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Almost Sure Exponential Stability of Stochastic Differential Delay Equations on Time Scales

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Abstract: The aim of this paper is to study the almost sure exponential stability of stochastic differential delay equations on time scales. This work can be considered as a unification and generalization of stochastic difference and stochastic differential delay equations.

Keywords: Delay equation, almost sure exponential stability, Ito formula, Lyapunov function.

1. Introduction

The stochastic differential/difference delay equations have come to play an important role in describing the evolution of eco-systems in random environment, in which the future state depends not only on the present state but also on its history. Therefore, their qualitative and quantitative properties have received much attention from many research groups (see [1, 2] for the stochastic differential delay equations and [3-6] for the stochastic difference one).

In order to unify the theory of differential and difference equations into a single set-up, the theory of analysis on time scales has received much attention from many research groups. While the deterministic dynamic equations on time scales have been investigated for a long history (see [7-11]), as far as we know, we can only refer to very few papers [12-15] which contributed to the stochastic dynamics on time scales. Moreover, there is no work dealing with the stochastic dynamic delay equations.

Recently, in [14], we have studied the exponential p-stability of stochastic ∇-dynamic equations on time scale, via Lyapunov function. Continuing the idea of this article [14], we study the almost sure exponential stability of stochastic dynamic delay equations on time scales.

Motivated by the aforementioned reasons, the purpose of this paper is to use Lyapunov function to consider the almost sure exponential stability of ∇-stochastic dynamic delay equations on time scale $\mathbb{T}$.

The organization of this paper is as follows. In Section 1 we survey some basic notation and properties of the analysis on time scales. Section 2 is devoted to giving definition and some theorems,

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corollaries for the almost sure exponential stability for ∇-stochastic dynamic delay equations on time scale and some examples are provided to illustrate our results.

2. Preliminaries on time scales

Let $\mathbb{T}$ be a closed subset of $\mathbb{R}$, enclosed with the topology inherited from the standard topology on $\mathbb{R}$. Let $\sigma(t)=\inf\{s\in\mathbb{T}: s>t\}, \mu(t)=\sigma(t)-t$ and $\rho(t)=\sup\{s\in\mathbb{T}: s<t\}, \nu(t)=t-\rho(t)$ (supplemented by $\sup\varnothing=\inf\mathbb{T}, \inf\varnothing=\sup\mathbb{T}$). A point $t\in\mathbb{T}$ is said to be *right-dense* if $\sigma(t)=t$, *right-scattered* if $\sigma(t)>t$, *left-dense* if $\rho(t)=t$, *left-scattered* if $\rho(t)<t$ and isolated if t is simultaneously right-scattered and left-scattered. The set ${}_k\mathbb{T}$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a right-scattered minimum; otherwise it is $\mathbb{T}$ without this right-scattered minimum. A function f defined on $\mathbb{T}$ is *regulated* if there exist the left-sided limit at every left-dense point and right-sided limit at every right-dense point. A regulated function is called *ld*-continuous if it is continuous at every left-dense point. Similarly, one has the notion of *rd*-continuous. For every $a,b\in\mathbb{T}$, by *[a,b]*, we mean the set $\{t\in\mathbb{T}: a\le t\le b\}$. Denote $\mathbb{T}_a=\{t\in\mathbb{T}: t\ge a\}$ and by $\mathcal{R}$ $(resp.\ \mathcal{R}^+)$ the set of all *rd*-continuous and regressive (resp. positive regressive) functions. For any function f defined on $\mathbb{T}$, we write f^ρ for the function $f\circ\rho$; i.e., $f_t^\rho=f(\rho(t))$ for all $t\in{}_k\mathbb{T}$ and $\lim_{\sigma(s)\uparrow t} f(s)$ by $f(t_-)$ or f_{t_-} if this limit exists. It is easy to see that if t is left-scattered then $f_{t_-}=f_t^\rho$. Let

$$I=\{\ t:\ t \text{ is left-scattered}\}.$$

Clearly, the set I of all left-scattered points of $\mathbb{T}$ is at most countable.

Throughout of this paper, we suppose that the time scale $\mathbb{T}$ has bounded graininess, that is $\nu_*=\sup\{\nu(t): t\in{}_k\mathbb{T}\}<\infty$.

