SPECIALIZATION OF INVERSE LIMITS

Dam Van Nhi

Pedagogical College of Thai Binh, Vietnam

Le Thi Phuong Dong

Pedagogical University, Hai Phong, Vietnam

Abstract. The ground field k is assumed to be infinite. We denote by K a field extension of k. Let $x = (x_1, \dots, x_n)$ be indeterminates and u an indeterminate, which is considered as a parameter. The specialization of an ideal I of $R = k(u)[x]$ with respect to the substitution $u \to \alpha$ was defined as the ideal I_α, which is generated by the set $\{f(\alpha, x) \mid f(u,x) \in I \cap k[u,x]\}$. The theory of specialization of ideals was introduced by W. Krull. Krull has showed that the ideal I_α inherits most of basic properties of I and it was used to prove many important results in algebra and in algebraic geometry. In this paper we introduce and study specializations of an inverse system of finitely generated S-modules over directed set $\mathbb{N}$ and of its inverse limit. Some basic properties of specializations of inverse limits and of completion of modules by specializations are developed.

Introduction

The purpose of this paper is to give the definition of specializations of modules which are inverse limits of a inverse system of finitely generated modules over directed set $\mathbb{N}$ of all non-negative integers and to show that the preservation of the index of reduibility of modules through specializations. First of all we fix some of notations that will be used throughout this paper. The groundfield k is always assumed to be infinite. We will refer to denote by K a fixed field extension of k. Let $x_1, \dots, x_n$ or $\alpha_1, \dots, \alpha_m$, where $\forall \alpha_i \in K$, will often be dnoted by x or α. Accordingly, the ring or field extensions $k[x_1, \dots, x_n]$ or $k(\alpha_1, \dots, \alpha_m)$ will be written $k[x]$ or $k(\alpha)$, with evident variants of these designations. Let $u = (u_1, \dots, u_m)$ be a family of indeterminates. We denote by R and R_α the polynomial rings $k(u)[x] := k(u)[x_1, \dots, x_n]$ and $k(\alpha)[x] := k(\alpha)[x_1, \dots, x_n]$. In this paper we shall say that a property holds for almost all α if it holds for all α except perhaps those lying on a proper algebraic subvariety of K^m.

The theory of specialization of ideals was introduced by W. Krull [4]. Following [8] and [10] the specialization of an ideal I of a polynomial ring $R = k(u)[x]$ with respect to the substitution $u \longrightarrow \alpha$ was defined as the ideal I_α generated by elements of the set $\{f(\alpha, x) \mid f(u,x) \in I \cap k[u,x]\}$. For almost all substitutions $u \longrightarrow \alpha$, the ideal I_α inherits most of the basic properties of I. Using specializations of finitely generated free modules and of homomorphisms between them, we developed in [5], [6], [7] the theory of specializations of finitely generated modules, and we showed that basic properties and operations on modules are preserved by specializations. So far we have concerned oursolves only with specializations of finitely generated modules. We turn now to describe definition

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only with specializations of finitely generated modules. We turn now to describe definition of specialization for modules which are not finitely generated. We also propose in this paper to give the definition of specializations of modules which are inverse limits of an inverse system of finitely generated modules over directed set $\mathbb{N}$. For the purpose of present paper, a more limited assumption of the ground-field k and the number of indeterminates u_i is used. Throughout this paper we shall restrict ourselves to the case k is an arbitrary infinite uncountless field and $m = 1$.

This paper is divided into three sections. In section 1 we want to give the definition of specializations of an inverse system of finitely generated S-modules over directed set $\mathbb{N}$ and of its inverse limit. Here, some basic properties of specializations of inverse limits are developed. Section 2 is devoted to the discussion of completion of modules by specializations. In section 3 we shall see that the index of reducibility of modules are unchanged through a specialization.

1. Specialization of an inverse system of modules

We propose in this section to describe the definition of specialization of an S-module which is an inverse limit of an inverse system of finitely generated S-modules indexed by $\mathbb{N}$ of all non-negative integers.

