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The Leading Eikonal of the Scattering Amplitude of Particles in Gravitational Field at High Energy Using the Partial Wave Method

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Abstract : The scattering amplitudes for two spineless particles colliding at Planckian center-of-mass energies are considered by the partial wave method in quantum gravity. In the framework of the partial method, a scheme for finding the leading eikonal scattering amplitudes is developed and constructed. The connection between the solutions obtained by partial wave method, quasi-potential and functional approaches is also discussed.

Keywords: Eikonal scattering, partial wave method, gravitational field.

1. Introduction

In recent years there have been important advances in our understanding of Planck scattering in quantum field theory and string theory ($M_{Pl} = G^{-1/2}$ is called Planck mass at the energy scale about $10^{19}\,GeV$) [1-6]. This understanding give us a scientific basis to investigate the singularity, the formation of black holes and the loss of information near black holes as well as the modification of the string theory in quantum gravity. The research results have confirmed that [1-6] if gravitational interaction is considred, Planck scattering amplitude of two particles at high energy, $\sqrt{s} \approx M_{Pl}$ (s is the square of the total energy of two particles in the center of mass system) and small fixed momentum transfer t (t is the square of momentum transfer) has the form Glauber with the scattering phase depends on energy in the limit $(t/s) \to \infty$ [1].

The calculation of high-level correction terms to the eikonal leading term of scattering amplitude has been studied by many authors, but this problem remains an issues. By using the integral method and quasi – potential equation we have obtained the analytical expression for the eikonal leading term in this problem [7]. To confirm this result, we revisit this problem using a new approach that is the partial wave method [8]. All results which have been obtained are compared.

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The paper is organized as follow. In section 2, we introduce briefly the method to find scattering amplitude and scattering phase by using the partial wave method. Section 3 is devoted to compute the leading term and correction terms of scattering amplitude at high energy and small momentum tranfer. Finally, in section 4, we discuss and compare the results that we obtained in previous sections and draw conclude.

2. Scattering of uncharged particles in the gravitational field

The covariant Klein-Gordon equation for the massless test particle with no electric charge and moving in the gravitational field is

$$\frac{1}{\sqrt{-g}}\partial^{\mu}(\sqrt{-g}\; g^{\mu\nu}\partial_{\mu}\psi)=0\,, \tag{2.1}$$

where $g = det\, g_{\mu\nu}(x) = \sqrt{-g}g^{\mu\nu}$.

The solution of classical Schwarzschild background field of slow target particle that is obtained by the Einstein equation has the form

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)\,, \tag{2.2}$$

where M is the mass of the target particle. In the center of mass frame of the particles, $M << \sqrt{s}$.

Main diagonal terms of the Schwarschild metric is determined by the expression

$$g_{\mu\nu} = \begin{pmatrix} \left(1-\frac{2GM}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix} \tag{2.3}$$

with

$$\sqrt{-g} = r^2 sin\theta\,,\; g^{tt} = -\left(1-\frac{2GM}{r}\right)^{-1},\; g^{rr} = \left(1-\frac{2GM}{r}\right),\; g^{\theta\theta} = r^{-2}, g^{\varphi\varphi} = (r\,sin\theta)^{-2}$$

Using expression (2.3), we rewrite equation (2.1) in the form

$$-r^2 sin\theta\left(1-\frac{2GM}{r}\right)^{-1}\partial_{tt}\Psi + 2r\,sin\theta\left(1-\frac{GM}{r}\right)\partial_r\Psi +$$

$$+r^2 sin\theta\left(1-\frac{2GM}{r}\right)\partial_{rr}\Psi + cos\theta\;\partial_{\theta}\Psi + sin\theta\;\partial_{\theta\theta}\Psi + \partial_{\varphi\varphi}\Psi = 0 \tag{2.4}$$

The wave function of the test particle that obtain from eq. (2.1) or (2.4) is assumed to have the form

$$\Psi(\vec{r},t) = \psi(\vec{r})e^{iEt} = \frac{f(r)}{r}.Y_{lm}(\theta,\phi)e^{iEt} \tag{2.5}$$

where E is energy of the test particle as measured by an asymptotic observer
From the Eqs. (2.5)-(2.6), we have

$$\partial_{tt}\Psi \equiv \partial_{tt}\Psi(\vec{r},t) = -E^2 \frac{f(r)}{r} Y_{lm}(\theta,\varphi) e^{iEt}$$

$$\partial_r \Psi = \frac{r\partial_r f(r) - f(r)}{r^2} Y_{lm}(\theta,\varphi).e^{iEt} = \left(\frac{\partial_r f(r)}{r} - \frac{f(r)}{r^2}\right) Y_{lm}(\theta,\varphi).e^{iEt};$$