Let A be an increasing right continuous function defined on $\mathbb{T}$. We denote by μ_∇^A the Lebesgue ∇-measure associated with A. For any μ_∇^A-measurable function $f:\mathbb{T}\to\mathbb{R}$ we write $\int_a^t f_\tau\nabla A_\tau$ for the integral of f with respect to the measures μ_∇^A on $(a,t]$. It is seen that the function $t\to\int_a^t f_\tau\nabla A_\tau$ is cadlag. It is continuous if A is continuous. In case $A(t)\equiv t$ we write simply $\int_a^t f_\tau\nabla\tau$ for $\int_a^t f_\tau\nabla A_\tau$. For details, we can refer to [7].

In general, there is no relation between the Δ-integral and ∇-integral. However, in case the integrand f is regulated one has

$$\int_a^b f(\tau_-)\nabla\tau=\int_a^b f(\tau)\Delta\tau,\ \forall a,b\in\mathbb{T}^k.$$

Indeed, by [7, Theorem 6.5],

$$\int_a^b f(\tau)\Delta\tau = \int_{[a;b)} f(\tau)d\tau + \sum_{a\le s<b} f(s)\mu(s)$$
$$= \int_{(a,b]} f(\tau_-)d\tau + \sum_{a<s\le b} f(s_-)\nu(s) = \int_a^b f(\tau_-)\nabla\tau.$$

Therefore, if $p \in \mathcal{R}$ then the exponential function $e_p(t,t_0)$ defined by [2, Definition 2.30, pp. 59] is solution of the initial value problem

$$y^\nabla(t) = p(t_-)y(t_-),\ y(t_0)=1,\ t>t_0.$$

Also if $p \in \mathcal{R}$, $e_{\ominus p}(t,t_0)$ is the solution of the equation

$$y^\nabla(t) = -p(t_-)y(t),\ y(t_0)=1,\ t>t_0,$$

where $\ominus p(t) = \dfrac{-p(t)}{1+\mu(t)p(t)}$.

Theorem 1.1 (Ito formula, [16]). Let $X=(X_1,\ldots,X_d)$ be a $d-$tuple of semimartingales, and let $V:\mathbb{R}^d\to\mathbb{R}^d$ be a twice continuously differentiable function. Then $V(X)$ is a semimartingale and the following formula holds

$$V(X(t)) = V(X(a)) + \sum_{i=1}^d \int_a^t \frac{\partial V}{\partial x_i}(X(\tau_-))\nabla X_i(\tau) \quad + \frac{1}{2}\sum_{i,j}\int_a^t \frac{\partial^2 V}{\partial x_i x_j}(X(\tau_-))\nabla[X_i,X_j]_\tau$$
$$+\sum_{s\in(a,t]}(V(X(s))-V(X(s_-))) - \sum_{s\in(a,t]}\sum_{i=1}^d \frac{\partial V}{\partial x_i}(X(s_-))\nabla^* X_i(s)$$
$$-\frac{1}{2}\sum_{s\in(a,t]}\sum_{i,j}\int_a^t \frac{\partial^2 V}{\partial x_i x_j}(X(s_-))(\nabla^* X_i(s))(\nabla^* X_j(s)),$$

where $\nabla^* X_i(s) = X_i(s) - X_i(s_-)$.

3. Almost sure exponential stability of stochastic dynamic delay equations

Let $\mathbb{T}$ be a time scale and with fixed $a\in\mathbb{T}$. We say that the *rd*-map $\delta(\cdot):\mathbb{T}\to\mathbb{T}$ is a delay function if $\delta(t)\le t-$ for all $t\in\mathbb{T}$ and $\sup\{t-\delta(t): t\in\mathbb{T}\}<\infty$. For any $s\in\mathbb{T}$, we see that $b_s := \min\{\delta(t): t\ge s\} > -\infty$. Denote $\Gamma_s = \{\delta(t): t\ge s\}\cap[b_s,s]$. We write simply Γ for Γ_s if there is no confusion. Let $C(\Gamma_s;\mathbb{R}^d)$ be the family of continuous functions from Γ_s to $\mathbb{R}^d$ with the norm $\|\varphi\|_s = \sup_{s\in\Gamma_s}|\varphi(s)|$. Fix $t_0\in\mathbb{T}$ and let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{T}_{t_0}},\mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}_{t_0}}$ satisfying the usual conditions (i.e., $\{\mathcal{F}_t\}_{t\in\mathbb{T}_{t_0}}$ is increasing and right continuous while $\mathcal{F}_{t_0}$

