k=2N_1+N_2+j\le 2(N_1+N_2)\\ a_{2(i+1)}b_{2(j+1)} & \text{if } 2(N_1+N_2)+(j-1)N_1<k=2(N_1+N_2)+\\ & (j-1)N_1+i\le 2(N_1+N_2)+jN_1,\text{ for } j=1,...,N_2.\end{cases}$$

Take $z=(c_1,c_2,...)\in A^\infty$. By putting $c_m=0$ for $m>2(N+1)$, we have $S(f(\alpha_1,\alpha_2,\alpha_3,\alpha_4))_A(z)=0$. Hence the element a is $S(f(\alpha_1,\alpha_2,\alpha_3,\alpha_4))$- quasi regular and the condition of Proposition 2 is satisfied. Thus $f(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ is a radical expression. The remaining assertions are proved similarly.

Now let us survey some following concrete cases:

The Jacobson radical [7] is defined by the radical expression $f(1,1,0,0)$. The Brown - Mecoy radical [3] is defined by the radical expression $f(1,1,1,1)$. The λ- regularity in the sense of De La Rose [10] is the $S(f(0,0,1,0))$- quasi regularity. The (p,q)- regularity in the sense of Musser [8] is a $S(\psi(1,p(t),q(t)))$- quasi regularity. The regularity in the sense of Von Newmann [9] is $S(\psi(1,t,t))$- quasi regularity. The strong regularity in the sense of Arens and Kaplansky [1] is a $\psi(1,t^2,1)$- quasi regularity. The left pseudo - regularity in the sense of Divinsky [4] is a $S(\psi(1,1,t+t^2))$- quasi regularity. The f- regularity in the sense of Blair [2] is the same $S(\varphi(3,1))$- quasi regularity. The $g(1,0,1)$- quasi regularity is a E_5- ring in the sense of Sza'sz [11]. The $g(1,1,1)$- quasi regularity is a E_6- ring in the sense of Sza'sz [11].

The open problems

Problem 1. Find a necessary and sufficient condition for the radical class determined by the S- quasi regularity is hereditary.

Problem 2. Find the conditions for the radical expression f and g such that $R(f) = R(g); R(f) \subset R(g)$.

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