$$\partial_{rr}\Psi = \left(\frac{\partial_{rr} f(r)}{r} - \frac{2\partial_r f(r)}{r^2} + \frac{2f(r)}{r^3}\right) Y_{lm}(\theta,\varphi).e^{iEt} \quad (2.6)$$

$$\partial_\theta \Psi = \frac{f(r)}{r}\partial_\theta Y_{lm}(\theta,\varphi).e^{iEt};$$

$$\partial_{\theta\theta}\Psi = \frac{f(r)}{r}\partial_{\theta\theta} Y_{lm}(\theta,\varphi).e^{iEt}; \ \partial_{\varphi\varphi}\Psi = \frac{f(r)}{r}\partial_{\varphi\varphi} Y_{lm}(\theta,\varphi).e^{iEt}$$

Substituting Eq.(2.5) into Eq. (2.1) and using Eq. (2.6), one obtains the equation for wave function of the radial coordinate

$$\left[r^2 \left(1 - \frac{2GM}{r}\right)^{-1} E^2 + \partial_r \left\{ r^2 (1 - \frac{2GM}{r})\partial_r \right\} - \hat{L}^2 \right] \psi(\vec{r}) = 0, \quad (2.7)$$

here $\hat{L}^2 \equiv -\left[\frac{1}{\sin\theta}\partial_\theta(\sin\theta)\partial_\theta + \frac{1}{\sin^2\theta}\partial_{\varphi\varphi}\right]$; và $\hat{L}^2 Y_{lm}(\theta,\phi) = l(l+1)Y_{lm}(\theta,\phi)$.

From eq. (2.7), linearizing the Schwarzschild metric, substituting $s = 2ME$ (s is the Mandelstam variable in quantum relativistic mechanics) and retaining terms up to order $(2GM/r)^2$, the radial equation of the l^{th} partial wave is (for large l)

$$\frac{d^2 f(r)}{dr^2} - \left[\frac{l(l+1) - G^2 s^2}{r^2} - \frac{2GsE}{r} - E^2\right] f(r) = 0. \quad (2.8)$$

Thus, we can find the solution of this equation without adding a further approximation while keeping in mind that at Planck scale (about 10^{-33} cm) very small impact parameter scattering cannot be probed

The radial equation (2.8) will be solved by using hypergeometric functions with its asymptotic form. It is By setting

$$p_l(s)\left(p_l(s)+1\right) \equiv l(l+1) - G^2 s^2 \quad (2.9)$$

The eq.(2.8) is rewritten in form

$$\frac{d^2 f(r)}{d^2 r} + \left[E^2 - \frac{p_l(s)\left(p_l(s)+1\right)}{r^2} + \frac{2GsE}{r}\right] f(r) = 0. \quad (2.10)$$

Equation (2.10) is the hypergeometric equation. At far enough distance, ρ_l, its solution has form [9]

$$\begin{aligned} f_l(r) &= sin\left(Er - \frac{p_l(s).\pi}{2}\right) + \frac{i}{2}(-i)^{p_l(s)+1}(e^{2i\delta_l} - 1)e^{iEr} \\ &= sin\left(Er - \frac{p_l(s).\pi}{2}\right) - (-i)^{p_l(s)+1}e^{i\delta_l}\, sin\,\delta_l.e^{iEr},\ \ Er >> p_l(s) \end{aligned} \tag{2.11}$$

the phase shift of the partial wave is

$$\begin{aligned} E.sin\,\delta_l &= \int_0^{\rho} \frac{2GsE}{r} f_l(r).f_{0l}(r)dr \\ &= \int_0^{\rho} \frac{2GsE}{r}\left[sin\left(Er - \frac{p_l(s).\pi}{2}\right) - (-i)^{p_l(s)+1}e^{i\delta_l}\, sin\,\delta_l.e^{iEr}\right].sin\left(Er - \frac{p_l(s).\pi}{2}\right)dr \end{aligned} \tag{2.12}$$

In the first order approximation, it has the form

$$\delta_l = 2Gs\int_0^{\rho}\frac{1}{r}.sin^2\left(Er - \frac{p_l(s).\pi}{2}\right)dr = \frac{\Gamma\left(p_l(s)+1-iGs\right)}{\left|\Gamma\left(p_l(s)+1-iGs\right)\right|} \tag{2.13}$$

If the particle is free motion, eq. (2.11) become to

$$f_{0l}(r) = sin\left(Er - \frac{p_l(s).\pi}{2}\right),\ \ Er >> p_l(s). \tag{2.14}$$

