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Mathematical Uncertainty Relations and Their Generalization for Multiple Incompatible Observables

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Abstract: We show that the famous Heisenberg uncertainty relation for two incompatible observables can be generalized elegantly to the determinant form for *N* arbitrary observables. To achieve this purpose, we propose a generalization of the Cauchy-Schwarz inequality for two sets of vectors. Simple consequences of the *N*-ary uncertainty relation are also discussed.

Keywords: Generalized uncertainty relation, Generalized uncertainty principle, Generalized Cauchy-Schwarz inequality.

1. Introduction

The uncertainty principle was introduced by Heisenberg [1] who demonstrated the impossibility of simultaneous precise measurement of the canonical quantum observables $\hat{x}$ (the coordinate) and $\hat{p}_x$ (the momentum) by positing an approximate relation $\delta x\,\delta p_x \sim \hbar$, where $\hbar$ is the Plank constant. A year after Heisenberg formulated his principle, Weyl [2] derived the more formal relation $\Delta x\,\Delta p \geq \frac{\hbar}{2}$. Robertson [3] generalized the Weyl's result for two arbitrary Hermitian operators $\hat{A}$ and $\hat{B}$:

$$\Delta A\,\Delta B \geq \left| \frac{1}{2i} \left\langle [\hat{A}, \hat{B}] \right\rangle \right|. \tag{1}$$

where ΔA and ΔB are the standard deviations and $[\hat{A}, \hat{B}]$ represents the commutator $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. The Robertson formula (1) has been recognized as the modern Heisenberg uncertainty relation.

Going further, Schrödinger [4] derived the following *stronger* uncertainty relation:

$$\Delta A\,\Delta B \geq \sqrt{\left(\frac{1}{2}\left\langle \{\hat{A}, \hat{B}\} \right\rangle - \left\langle \hat{A} \right\rangle \left\langle \hat{B} \right\rangle\right)^2 + \left(\frac{1}{2i}\left\langle [\hat{A}, \hat{B}] \right\rangle\right)^2}. \tag{2}$$

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The difference between Eqs (1) and (2) is the first squared term under the square root, analogously known as the *covariance* in the theory of probability and statistics, consisting of the anti-commutator $\{\hat{A},\hat{B}\}$, defined as $\{\hat{A},\hat{B}\}=\hat{A}\hat{B}+\hat{B}\hat{A}$, and the product of two expectation values $\langle\hat{A}\rangle\langle\hat{B}\rangle$. These extra terms lead to substantial differences between the two uncertainty relations (1) and (2) in many cases.
All uncertainty relations mentioned above are *binary*, that means only two observables are involved in such relations. In this article, we propose a novel generalized uncertainty relation in which N arbitrary observables simultaneously participate. In order to achieve this goal, we need to establish new generalized Cauchy-Schwarz inequality.
The paper is organized as follow. In section 2, we introduce notation and derive the Robertson and Schrödinger uncertainty relations. In section 3, we propose a generalization of the Cauchy-Schwarz inequality and subsequently formulate a novel uncertainty relation for arbitrary incompatible observables. Section 4 is devoted to present simple consequences of the generalized uncertainty relations presented in previous section. Finally, in section 5 we briefly discuss related results and conclude.

2. Mathematical derivation of Schrödinger uncertainty relation

Throughout this article we consider a certain physical state $|\Psi\rangle$ (in a Hilbert space H), all observables $\hat{A},\hat{B},\hat{C},...$ act on that state, and all observables are assumed to be Hermitian operators. For each operator $\hat{A}$ we define the *expectation* (which depends on Ψ): $\langle\hat{A}\rangle=\langle\Psi|\hat{A}|\Psi\rangle$, the operator $\delta\hat{A}$ defined by $\delta\hat{A}=\hat{A}-\langle\hat{A}\rangle Id$, the associated vector is given by $|\delta A\rangle=\delta\hat{A}|\Psi\rangle$, the *variance* or the *dispersion* of $\hat{A}$: $(\Delta A)^2=(\sigma_A)^2=\langle(\delta\hat{A})^2\rangle=\langle\hat{A}^2\rangle-\langle\hat{A}\rangle^2$. One easily finds that: $[\delta\hat{A},\delta\hat{B}]=[\hat{A},\hat{B}]$. The *symmetrized covariance* of $\hat{A}$ and $\hat{B}$ can be defined as: $Cov(\hat{A},\hat{B})=\frac{1}{2}\langle\hat{A}\hat{B}+\hat{B}\hat{A}\rangle-\langle\hat{A}\rangle\langle\hat{B}\rangle=Cov(\hat{B},\hat{A})$.