contains all P-null sets). Denote by $\mathcal{M}_2$ the set of the square integrable $\mathcal{F}_t$-martingales and by $\mathcal{M}_2^r$ the subspace of the space $\mathcal{M}_2$ consisting of martingales with continuous characteristics. Let $M \in \mathcal{M}_2$ with the characteristic $\langle M\rangle_t$ (see [5]). We write $\mathcal{L}_2([t_0,T],\mathbb{R}^d,M)$ for the set of the processes $h(t)$, valued in $\mathbb{R}^d$, $\mathcal{F}_t$-adapted such that

$$\mathbb{E}\int_{t_0}^{T} h^2(t)\nabla\langle M\rangle_t < \infty.$$

For any $f \in \mathcal{L}_2([t_0,T],\mathbb{R}^d,M)$ we can define the stochastic integral

$$\int_{t_0}^{b} f(s)\nabla M_s$$

(see [5] in detail).

Denote also by $\mathcal{L}_1([t_0,T];\mathbb{R}^d)$ the set of functions $f:[t_0,T]\to\mathbb{R}^d$ such that

$$\int_{t_0}^{T} f(t)\nabla t < \infty.$$

We now consider the ∇-stochastic dynamic delay equations on time scale

$$\begin{cases} d^{\nabla}X(t) = f(t, X(t_-), X(\delta(t)))d^{\nabla}t + g(t, X(t_-), X(\delta(t)))d^{\nabla}M(t), t \in \mathbb{T}_{t_0} \\ X(s = \xi(s) \ \ \forall s \in \Gamma_{t_0}, \end{cases} \quad (2.1)$$

where $f:\mathbb{T}\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d;$ $g:\mathbb{T}\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ are two Borel functions and and $\xi=\{\xi(s): b_{t_0} \le s \le t_0\}$ is a $C(\Gamma_{t_0};\mathbb{R}^d)$-valued, $\mathcal{F}_{t_0}$-measurable random variable with $E\|\xi\|_{t_0}^2 < +\infty$.

Definition 2.1. An stochastic process $\{X(t)\}_{t\in[b_{t_0},T]}$, valued in $\mathbb{R}^d$, is called a solution of the equation (2.1) if

(i) $\{X(t)\}$ is $\mathcal{F}_t$-adapted;

(ii) $f(\cdot, X(\cdot_-), X(\delta(\cdot))) \in \mathcal{L}_1([t_0,T];\mathbb{R}^d)$ and $g(\cdot, X(\cdot_-), X(\delta(\cdot))) \in \mathcal{L}_2([t_0,T],\mathbb{R}^d,M)$;

(iii) $X(t) = \xi(t) \ \forall t \in \Gamma_{t_0}$ and for any $t \in [t_0,T]$ and there holds the equation

$$X(t) = \xi(t_0) + \int_{t_0}^{t} f(s, X(s_-), X(\delta(s)))\nabla s + \int_{t_0}^{t} g(s, X(s_-), X(\delta(s)))\nabla M_s, \ \forall t \in [t_0,T], \quad (2.2)$$

The equation (2.1) is said to have the uniqueness of solutions on $[b_{t_0}, T]$ if $X(t)$ and $\bar{X}(t)$ are two processes satisfying (2.2) then

$$P\{X(t) = \bar{X}(t) \quad \forall t \in [b_{t_0}, T]\} = 1.$$

It is seen that $\int_{t_0}^{t} g(s, X(s_-), X(\delta(s)))\nabla M_s$ is $\mathcal{F}_t$-martingale so it has a cadlag modification. Hence, if $X(t)$ satisfies (2.2) then $X(t)$ is cadlag. In addition, if M_t is *rd*-continuous, so is $X(t)$.

For any $M \in \mathcal{M}_2$, set

$$\hat{M}_t = M_t - \sum_{s\in(t_0,t]}\left(M_s - M_{\rho(s)}\right).$$

It is clearly that

$$\langle\hat{M}\rangle_t = \langle M\rangle_t - \sum_{s\in(t_0,t]}\left(\langle M\rangle_s - \langle M\rangle_{\rho(s)}\right). \quad (2.3)$$

Denote by $\mathcal{B}$ the class of Borel sets in $\mathbb{R}$ whose closure does not contain the point 0. Let $\delta(t,A)$ be the number of jumps of M on the $(t_0, t]$ whose values fall into the set $A \in \mathcal{B}$. Since the sample functions of the martingale M are cadlag, the process $\delta(t,A)$ is defined with probability 1 for all $t \in \mathcal{T}_{t_0}, A \in \mathcal{B}$. We extend its definition over the whole Ω by setting $\delta(t,A) \equiv 0$ if the sample $t \to M_t(\omega)$ is not cadlag. Clearly the process $\delta(t,A)$ is $\mathcal{F}_t$-adapted and its sample functions are nonnegative, monotonically nondecreasing, continuous from the right and take on integer values.