Given a indeterminate u and a element α of an extension K of k. Set $R = k(u)[x]$ and $R_\alpha = k(\alpha)[x]$. Let P be an arbitrary separable prime ideal of R. By [4, Satz 14], P_α is a radical ideal of R_α. Assume that $\mathfrak{p}$ is an arbitrary associated prime ideal of P_α. For short we will put $S = R_P$ and $S_\alpha = (R_\alpha)_\mathfrak{p}$. We denote PS and $\mathfrak{p}S_\alpha$ by $\mathfrak{m}$ and $\mathfrak{m}_\alpha$. An arbitrary element $f \in R$ may be written in the form

$$f = \frac{p(u,x)}{q(u)},\ p(u,x) \in k[u,x],\ q(u) \in k[u] \setminus \{0\}.$$

In [7], for any α such that $q(\alpha) \neq 0$ we define $f_\alpha := p(\alpha, x)/q(\alpha)$. For every element

$$a = \frac{f}{g} \in S,\ f, g \in R,\ g \neq 0,$$

we define $a_\alpha := f_\alpha/g_\alpha$ if $g_\alpha \neq 0$. Then a_α is uniquely determined and belongs to S_α for almost all α, (see [6]). First we will recall the definition of specialization of a finitely generated S-module.

Let L be a finitely generated S-module. Let $S^s \xrightarrow{\varphi} S^t \longrightarrow L \longrightarrow 0$ be a finite free presentation of L, where φ is represented by the matrix $A = (a_{ij}(u,x))$ with $a_{ij}(u,x) \in S$. Set $A_\alpha = (a_{ij}(\alpha, x))$, and the homomorphism

$$\varphi_\alpha : S_\alpha^s \longrightarrow S_\alpha^t$$

is represented by the matrix A_α. The specialization of L, which is denoted by L_α, is defined as $L_\alpha = \operatorname{Coker} \varphi_\alpha$.

Let $\varphi : L \longrightarrow M$ be a homomorphism of finitely generated S-modules. Consider a commutative diagram

$$\begin{array}{ccccccc} F_1 & \xrightarrow{\phi} & F_0 & \longrightarrow & L & \longrightarrow & 0 \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi} & & \\ G_1 & \xrightarrow{\psi} & G_0 & \longrightarrow & M & \longrightarrow & 0, \end{array}$$

where the rows are finite free presentations of L, M. There is an induced homomorphism $\varphi_\alpha : L_\alpha \longrightarrow M_\alpha$, which makes the diagram

$$\begin{array}{ccccccc} (F_1)_\alpha & \xrightarrow{\phi_\alpha} & (F_0)_\alpha & \longrightarrow & L_\alpha & \longrightarrow & 0 \\ \downarrow{\scriptstyle (\varphi_1)_\alpha} & & \downarrow{\scriptstyle (\varphi_0)_\alpha} & & \downarrow{\scriptstyle \varphi_\alpha} & & \\ (G_1)_\alpha & \xrightarrow{\psi_\alpha} & (G_0)_\alpha & \longrightarrow & M_\alpha & \longrightarrow & 0 \end{array}$$

commutative for almost all α. The homomorphism φ_α is called a specialization of $\varphi : L \longrightarrow M$ with respect to (ϕ, ψ), see [6]. To establish basic properties of specializations of S-modules the following theorem is often used.

Theorem 1.1. [6, Theorem 2.2] *Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of finitely generated S-modules. Then $0 \longrightarrow L_\alpha \longrightarrow M_\alpha \longrightarrow N_\alpha \longrightarrow 0$ is exact for almost all α.*

The definitions of L_α and of φ depend on the chosen presentations of L and M. From Theorem 1.1 it follows that that L_α and φ_α are uniquely determined up to isomorphisms.

Modifying the definition of specialization of finitely generated S-module, we can give the definition of specialization of an S-module which is an inverse limit of an inverse system of finitely generated S-modules indexed by $\mathbb{N}$. Now we will recall the basic facts about the inverse limits [1].

Let $(L_i)_{i \in \mathbb{N}}$ be a family of finitely generated S-modules indexed by $\mathbb{N}$. For each pair $i, j \in \mathbb{N}$ such that $i \leq j$, let $f_{ij} : L_j \longrightarrow L_i$ be an S-homomorphism, and suppose that the following conditions are satisfied:

(1) f_{ii} is the identity mapping of L_i for all $i \in \mathbb{N}$,

(2) $f_{ih} = f_{ij} \circ f_{jh}$ whenever $i \leq j \leq h$.