In an inner product space, the Cauchy-Schwarz inequality states that for any vectors u and v

$\|u\|^2\cdot\|v\|^2 \;\geq\; |\langle u|v\rangle|^2$, the equality holds if and only if $u=\alpha\, v$ for some complex α. (3)

On another side, the imaginary and real part of $\langle\delta A|\delta B\rangle$ can be calculated as

$$Im\langle\delta A|\delta B\rangle=\frac{\langle\delta A|\delta B\rangle-\langle\delta B|\delta A\rangle}{2i}=\frac{\langle\Psi|\delta\hat{A}\,\delta\hat{B}-\delta\hat{B}\,\delta\hat{A}|\Psi\rangle}{2i}=\left\langle\frac{1}{2i}[\hat{A},\hat{B}]\right\rangle, \tag{4}$$

$$Re\langle\delta A|\delta B\rangle=\frac{\langle\delta A|\delta B\rangle+\langle\delta B|\delta A\rangle}{2}=\frac{\langle\{\hat{A},\hat{B}\}\rangle}{2}-\langle\hat{A}\rangle\langle\hat{B}\rangle=Cov(\hat{A},\hat{B}). \tag{5}$$

Combining (3), (4) and (5) we obtain the following inequality:

$(\Delta A)^2(\Delta B)^2=\|\delta A\|^2\cdot\|\delta B\|^2\geq|\langle\delta A|\delta B\rangle|^2=(Re\langle\delta A|\delta B\rangle)^2+(Im\langle\delta A|\delta B\rangle)^2$, or

$$(\Delta A)^2(\Delta B)^2\geq\left(\frac{1}{2}\langle\{\hat{A},\hat{B}\}\rangle-\langle\hat{A}\rangle\langle\hat{B}\rangle\right)^2+\left\langle\frac{1}{2i}[\hat{A},\hat{B}]\right\rangle^2 \tag{6}$$

$$(\Delta A)^2(\Delta B)^2 \geq \left\langle \frac{1}{2i}[\hat{A},\hat{B}] \right\rangle^2. \tag{7}$$

The inequalities (6) and (7) are exactly the Schrödinger and Robertson uncertainty relations, respectively. Equality in (6) holds if and only if $|\delta A\rangle = s|\delta B\rangle$ for some $s \in C$ (complex number), while equality in (7) holds if and only if $|\delta A\rangle = s|\delta B\rangle$ for some $s \in iR$ (imaginary number).

Uncertainty relations also apply to the case of mixed states. The Robertson uncertainty relation for mixed state can be easily found [5]:

$$\Delta A\,\Delta B \geq \left| \frac{1}{2i} Tr(\varrho[\hat{A},\hat{B}]) \right|, \tag{8}$$

where ϱ is the density operator that describes the mixed state and Tr denotes the trace. Similarly, the Schrödinger uncertainty relation for mixed state follows [5]:

$$(\Delta A)^2\,(\Delta B)^2 \geq \left(\frac{1}{2} Tr(\varrho\{\hat{A},\hat{B}\}) - Tr(\varrho\hat{A}) Tr(\varrho\hat{B}) \right)^2 + \left(\frac{1}{2i} Tr(\varrho[\hat{A},\hat{B}]) \right)^2. \tag{9}$$

3. Uncertainty relations in multiple simultaneous measurements

As we have seen in previous section, the Cauchy-Schwarz inequality (3) is the mathematical foundation of the Heisenberg uncertainty relation (7). In this section, we first propose a novel generalized Cauchy-Schwarz inequality for multiple vectors, and subsequently, using this inequality we formulate a generalized uncertainty relation for multiple incompatible observables.