We also define $\hat{\delta}(t,A)$ for $\hat{M}_t$ by a similar way. Let $\tilde{\delta}(t,A) = \#\{s \in (t_0, t] : M_s - M_{\rho(s)} \in A\}$. It is evident that

$$\delta(t,A) = \hat{\delta}(t,A) + \tilde{\delta}(t,A). \quad (2.4)$$

Further, for fixed t, $\delta(t,\cdot), \hat{\delta}(t,\cdot)$ and $\tilde{\delta}(t,.)$ are measures.

The functions $\delta(t,A), \hat{\delta}(t,A)$ and $\tilde{\delta}(t,A), t \in \mathcal{T}_{t_0}$ are $\mathcal{F}_t$-regular submartingales for fixed A. By Doob-Meyer decomposition, each process has a unique representation of the form

$$\delta(t,A) = \zeta(t,A) + \pi(t,A), \quad \hat{\delta}(t,A) = \hat{\zeta}(t,A) + \hat{\pi}(t,A), \quad \tilde{\delta}(t,A) = \tilde{\zeta}(t,A) + \tilde{\pi}(t,A),$$

where $\pi(t,A), \hat{\pi}(t,A)$ and $\tilde{\pi}(t,A)$ are natural increasing integrable processes and $\zeta(t,A), \hat{\zeta}(t,A)$, $\tilde{\zeta}(t,A)$ are martingales. We find a version of these processes such that they are measures when t is fixed. By denoting

$$\hat{M}_t^c = \hat{M}_t - \hat{M}_t^d,$$

Where

$$\hat{M}_t^d = \int_{t_0}^t \int_{\mathbb{R}} u\hat{\zeta}(\nabla\tau, du),$$

we get

$$\langle\hat{M}\rangle_t = \langle\hat{M}^c\rangle_t + \langle\hat{M}^d\rangle_t,\ \ \langle\hat{M}^d\rangle_t = \int_{t_0}^t \int_{\mathbb{R}} u^2\hat{\pi}(\nabla\tau, du).\ (2.5)$$

Throughout this paper, we suppose that $\langle M\rangle_t$ is absolutely continuous with respect to Lebesgue measure μ_∇, i.e., there exists $\mathcal{F}_t$-adapted progressively measurable process K_t such that

$$\langle M\rangle_t = \int_{t_0}^t K_\tau \nabla\tau.\ \ (2.6)$$

Further, for any $T \in \mathbb{T}_{t_0}$,

$$\mathbb{P}\{\sup_{t_0 \le t \le T} |K_t| \le N\} = 1,\ (2.7)$$

where N is a constant (possibly depending on T).

The relations (2.3), (2.5) imply that $\langle\hat{M}^c\rangle_t$ and $\langle\hat{M}^d\rangle_t$ are absolutely continuous with respect to μ_∇ on $\mathbb{T}$. Thus, there exists $\mathcal{F}_t$-adapted, progressively measurable bounded process $\hat{K}_t^c$ and $\hat{K}_t^d$ satisfying

$$\langle\hat{M}^c\rangle_t = \int_{t_0}^t \hat{K}_\tau^c \nabla\tau,\ \ \langle\hat{M}^d\rangle_t = \int_{t_0}^t \hat{K}_\tau^d \nabla\tau,$$

and the following relation holds

$$\mathbb{P}\{\sup_{t_0 \le t \le T} \hat{K}_t^c + \hat{K}_t^d \le N\} = 1.$$

Moreover, it is easy to show that $\hat{\pi}(t, A)$ is absolutely continuous with respect to μ_∇ on $\mathbb{T}$, that is, it can be expressed as

$$\hat{\pi}(t, A) = \int_{t_0}^t \hat{\Upsilon}(\tau, A)\nabla\tau, (2.8)$$

with an $\mathcal{F}_t$-adapted, progressively measurable process $\hat{\Upsilon}(t, A)$. Since $\mathcal{B}$ is generated by a countable family of Borel sets, we can find a version of $\hat{\Upsilon}(t, A)$ such that the map $t \to \hat{\Upsilon}(t, A)$ is measurable and for t fixed, $\hat{\Upsilon}(t, \cdot)$ is a measure.Hence, from [2.5] we see that

$$\langle\hat{M}^d\rangle_t = \int_{t_0}^t \int_{\mathbb{R}} u^2 \hat{\Upsilon}(\tau, du)\nabla\tau.$$