Then we have an inverse system (L_i, f_{ij}) over directed set $\mathbb{N}$. An inverse system (L_i, f_{ij}) is called a *surjective system* if every homomorphism $d_{i+1} : L_{i+1} \longrightarrow L_i$ is an epimomorphism for all $i \in \mathbb{N}$, where $d_{i+1} := f_{ij}$ with $j = i + 1$, see [1].

We consider an inverse system (L_i, f_{ij}) of finitely generated S-modules indexed by $\mathbb{N}$. By definition of specialization of a finitely generated S-module, there is a polynomial $t_i(u) \in k[u]$ such that if $t_i(\alpha) \neq 0$ then we have a specialization $(L_i)_\alpha$; and there are polynomials $t_i(u), t_j(u), t_{ij}(u) \in k[u]$ such that if $t_i(\alpha) \neq 0, t_j(\alpha) \neq 0, t_{ij}(\alpha) \neq 0$, then we have

$$(L_i)_\alpha, (L_j)_\alpha, \text{ and a homomorphism } (f_{ij})_\alpha : (L_j)_\alpha \longrightarrow (L_i)_\alpha$$

for each pair $i, j \in \mathbb{N}$ with $i \leq j$. By Theorem 1.1, it is easy to show that if (L_i, f_{ij}) is a surjective system, then $((L_i)_\alpha, (f_{ij})_\alpha)$ is also a surjective system. Basing on the fact that the set $\{t_i(u), t_{ij}(u) \mid i, j \in \mathbb{N}\}$ is countable, and therefore the collection A of all solutions of $t_i(u) = 0$ and of $t_{ij}(u) = 0$ for all $i, j \in \mathbb{N}$ is countable, one can obtain an inverse system $((L_i)_\alpha, (f_{ij})_\alpha)$ over the directed set $\mathbb{N}$ for all $\alpha \in k \setminus A \neq \emptyset$.

From now on, we shall say that the property T is true for almost α, that is, there exists a countable set A such that T is true for all $\alpha \notin A$. But, for the sake of simplicity, the phrase "for almost α" will be deleted in the proofs of all results.

Definition. *Let (L_i, f_{ij}) be an inverse system of finitely generated S-modules over directed set $\mathbb{N}$. The inverse system $((L_i)_\alpha, (f_{ij})_\alpha)$ of finitely generated R_α-modules $(L_i)_\alpha$ over the directed set $\mathbb{N}$ is called a* specialization *of (L_i, f_{ij}), and it will be denoted by $(L_i, f_{ij})_\alpha$. We call $\varprojlim_i (L_i)_\alpha$ a* specialization *of $\varprojlim_i L_i$, and denote by $(\varprojlim_i L_i)_\alpha$.*

In the case, an S-module L is finitely generated, we can construct the inverse system of the form ($L, f_{ij} : L \longrightarrow L$ is an identical map). There is $L = \varprojlim_i L$, and $L_\alpha = \varprojlim_i L_\alpha$. Thus, the above definition is of course an extension of the definition in [6].

The definition of an inverse system $((L_i)_\alpha, (f_{ij})_\alpha)$ of finitely generated S_α-modules $(L_i)_\alpha$ over the directed set $\mathbb{N}$ depends on the chosen presentations of L_i and f_{ij}. To show that $(L_i, f_{ij})_\alpha$ is uniquely determined up isomorphisms we need to define the specialization of a homomorphism of inverse systems.

Let (L_i, f_{ij}) and (M_i, g_{ij}) be inverse systems of finitely generated S-modules over the same directed set $\mathbb{N}$, and $\{\varphi_i : L_i \longrightarrow M_i\}$ a family of homomorphisms which define a homomorphism

$$\varphi : (L_i, f_{ij}) \longrightarrow (M_i, g_{ij})$$

such that there are commutative diagrams

$$\begin{array}{ccc} L_j & \xrightarrow{f_{ij}} & L_i \\ \downarrow{\scriptstyle \varphi_j} & & \downarrow{\scriptstyle \varphi_i} \\ M_j & \xrightarrow{g_{ij}} & M_i, \end{array}$$

for all $i, j \in \mathbb{N}$ with $i \leq j$. For almost α we have a family of S_α-module homomorphisms $\{(\varphi_i)_\alpha : (L_i)_\alpha \longrightarrow (M_i)_\alpha\}$ and commutative diagrams

$$\begin{array}{ccc} (L_j)_\alpha & \xrightarrow{(f_{ij})_\alpha} & (L_i)_\alpha \\ \downarrow{\scriptstyle (\varphi_j)_\alpha} & & \downarrow{\scriptstyle (\varphi_i)_\alpha} \\ (M_j)_\alpha & \xrightarrow{(g_{ij})_\alpha} & (M_i)_\alpha, \end{array}$$