Consider two sets of m and n complex vectors from a Hilbert space H : $X = \{|x_1\rangle, |x_2\rangle, \ldots, |x_m\rangle\}$ and $Y = \{|y_1\rangle, |y_2\rangle, \ldots, |y_n\rangle\}$. We introduce the following 4 complex matrices:

$$M(X) = \begin{bmatrix} \langle x_1|x_1\rangle & \cdots & \langle x_1|x_m\rangle \\ \vdots & & \vdots \\ \langle x_m|x_1\rangle & \cdots & \langle x_m|x_m\rangle \end{bmatrix}, \quad M(Y) = \begin{bmatrix} \langle y_1|y_1\rangle & \cdots & \langle y_1|y_n\rangle \\ \vdots & & \vdots \\ \langle y_n|y_1\rangle & \cdots & \langle y_n|y_n\rangle \end{bmatrix}, \tag{10}$$

$$M(XY) = \begin{bmatrix} \langle x_1|y_1\rangle & \cdots & \langle x_1|y_n\rangle \\ \vdots & & \vdots \\ \langle x_m|y_1\rangle & \cdots & \langle x_m|y_n\rangle \end{bmatrix}, M(YX) = \begin{bmatrix} \langle y_1|x_1\rangle & \cdots & \langle y_1|x_m\rangle \\ \vdots & & \vdots \\ \langle y_n|x_1\rangle & \cdots & \langle y_n|x_m\rangle \end{bmatrix}. \tag{11}$$

We are able to prove the following determinant inequality [6]:

Theorem 1. Suppose that the matrix $M(Y)$ is invertible. Then we have the inequality:

$$\det M(X) \geq \det[M(XY) \cdot M(Y)^{-1} \cdot M(YX)]. \tag{12}$$

The equality holds if and only if X is linearly dependent or $X = A \cdot Y$ for some matrix A of size $m \times n$.

In the particular, if $m = n$ we get the following form:

$$\det M(X) \cdot \det M(Y) \geq |\det M(YX)|^2. \tag{13}$$

We remark that for $m=n=1$, the inequality (12) becomes: $||x||^2||y||^2 \geq |\langle x|y\rangle|^2$, which is the Cauchy-Schwarz inequality (3). For this reason, we shall refer to the inequality (12) as "*generalized Cauchy-Schwarz inequality*".

For two sets of Hermitian operators $\hat{X}=\{\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_m\}$ and $\hat{Y}=\{\hat{y}_1,\hat{y}_2,\ldots,\hat{y}_m\}$, there are two sets of associated vectors $\delta X=\{|\delta x_1\rangle,|\delta x_2\rangle,\ldots,|\delta x_m\rangle\}$ and $\delta Y=\{|\delta y_1\rangle,|\delta y_2\rangle,\ldots,|\delta y_n\rangle\}$, defined as in previous section. Following the inequality (12), the natural *generalized* uncertainty relation for $m+n$ observables $\{\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_m;\hat{y}_1,\hat{y}_2,\ldots,\hat{y}_n\}$ should be:

$$det\,M(\delta X) \geq det[M(\delta X\ \delta Y)\cdot M(\delta Y)^{-1}\cdot M(\delta Y\ \delta X)]. \tag{14}$$

Uncertainty relations for mixed states can be derived in a similar way. Below we consider particular interesting cases, for several observables.