This means that

$$\hat{K}_t^d = \int_{\mathbb{R}} u^2 \hat{\Upsilon}(t, du).$$

The process $\tilde{\pi}(t, A)$ is written in the specific form as following

$$\dot{\pi}(t,A)=\sum_{s\in(t_0,t]}\mathbb{E}[1_A(M_s-M_{\rho(s)})\,|\,\mathcal{F}_{\rho(s)}].$$

Putting $\dot{\Upsilon}(t,A)=\dfrac{\mathbb{E}[1_A(M_t-M_{\rho(t)})\,|\,\mathcal{F}_{\rho(t)}]}{\nu(t)}$ if $\nu(t)>0$ and $\tilde{\Upsilon}(t,A)=0$ if $\nu(t)=0$

yields

$$\tilde{\pi}(t,A)=\int_{t_0}^{t}\tilde{\Upsilon}(\tau,A)\nabla\tau. \quad (2.9)$$

Further, by the definition if $\nu(t)>0$ we have

$$\int_{\mathbb{R}}u\tilde{\Upsilon}(t,du)=\frac{\mathbb{E}\left[M_t-M_{\rho(t)}\,|\,\mathcal{F}_{\rho(t)}\right]}{\nu(t)}=0, \quad (2.10)$$

and

$$\int_{\mathbb{R}}u^2\tilde{\Upsilon}(t,du)=\frac{\mathbb{E}\left[\left(M_t-M_{\rho(t)}\right)^2|\,\mathcal{F}_{\rho(t)}\right]}{\nu(t)}=\frac{\langle M\rangle_t-\langle M\rangle_{\rho(t)}}{\nu(t)}.$$

Let $\Upsilon(t,A)=\hat{\Upsilon}(t,A)+\tilde{\Upsilon}(t,A)$. We see from (2.4) that

$$\pi(t,A)=\int_{t_0}^{t}\Upsilon(\tau,A)\nabla\tau.$$

Let $C^{1,2}(\mathbb{T}_{t_0}\times\mathbb{R}^d;\mathbb{R})$ be the set of all functions $V(t,x)$ defined on $\mathbb{T}_{t_0}\times\mathbb{R}^d$, having continuous ∇-derivative in t and continuous second derivative in x. For any $V\in C^{1,2}(\mathbb{T}_{t_0}\times\mathbb{R}^d;\mathbb{R})$, define the operators $\mathcal{A}V:\mathbb{T}_{t_0}\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ with respect to (2.1) is defined by

$$\mathcal{A}V(t,x,y)=\sum_{i=1}^{d}\frac{\partial V(t,x)}{\partial x_i}(1-1_I(t))f_i(t,x,y)+(V(t,x+f(t,x,y)\nu(t))-V(t,x))\Phi(t)$$

$$+\frac{1}{2}\sum_{i,j}\frac{\partial^2 V(t,x)}{\partial x_i x_j}g_i(t,x,y)g_j(t,x,y)\hat{K}_t^c-\sum_{i=1}^{d}\frac{\partial V(t,x)}{\partial x_i}g_i(t,x,y)\int_{\mathbb{R}}u\hat{\Upsilon}(t,du)$$

$$+\int_{\mathbb{R}}(V(t,x+f(t,x,y)\nu(t)+g(t,x,y)u)-V(t,x+f(t,x,y)\nu(t)))\Upsilon(t,du), \quad (2.11)$$

where

$$\Phi(t)=\begin{cases}0 & \text{if } t \text{ left-dense}\\ \dfrac{1}{\nu(t)} & \text{if } t \text{ left-scattered}\end{cases}$$

Theorem 2.2 (Ito formula, [13]). Let $X=(X_1,\ldots,X_d)$ be a $d-$tuple of semimartingales, and let $V:\mathbb{R}^d\to\mathbb{R}^d$ be a twice continuously differentiable function. Then $V(X)$ is a semimartingale and the following formula holds

$$V(t,X(t))=V(t_0,X(t_0))+\int_{t_0}^{t}\mathcal{L}V(\tau,X(\tau_-),X(\delta(\tau)))\nabla\tau+H_t.\quad(2.12)$$