Definition. *Let (L_i, f_{ij}) and (M_i, g_{ij}) be inverse systems of finitely generated S-modules over the same directed set $\mathbb{N}$, and $\{\varphi_i : L_i \longrightarrow M_i\}$ a family of homomorphisms determine a homomorphism $\varphi : (L_i, f_{ij}) \longrightarrow (M_i, g_{ij})$. We call the family of S_α-module homomorphisms $\{(\varphi_i)_\alpha : (L_i)_\alpha \longrightarrow (M_i)_\alpha\}$ a* specialization *of φ, and denote by φ_α.*

Now we want to prove the main theorem about specializations of inverse systems of finitely generated S-modules over the directed set $\mathbb{N}$.

Theorem 1.2. *Let (L_i, f_{ij}), (M_i, g_{ij}), (N_i, h_{ij}) be inverse systems of finitely generated S-modules over the directed set $\mathbb{N}$. If the sequence*

$$0 \longrightarrow (L_i, f_{ij}) \xrightarrow{\varphi} (M_i, g_{ij}) \xrightarrow{\psi} (N_i, h_{ij}) \longrightarrow 0$$

is an exact sequence, then the sequence

$$0 \longrightarrow (L_i, f_{ij})_\alpha \varphi_\alpha \longrightarrow (M_i, g_{ij})_\alpha \xrightarrow{\psi_\alpha} (N_i, h_{ij})_\alpha \longrightarrow 0$$

is exact for almost α.

Proof. Assume that the sequence

$$0 \longrightarrow (L_i, f_{ij}) \xrightarrow{\varphi} (M_i, g_{ij}) \xrightarrow{\psi} (N_i, h_{ij}) \longrightarrow 0$$

is an exact sequence of inverse systems of finitely generated S-modules over the same directed set $\mathbb{N}$. Then the sequences

$$0 \longrightarrow L_i \xrightarrow{\varphi_i} M_i \xrightarrow{\psi_i} N_i \longrightarrow 0$$

are exact and the diagrams

$$\begin{array}{ccccccccc} 0 \longrightarrow & L_j & \xrightarrow{\varphi_j} & M_j & \xrightarrow{\psi_j} & N_j & \longrightarrow & 0 \\ & \downarrow f_{ij} & & \downarrow g_{ij} & & \downarrow h_{ij} & & \\ 0 \longrightarrow & L_i & \xrightarrow{\varphi_i} & M_i & \xrightarrow{\psi_i} & N_i & \longrightarrow & 0 \end{array}$$

are commutative for all $i, j \in \mathbb{N}$ with $i \leq j$. By Theorem 1.1, the sequences

$$0 \longrightarrow (L_i)_\alpha \xrightarrow{(\varphi_i)_\alpha} (M_i)_\alpha \xrightarrow{(\psi_i)_\alpha} (N_i)_\alpha \longrightarrow 0$$

are exact and by the above definition the following diagrams

$$\begin{array}{ccccccccc} 0 \longrightarrow & (L_j)_\alpha & \xrightarrow{(\varphi_j)_\alpha} & (M_j)_\alpha & \xrightarrow{(\psi_j)_\alpha} & (N_j)_\alpha & \longrightarrow & 0 \\ & \downarrow (f_{ij})_\alpha & & \downarrow (g_{ij})_\alpha & & \downarrow (h_{ij})_\alpha & & \\ 0 \longrightarrow & (L_i)_\alpha & \xrightarrow{(\varphi_i)_\alpha} & (M_i)_\alpha & \xrightarrow{(\psi_i)_\alpha} & (N_i)_\alpha & \longrightarrow & 0, \end{array}$$

are commutative for all $i, j \in \mathbb{N}$ with $i \leq j$. Therefore the sequence

$$0 \longrightarrow (L_i, f_{ij})_\alpha \xrightarrow{\varphi_\alpha} (M_i, g_{ij})_\alpha \xrightarrow{\psi_\alpha} (N_i, h_{ij})_\alpha \longrightarrow 0$$

is exact for almost α. As the immediate consequences of Theorem 1.2 we have some following corollaries.