1. Three Observables $m=1;n=2$:

For $\delta\hat{X}=\{\delta\hat{x}\}$ and $\delta\hat{Y}=\{\delta\hat{y},\delta\hat{z}\}$, the uncertainty relation (14) becomes *ternary*:

$$(\Delta x\Delta y\Delta z)^2 \geq (\Delta x)^2|\langle\delta y|\delta z\rangle|^2+(\Delta y)^2|\langle\delta z|\delta x\rangle|^2+(\Delta z)^2|\langle\delta x|\delta y\rangle|^2 - 2\,Re\big[\langle\delta x|\delta y\rangle\langle\delta y|\delta z\rangle\langle\delta z|\delta x\rangle\big]. \tag{15}$$

We remark that the inequality (15) is stronger than inequality (7) which is the Robertson uncertainty relation. Indeed, $|Re\big[\langle\delta x|\delta y\rangle\langle\delta y|\delta z\rangle\langle\delta z|\delta x\rangle\big]| \leq |\langle\delta x|\delta y\rangle\langle\delta y|\delta z\rangle\langle\delta z|\delta x\rangle| \leq (\Delta x\Delta y\Delta z)^2$, then (15) is stronger than:

$$3(\Delta x\Delta y\Delta z)^2 \geq (\Delta x)^2|\langle\delta y|\delta z\rangle|^2+(\Delta y)^2|\langle\delta z|\delta x\rangle|^2+(\Delta z)^2|\langle\delta x|\delta y\rangle|^2, \tag{16}$$

which can be easily derived from the Robertson uncertainty relation.

2. Four Observables $m=2;n=2$:

For $\delta\hat{X}=\{\delta\hat{x}_1,\delta\hat{x}_2\}$ and $\delta\hat{Y}=\{\delta\hat{x}_3,\delta\hat{x}_4\}$, Eq. (14) forms a *quaternary* uncertainty relation:

$$\Big((\Delta x_1\Delta x_2)^2-|\langle\delta x_1|\delta x_2\rangle|^2\Big)\Big((\Delta x_3\Delta x_4)^2-|\langle\delta x_3|\delta x_4\rangle|^2\Big) \geq |\langle\delta x_1|\delta x_3\rangle\langle\delta x_2|\delta x_4\rangle-\langle\delta x_1|\delta x_4\rangle\langle\delta x_2|\delta x_3\rangle|^2\,. \tag{17}$$

We need to note that the inequality (17) is stronger than the estimation derived from the Robertson uncertainty relation for two pairs of operators $(\hat{x}_1,\hat{x}_2)$ and $(\hat{x}_3,\hat{x}_4)$, which has zero lower bound:

$$\Big((\Delta x_1\Delta x_2)^2-|\langle\delta x_1|\delta x_2\rangle|^2\Big)\Big((\Delta x_3\Delta x_4)^2-|\langle\delta x_3|\delta x_4\rangle|^2\Big)\geq 0. \tag{18}$$

3. Five Observables $m=3;n=2$ or $m=4;n=1$:

Inequality (14) leads to the same relations as for three and four observables.

4. Applications

The uncertainty relation (14) can be used in different areas of quantum physics. Below, for simplicity, we limit to several consequences of the generalized uncertainty relation in quantum mechanics and noncommutative quantum fields.

A. Consider three incompatible components of angular momentum. Their commutators read [5]:

$$[\hat{J}_1,\hat{J}_2]=i\hbar\hat{J}_3,\ [\hat{J}_2,\hat{J}_3]=i\hbar\hat{J}_1,\ [\hat{J}_3,\hat{J}_1]=i\hbar\hat{J}_2. \tag{19}$$

The uncertainty relation (15) takes the form:

$$\begin{aligned}(\Delta J_1\Delta J_2\Delta J_3)^2 \ \ge\ & (\Delta J_1)^2|\langle\delta J_2|\delta J_3\rangle|^2+(\Delta J_2)^2|\langle\delta J_3|\delta J_1\rangle|^2+(\Delta J_3)^2|\langle\delta J_1|\delta J_2\rangle|^2\\ & -2Re\left[\langle\delta J_1|\delta J_2\rangle\langle\delta J_2|\delta J_3\rangle\langle\delta J_3|\delta J_1\rangle\right]\\ \ge & \frac{\hbar^2}{4}\left((\Delta J_1)^2|\langle J_1\rangle|^2+(\Delta J_2)^2|\langle J_2\rangle|^2+(\Delta J_3)^2|\langle J_3\rangle|^2\right)-\\ & 2Re\left[\langle\delta J_1|\delta J_2\rangle\langle\delta J_2|\delta J_3\rangle\langle\delta J_3|\delta J_1\rangle\right].\end{aligned} \tag{20}$$