Where

$$\mathcal{L}V(t,x,y)=V^{\nabla_t}(t,x)+\mathcal{A}V(t,x,y),\quad(2.13)$$

and

$$\Psi(\tau)=V(\tau,X(\tau_-)+f(\tau,X(\tau_-),X(\delta(\tau)))\nu(\tau)+g(\tau,X(\tau_-),X(\delta(\tau)))u)$$
$$-V(\tau,X(\tau_-)+f(\tau,X(\tau_-),X(\delta(\tau)))\nu(\tau)).$$

$$H_t=\sum_{i=1}^{d}\int_{t_0}^{t}\frac{\partial V(\tau,X(\tau_-))}{\partial x_i}g_i(\tau,X(\tau_-),X(\delta(\tau)))\nabla M_\tau+\int_{t_0}^{t}\int_{\mathbb{R}}\Psi(\tau)\tilde{\zeta}(\nabla\tau,du)$$
$$+\int_{t_0}^{t}\int_{\mathbb{R}}\Big(\Psi(\tau)-\sum_{i=1}^{d}u\frac{\partial V(\tau,X(\tau_-))}{\partial x_i}g_i(\tau,X(\tau_-),X(\delta(\tau)))\Big)\hat{\zeta}(\nabla\tau,du).\quad(2.14)$$

Using the Ito formula in [13], we see that for any $V\in C^{1,2}(\mathbb{T}_{t_0}\times\mathbb{R}^d;\mathbb{R}_+)$

$$V(t,X(t))-V(t_0,X(t_0))-\int_{t_0}^{t}\big(V^{\nabla_\tau}(\tau,X(\tau_-))+\mathcal{A}V(\tau,X(\tau_-),X(\delta(\tau)))\big)\nabla\tau\quad(2.15)$$

is a locally integrable martingale, where V^{∇_t} is partial ∇-derivative of $V(t,x)$ in t.

We now give conditions guaranteeing the existence and uniqueness of the solution to the equation (2.1).

Theorem 2.3. (Existence and uniqueness of solution). Assume that there exist two positive constants $\bar{K}$ and K such that

(i) (Lipschitz condition) for all $x_i,y_i\in\mathbb{R}^d$ $i=1,2$ and $t\in[t_0,T]$

$$\|f(t,x_1,y_1)-f(t,x_2,y_2)\|^2\vee\|g(t,x_1,y_1)-g(t,x_2,y_2)\|^2$$
$$\le\bar{K}(\|x_2-x_1\|^2+\|y_2-y_1\|^2).\quad(2.16)$$

(ii) (Linear growth condition) for all $(t,x,y)\in[t_0,T]\times\mathbb{R}^d\times\mathbb{R}^d$

$$\|f(t,x,y)\|^2\vee\|g(t,x,y)\|^2\le K(1+\|x\|^2+\|y\|^2).\quad(2.17)$$

Then, there exists a unique solution $X(t)$ to equation (2.1) and this solution is a square integrable semimartingale.

We suppose that for any $s > t_0$ and $\xi \in C(\Gamma_s; \mathbb{R}^d)$, there exists a unique solution $X(t,s,\xi), t \geq b_s$ of the equation 2.1 satisfying $X(t,s,\xi) = \xi(t)$ for any $t \in \Gamma_s$. Further,

$$f(t,0,0) \equiv 0; g(t,0,0) \equiv 0, \ \forall t \in \mathbb{T}_a. (2.18)$$

Definition 2.4. The trivial solution $X(t) \equiv 0$ of the equation (2.1) is said to be almost surely exponentially stable if for any $s \in \mathbb{T}_{t_0}$ the relation

$$\limsup_{t\to\infty} \frac{\log \| X(t,s,\xi) \|}{t} < 0 \, (2.19)$$

holds for any $\xi \in C(\Gamma_s; \mathbb{R}^d)$.

Theorem 2.5. Let $\alpha_1, \alpha_2, p, c_1$ be positive numbers with $\alpha_1 > \alpha_2$. Let α be a positive number satisfying $\frac{\alpha}{1+\alpha\nu(t)} < \alpha_3$ and let η be a non-negative ld-continuous function defined on $\mathbb{T}_{t_0}$ such that

$$\int_{t_0}^{\infty} e_\alpha(t_-, t_0)\eta_t \nabla t < \infty \ a.s..$$

Suppose that there exists a positive definite function $V \in C^{1,2}(\mathbb{T}_{t_0} \times \mathbb{R}^d; \mathbb{R}_+)$ satisfying

$$c_1 \| x \|^p \leq V(t,x) \quad \forall (t,x) \in \mathbb{T}_{t_0} \times \mathbb{R}^d, (2.20)$$

and for all $t \geq t_0$,

$$V^{\nabla_t}(t,x) + \mathrm{A}V(t,x,y) \leq -\alpha_1 V(t_-,x) + \alpha_2 V(\delta(t),y) + \eta_t \qquad a.s., (2.21)$$

for all $x \in \mathbb{R}^d$ and $t \geq t_0$. Then, the trivial solution of equation (2.1) is almost surely exponentially stable.