Corollary 1.3. *Let (L_i, f_{ij}) and (M_i, g_{ij}) be direct systems of finitely generated S-modules over the same directed set $\mathbb{N}$. Let*

$$\varphi : (L_i, f_{ij}) \longrightarrow (M_i, g_{ij})$$

be a homomorphism. Then φ_α is injective (resp. surjective) if φ is injective (resp. surjective) for almost α.

Corollary 1.4. *The specializations $(L_i, f_{ij})_\alpha$ of (L_i, f_{ij}) is uniquely defined up to an isomorphism.*

The following proposition follows from applying Theorem 1.2.

Proposition 1.5. *Let (L_i, f_{ij}), (M_i, g_{ij}), (N_i, h_{ij}) be surjective systems of finitely generated S-modules over the directed set $\mathbb{N}$. Set $L = \varprojlim_i L_i$, $M = \varprojlim_i M_i$, $N = \varprojlim_i N_i$. If the sequence*

$$0 \longrightarrow (L_i, f_{ij}) \xrightarrow{\varphi} (M_i, g_{ij}) \xrightarrow{\psi} (N_i, h_{ij}) \longrightarrow 0$$

is an exact sequence, then the sequence

$$0 \longrightarrow L_\alpha \longrightarrow M_\alpha \longrightarrow N_\alpha \longrightarrow 0$$

is exact for almost α.

2. $\mathfrak{m}$-adic completion by specialization

As an application, we shall restrict ourselves to the specializations of $\mathfrak{m}$-adic completion of modules. From now on the ground-field k will be assumed perfect.

Let L be a finitely generated S-module. We proceed to establish the fundamental result about a specialization of the $\mathfrak{m}$-adic completion of L. Denote by $\hat{L}$ the $\mathfrak{m}$-adic completion of L and by $(\hat{L})_\alpha$ a specialization of $\hat{L}$. Let L_α be a specialization of L. The $\mathfrak{m}_\alpha$-adic completion of L_α will be denoted by $\widehat{L_\alpha}$. We shall see that completion and specialization are commute.

Theorem 2.1. *Let L be a finitely generated S-module. Then, for almost α, we get*

$$\widehat{L_\alpha} \cong (\hat{L})_\alpha.$$

Proof. By definition of the $\mathfrak{m}$-adic completion of L we have $\hat{L} = \varprojlim_j L/\mathfrak{m}^j L$. By [6, Proposition 3.2 and 3.6], we have $L_\alpha/\mathfrak{m}_\alpha^j L_\alpha \cong (L/\mathfrak{m}^j L)_\alpha$ for every $j \in \mathbb{N}$. Let $d_{j+1} : L/\mathfrak{m}^{j+1}L \longrightarrow L/\mathfrak{m}^j L$ be the natural map induced by identity map on S. Then $(d_{j+1})_\alpha : L_\alpha/\mathfrak{m}_\alpha^{j+1} L_\alpha \longrightarrow L_\alpha/\mathfrak{m}_\alpha^j L_\alpha$ is the natural map induced by identity map on S_α and $(L_\alpha/\mathfrak{m}_\alpha^j L_\alpha, (d_{j+1})_\alpha)$ is a specialization of $(L/\mathfrak{m}^j L, d_{j+1})$. By Proposition 1.5, we get $(\hat{L})_\alpha \cong \varprojlim_j L_\alpha/\mathfrak{m}_\alpha^j L_\alpha = \widehat{L_\alpha}$ for almost α.

Corollary 2.2. *Let L be a finitely generated S-module. Then, for almost α, we have*

$$\dim(\hat{L})_\alpha = \dim \hat{L} \text{ and } \operatorname{depth}(\hat{L})_\alpha = \operatorname{depth} \hat{L}.$$

Proof. By [3, Corollary 2.1.8], there are $\dim \hat{L} = \dim L$ and $\dim \widehat{L_\alpha} = \dim L_\alpha$. In view of Theorem 2.1, there is $\dim(\hat{L})_\alpha = \dim \widehat{L_\alpha}$. Since $\dim L_\alpha = \dim L$ by [6, Theorem 2.6], $\dim(\hat{L})_\alpha = \dim \hat{L}$. The equality $\operatorname{depth}(\hat{L})_\alpha = \operatorname{depth} \hat{L}$ will be proved in the same way as in the proof above.