Following (16) the weaker estimation of (20) has the form:

$$(\Delta J_1\Delta J_2\Delta J_3)^2\ge\frac{\hbar^2}{12}\left((\Delta J_1)^2|\langle J_1\rangle|^2+(\Delta J_2)^2|\langle J_2\rangle|^2+(\Delta J_3)^2|\langle J_3\rangle|^2\right). \tag{21}$$

We note that in eigenstates of the operators J_3 and J^2, both sides of (21) are zeros.

B. Consider canonical noncommutative coordinates in a noncommutative space:

$$[\hat{x},\hat{y}]=i\hbar\theta_3,\ [\hat{y},\hat{z}]=i\hbar\theta_1,\ [\hat{z},\hat{x}]=i\hbar\theta_2. \tag{22}$$

The *ternary* uncertainty relation (15) becomes:

$$\begin{aligned}&(\Delta x\Delta y\Delta z)^2+2C(\delta\hat{x},\delta\hat{y})C(\delta\hat{y},\delta\hat{z})C(\delta\hat{z},\delta\hat{x})-\frac{\hbar^2}{2}\left(C(\delta\hat{x},\delta\hat{y})+C(\delta\hat{y},\delta\hat{z})+C(\delta\hat{z},\delta\hat{x})\right)\\ &\ge(\Delta x)^2\left(\frac{\hbar^2\theta_1^2}{4}+|C(\delta\hat{y},\delta\hat{z})|^2\right)+(\Delta y)^2\left(\frac{\hbar^2\theta_2^2}{4}+|C(\delta\hat{z},\delta\hat{x})|^2\right)+(\Delta z)^2\left(\frac{\hbar^2\theta_3^2}{4}+|C(\delta\hat{x},\delta\hat{y})|^2\right)\\ &\ge\frac{\hbar^2}{4}\left((\Delta x)^2\theta_1^2+(\Delta y)^2\theta_2^2+(\Delta z)^2\theta_3^2\right)\ge\frac{3\hbar^3}{8}\theta_1\theta_2\theta_3,\end{aligned} \tag{23}$$

where $C(\hat{A},\hat{B})=Cov(\hat{A},\hat{B})=Cov(\hat{B},\hat{A})$ the symmetrized covariance of $\hat{A}$ and $\hat{B}$.

C. Similarly, consider canonical noncommutative space with four coordinates satisfying $[\hat{x}_j,\hat{x}_k]=ic_{jk}$ for $j,k=1,\ldots,4$ and $c_{j,k}$ are real, the *quaternary* uncertainty relation (16) reads:

$$(\Delta x_1\Delta x_2\Delta x_3\Delta x_4)^2+2Re\left[\langle\delta x_1|\delta x_3\rangle\langle\delta x_3|\delta x_2\rangle\langle\delta x_2|\delta x_4\rangle\langle\delta x_4|\delta x_1\rangle\right]\ge\frac{c_{12}^2c_{34}^2+c_{13}^2c_{24}^2+c_{14}^2c_{23}^2}{16}. \tag{24}$$

5. Conclusion

In this article, we have proposed a novel uncertainty relations for $N\,(N\ge 2)$ incompatible observables. The uncertainty relations for three and four observables have been derived explicitly, which have been shown stronger than the ones derived from the Schrödinger (or Heisenberg) binary uncertainty relations. Moreover, we have formulated a determinant form of N-ary uncertainty relation for arbitrary N incompatible observables. Our results have been derived from generalizing the classical Cauchy-Schwarz inequality. Alternative stronger uncertainty relations for multi observables, their associative lower bounds and minimal states have been investigated recently [7-10]. These uncertainty relations, based on different inequalities, are not equivalent to the one discussed in this article. The differences of such uncertainty relations and the corresponding minimal states will be analyzed in details elsewhere.

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