Proof. Let $\alpha_3 = \alpha_1 - \alpha_2$. By (2.12), (2.21) and calculating expectations we get

$$e_\alpha(t,t_0)V(t,X(t)) = V(t_0,\xi(t_0)) + \int_{t_0}^{t} e_\alpha(\tau_-,t_0)[\alpha V(\tau_-,X(\tau_-))$$

$$+(1+\alpha\nu(\tau))(V^{\nabla_\tau}(\tau,X(\tau_-)) + \mathrm{A}V(\tau_-,X(\tau_-),X(\delta(\tau))))]\nabla\tau + \int_{t_0}^{t} e_\alpha(\tau,t_0)\nabla H_\tau$$

$$\leq V(t_0,\xi(t_0)) + \int_{t_0}^{t} e_\alpha(\tau_-,t_0)[\alpha V(\tau_-,X(\tau_-))$$

$$+(1+\alpha\nu(\tau))(-\alpha_1 V(\tau_-,X(\tau_-)) + \alpha_2 V(\delta(\tau),X(\delta(\tau))) + \eta_\tau)]\nabla\tau + \int_{t_0}^{t} e_\alpha(\tau,t_0)\nabla H_\tau$$

$$\leq[1+(1+\alpha(t_0-b_{t_0}))(t_0-b_{t_0})]\max_{b_{t_0}\leq s\leq t_0} V(s,\xi(s))$$
$$+\int_{t_0}^{t} e_\alpha(\tau_-,t_0)[\alpha V(\tau_-,X(\tau_-))-(1+\alpha\nu(\tau))(\alpha_3 V(\tau_-,X(\tau_-))+\eta_\tau)]\nabla\tau+\int_{t_0}^{t} e_\alpha(\tau,t_0)\nabla H_\tau.$$

Using the inequality $\frac{\alpha}{1+\alpha\nu(t)}<\alpha_3$ gets

$$e_\alpha(t,t_0)V(t,X(t))\leq[1+(1+\alpha(t_0-b_{t_0}))(t_0-b_{t_0})]\max_{b_{t_0}\leq s\leq t_0} V(s,\xi(s))+F_t+G_t,$$

where

$$F_t=\int_{t_0}^{t}(1+\alpha\nu(\tau))e_\alpha(\tau_-,t_0)\eta_\tau\nabla\tau; G_t=\int_{t_0}^{t} e_\alpha(\tau,t_0)\nabla H_\tau.$$

In view of the hypotheses we see that

$$F_\infty=\lim_{t\to\infty}F_t<\infty.$$

Define $Y_t=[1+(1+\alpha(t_0-b_{t_0}))(t_0-b_{t_0})]\max_{b_{t_0}\leq s\leq t_0} V(s,\xi(s))+F_t+G_t$ for all $t\in\mathbb{T}_{t_0}$.

Then Y is a nonnegative special semimartingale. By Theorem 7 on page 139 in [17], one sees that

$$\{F_\infty<\infty\}\subset\{\lim_{t\to\infty}Y_t \text{ exists and finite}\}\ a.s..$$

By $P\{F_\infty<\infty\}=1$. So we must have

$$P\{\lim_{t\to\infty}Y_t \text{ exists and finite}\}=1.$$

Note that $0\leq e_\alpha(t,t_0)V(t,X(t))\leq Y_t$ for all $t\geq t_0$ *a.s.*. It then follows that

$$P\{\limsup_{t\to\infty}e_\alpha(t,t_0)V(t,X(t))<\infty\}=1.$$

So

$$\limsup_{t\to\infty}\left[e_\alpha(t,t_0)V(t,X(t))\right]<\infty \quad a.s..(2.22)$$

Consequently, there exists a pair of random variables $\upsilon>t_0$ and $\xi>0$ such that

$$e_\alpha(t,t_0)V(t,X(t))\leq\xi \quad \text{for all } t\geq\upsilon \ a.s..$$

Using (2.20), we have

$$c_1 e_\alpha(t,t_0)\|X(t)\|^p\leq e_\alpha(t,t_0)V(t,X(t))\leq\xi \quad \text{for all } t\geq\upsilon \ a.s..$$