Corollary 2.3. *Let L be a finitely generated S-module of positive dimension. If $\hat{L}$ is Buchsbaum, so is $(\hat{L})_\alpha$ for almost α.*

Proof. Assume that $\hat{L}$ is a Buchsbaum module. Then L is Buchsbaum by [9, Chapter 1 Lemma 1.13]. Since L_α is again Buchsbaum by [6, Corollary 3.8], $\widehat{L_\alpha}$ is Buchsbaum by [9, Chapter 1 Lemma 1.13]. Hence $(\hat{L})_\alpha$ is a Buchsbaum module by Theorem 2.1.

Corollary 2.4. *Let L be a finitely generated S-module. For almost α,*

$$(L \otimes_S \hat{S})_\alpha \cong L_\alpha \otimes_{S_\alpha} \widehat{S_\alpha}.$$

Proof. There are $L \otimes_S \hat{S} \cong \hat{L}$ and $L_\alpha \otimes_{S_\alpha} \widehat{S_\alpha} \cong \widehat{L_\alpha}$ by [1, Proposition 10.13]. Since $(L \otimes_S \hat{S})_\alpha \cong (\hat{L})_\alpha$, we have $(L \otimes_S \hat{S})_\alpha \cong L_\alpha \otimes_{S_\alpha} \widehat{S_\alpha}$ by Theorem 2.1.

3. Preservation of index of reducibility of modules by specialization

The present section will be devoted to proving the preservation of indexes of reducibility through a specialization. The module $\mathrm{Soc}(L) = 0_L : \mathfrak{m}$ is called the *socle* of L. In [2], for an ideal $\mathfrak{q}$ of S such that the length $\ell(L/\mathfrak{q}L)$ of $L/\mathfrak{q}L$ is finite, the *index of reducibility* of $\mathfrak{q}$ relative to L is defined as

$$\mathrm{N}_S(\mathfrak{q}, L) = \dim_{S/\mathfrak{m}} \mathrm{Soc}(L/\mathfrak{q}L).$$

The *type* of L is defined as the number

$$r_S(L) = \sup\{\mathrm{N}_S(\mathfrak{q}, L) \mid \mathfrak{q} \text{ is a parameter ideal for } L\}.$$

We note that, for an $\mathfrak{m}$-primary ideal $\mathfrak{q}$, $\mathrm{N}_S(\mathfrak{q}, S/\mathfrak{q})$ is the number of irreducible ideals that appear in an irredundant irreducible decomposition of $\mathfrak{q}$. The following lemma shows that the index of reducibility of $\mathfrak{q}$ relative to L is preserved through specialization.

Lemma 3.1. *Let L be a finitely generated S-module and let $\mathfrak{q}$ be an ideal of S such that the length $\ell(L/\mathfrak{q}L)$ of $L/\mathfrak{q}L$ is finite. For almost α, we have*

$$\mathrm{N}_S(\mathfrak{q}, L) = \mathrm{N}_{S_\alpha}(\mathfrak{q}_\alpha, L_\alpha).$$

Proof. By [6, Lemma 2.3 and Lemma 2.5], we get $(L/\mathfrak{q}L)_\alpha \cong L_\alpha/\mathfrak{q}_\alpha L_\alpha$. Then

$$(\mathrm{Soc}(L/\mathfrak{q}L))_\alpha = (0_{L/\mathfrak{q}L} : \mathfrak{m})_\alpha \cong 0_{L_\alpha/\mathfrak{q}_\alpha L_\alpha} : \mathfrak{m}_\alpha = \mathrm{Soc}(L_\alpha/\mathfrak{q}_\alpha L_\alpha).$$