Since the time scale $\mathbb{T}$ has bounded graininess, there is a constant $\beta>0$ such that $e_\alpha(t,t_0)>e^{\beta(t-t_0)}$ for any $t\in\mathbb{T}$. Therefore,

$$\beta + p \lim_{t\to\infty} \frac{\ln\|X(t)\|}{t} \le 0 \quad \text{for all } t \ge \upsilon \quad a.s..$$

Thus,

$$\lim_{t\to\infty} \frac{\ln\|X(t)\|}{t} \le -\frac{\beta}{p} \quad \text{for all } t \ge \upsilon \quad a.s..$$

The proof is completed.

We now consider a special case where $V(t,x) = \|x\|^2$. Using (2.13) we have

$$\mathrm{L}V(t,x,y) = 2(1-1_{\mathrm{I}}(t))x^T f(t,x,y) + (\|x + f(t,x,y)\nu(t)\|^2 - \|x\|^2)\Phi(t)$$
$$+\|g(t,x,y)\|^2 \hat{K}_t^c - 2x^T g(t,x,y)\int_{\mathbb{R}} u\hat{\Upsilon}(t,du)$$
$$+\int_{\mathbb{R}} (\|x + f(t,x,y)\nu(t) + g(t,x,y)u\|^2 - \|x + f(t,x,y)\nu(t)\|^2)\Upsilon(t,du).$$

We have

$$2(1-1_{\mathrm{I}}(t))x^T f(t,x,y) + (\|x + f(t,x,y)\nu(t)\|^2 - \|x\|^2)\Phi(t) = 2x^T f(t,x,y) + \|f(t,x,y)\|^2 \nu(t). \quad (2.23)$$

Paying attention that $\nu(t)\int_{\mathbb{R}} u\hat{\Upsilon}(t,du) = 0$ and $\Upsilon(t,du) = \hat{\Upsilon}(t,du) + \tilde{\Upsilon}(t,du)$ yields

$$\int_{\mathbb{R}} (\|x + f(t,x,y)\nu(t) + g(t,x,y)u\|^2 - \|x + f(t,x,y)\nu(t)\|^2)\Upsilon(t,du)$$
$$= \int_{\mathbb{R}} \|g(t,x,y)\|^2 u^2 \Upsilon(t,du) + 2x^T g(t,x,y)\int_{\mathbb{R}} u\hat{\Upsilon}(t,du) \quad (2.24)$$

Since $K_t = \hat{K}_t^c + \hat{K}_t^d + \tilde{K}t$ and $\hat{K}_t^d + \tilde{K}t = \int_{\mathbb{R}} u^2 \Upsilon(t,du)$, we can combine (2.23) and (2.24) to obtain

$$\mathrm{L}V(t,x,y) = 2x^T f(t,x,y) + \|f(t,x,y)\|^2 \nu(t) + \|g(t,x,y)\|^2 K_t. \quad (2.25)$$

We can impose conditions on the functions f and g such that there are positive numbers α_1, α_2 with $\alpha_1 > \alpha_2$ and a non-negative ld-continuous function η satisfying

$$2x^T f(t,x,y) + \|f(t,x,y)\|^2 \nu(t) + \|g(t,x,y)\|^2 K_t \le -\alpha_1 \|x\|^2 + \alpha_2 \|y\|^2 + \eta_t$$

Example 2.6. Let T be a time scale $t_{-1} < 0 = t_0 < t_1 < \ldots t_n < \ldots$ where $t_n \uparrow \infty$. Consider the stochastic dynamic equation on time scale T

$$\begin{cases} d^\nabla X(t) = -X(t_-)d^\nabla t + \frac{1}{2}X(\rho(\rho(t)))d^\nabla W(t), t \in \mathrm{T} \\ X(t_{-1}) = a, X(0) = d, \end{cases} \quad (2.26)$$

where $W(t)$ is an one dimensional Brownian motion on time scale defined as in [9]. We can choose $\alpha_1 = 1, \alpha_2 = \frac{1}{4}, \mu = 0,$ By directly calculating, we obtain

$$\mathrm{LV}(t,x,y)=(\nu(t)-2)x^2+\frac{1}{4}y^2,(2.27)$$

where $f(t,x,y)=-x, g(t,x,y)=\frac{1}{2}y.$ If $\nu(t)\le 1; \forall t\in \mathrm{T}.$ By Theorem 2.5, the trivial solution of equation (2.26) is almost surely exponentially stable.

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