Since the length $\ell(L/\mathfrak{q}L)$ of $L/\mathfrak{q}L$ is finite, by using [6, Proposition 2.8] we obtain $\ell(L_\alpha/\mathfrak{q}_\alpha L_\alpha) = \ell(L/\mathfrak{q}L)$. Hence $\ell(L_\alpha/\mathfrak{q}_\alpha L_\alpha)$ is finite. Since $\mathrm{Soc}(L/\mathfrak{q}L)$ is a submodule of $L/\mathfrak{q}L$, the length $\ell(\mathrm{Soc}(L/\mathfrak{q}L))$ is finite. Because

$$\ell(\mathrm{Soc}(L_\alpha/\mathfrak{q}_\alpha L_\alpha)) = \ell(\mathrm{Soc}(L/\mathfrak{q}L))$$

by [6, Proposition 2.8], there is $\mathrm{N}_S(\mathfrak{q}, L) = \mathrm{N}_{S_\alpha}(\mathfrak{q}_\alpha, L_\alpha)$ for almost α.

Let $H^i_{\mathfrak{m}}(L)$ denote the ith local cohomology module of L with respect to $\mathfrak{m}$. Recall that L is called a *quasi Buchsbaum module* if $\mathfrak{m}H^i_{\mathfrak{m}}(L) = 0$ for all $i \neq d = \dim L$. Put

$$K_L = \mathrm{Hom}_S(H^d_{\mathfrak{m}}(L), E(S/\mathfrak{m})),$$

where $E(S/\mathfrak{m})$ denotes the injective hull of the residue field $S/\mathfrak{m}$. This module K_L is again a finitely generated S-module and is said to be the *canonical module* of L, see [2].

Lemma 3.2. *Let L be a finitely generated S-module. For almost α, $(K_L)_\alpha \cong K_{L_\alpha}$.*

Proof. By [6, Theorem 2.6], $\dim L_\alpha = \dim L = d$. Let $r = \dim S = \dim S_\alpha$. Since S and S_α are regular rings, they are Gorenstein rings. Therefore $K_L = \operatorname{Ext}_S^{r-d}(M, S)$ and $K_{L_\alpha} = \operatorname{Ext}_{S_\alpha}^{r-d}(L_\alpha, S_\alpha)$ by Matlis duality. Since $\operatorname{Ext}_S^{r-d}(L, S)_\alpha \cong \operatorname{Ext}_{S_\alpha}^{r-d}(L_\alpha, S_\alpha)$ by [6, Proposition 3.3], we have $(K_L)_\alpha \cong K_{L_\alpha}$ for almost α.

We now will prove that the type of a quasi Buchsbaum module is preserved through a specialization. In our proof, the minimal number of generators for L will be denoted by $\mu(L)$.

Theorem 3.3. *Let L be a finitely generated S-module. If L is a quasi Buchsbaum module, then $r_{S_\alpha}(L_\alpha) = r_S(L)$ for almost α.*

Proof. Assume that L is a quasi Buchsbaum module of dimension d. By [2, Theorem 2.5], we have

$$r_S(L) = \sum_{i=0}^{d-1} \binom{d}{i} \ell(H^i_{\mathfrak{m}}(L)) + \mu(K_{\hat{L}}).$$

By [6, Lemma 3.5], L_α is again a quasi Buchsbaum module. Since

$$K_{\hat{L}} = \operatorname{Ext}_S^{r-d}(L, S) \otimes_S \hat{S} = K_L \otimes_S \hat{S},$$

it follows that

$$(K_{\hat{L}})_\alpha \cong (K_L \otimes_S \hat{S})_\alpha.$$

Since $(K_L \otimes_S \hat{S})_\alpha \cong (K_L)_\alpha \otimes_{S_\alpha} \widehat{S_\alpha} \cong K_{L_\alpha} \otimes_{S_\alpha} \widehat{S_\alpha}$ by Corollary 2.4 and Lemma 3.2, it holds that $(K_{\hat{L}})_\alpha \cong K_{\widehat{L_\alpha}}$. We have $\mu(K_{\widehat{L_\alpha}}) = \mu(K_{L_\alpha}) = \mu(K_L) = \mu(K_{\hat{L}})$. Because $\ell(H^i_{\mathfrak{n}}(L_\alpha)) = \ell(H^i_{\mathfrak{m}}(L))$ for all $i \neq d$ by [6, Theorem 3.6], we get $r_{S_\alpha}(L_\alpha) = r_S(L)$.

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