Wave function is expressed in terms of a partial wave expansion

$$\Psi(\vec{r}) = \Psi(r,\theta,\phi) = \sum_{l=0}^{\infty}(2l+1).i^{p_l(s)}\frac{f_l(r)}{Er}P_l(cos\,\theta) \tag{2.15}$$

in view of the spherical symmetry

$$\Psi(\vec{r}) = \psi_0(r) + f(\theta).\frac{e^{iEr}}{r} = f_{0l}(r) + f(\theta).\frac{e^{iEr}}{r} \tag{2.16}$$

The scattering amplitude in the gravitational field is found in terms of a partial wave expansion

$$f(\theta) = -\frac{i}{2E}\sum_{l=0}^{\infty}(2l+1)\left[e^{2i\delta_l} - 1\right]P_l(cos\,\theta)\qquad , \tag{2.17}$$

where, the phase shift of the partial wave, characterized by a fixed angular momentum quantum number $l \gg 1$, is determined by eq. (2.16) and $p_l(s)$ is determined by eq. (2.9).

In the centre of mass frame (cms) of the particles, because $\sqrt{s} = 2E$ is the total energy (see eq. (9) in ref.[10]), the scattering amplitude is

$$f(\theta) = \frac{1}{i\sqrt{s}}\sum_{l=0}^{\infty}(2l+1)\left[e^{2i\delta_l} - 1\right]P_l(cos\,\theta) \tag{2.18}$$

It is not difficult to show from eq. (2.13) that, for fixed l, the phase shift has singularities at center mass energies

$$p_l(s) + 1 - iGs = -N \quad \text{with } \forall N > 0 \tag{2.19}$$

or $$Gs = \frac{i}{(2N+1)}\left[l(l+1) - N(N+1)\right] \tag{2.20}$$

for any non-negative integer N. Although still located on the imaginary axis of the complex s-plane, clearly the locations of these poles are quite distinct from those seen in the eikonal limit [2], viz., at $Gs = -iN$.

Here, we only consider the first order correction term for the leading term (leading eikonal) of the scattering amplitude in the limit $l \to \infty$. In case of large and fixed *l*, the explanation of the existence above poles outside the eikonal limit according to string theory was disscussed in [8].

The formula above also permits us to extract the leading order corrections to the eikonal limit $l \to \infty$, by using the asymptotic expansion of the argument of the gamma function in increasing inverse powers of *l*. We obtain

$$\delta_l \approx -Gs\left[\log l - \frac{1}{2l}\right] + \frac{(Gs)^2}{2l^2} + O\left(\frac{1}{l^3}\right). \tag{2.21}$$

The first term in eq. (2.21) obviously corresponds to the eikonal result, and the sub-leading corrections have been anticipated from reggeized string exchange diagrams [3]. The leading correction above to the eikonal phase shift behaves as $\left((Gs)^2 / l^2\right)\log s$.

By using quantum mechanics, we will not receive correction terms which are logarithmic functions [8]. Therefore, we need to use formalism of quantum field theory to achieve this aim.

3. The correction terms of leading term for scattering amplitude

For the scattering of particles at high energy and small momentum transfer, we can convert the sum into an integral in l in the expression of the scattering amplitude (2.17):

$$f(s,t) \equiv f(\theta) = -\frac{i}{\sqrt{s}}\int_0^\infty dl(2l+1)P_l(\cos\theta)\left[e^{2i\delta_l} - 1\right] \tag{3.1}$$

Set $b = \frac{(2l+1)}{2E} = \frac{2l+1}{\sqrt{s}}$ so that $dl = Edb = \frac{\sqrt{s}}{2}db$, here b is called impact parameter and the Legendre polynomial convert to the Bessel function of zezoth order

$$P_l(\cos\theta)\xrightarrow[\theta-small]{k-high} J_0\left((2l+1)\sin\frac{\theta}{2}\right). \tag{3.2}$$

The expression for the scattering amplitude is found

$$f(s,t) = -\frac{i\sqrt{s}}{2}\int_0^\infty bdb.J_0\left((2l+1)\sin\frac{\theta}{2}\right)\left[e^{2i\delta_l} - 1\right] \tag{3.3}$$

When small angle θ then $\sin\left(\frac{\theta}{2}\right) \approx \frac{\theta}{2}$, $(2l+1)\sin\left(\frac{\theta}{2}\right) \approx (2l+1)\frac{\theta}{2} = Eb\theta$, Eq. (3.3) becomes to

$$f(s,t) = -\frac{i\sqrt{s}}{2}\int_0^\infty bdbJ_0(Eb\theta)\left[e^{2i\delta_l} - 1\right] \tag{3.4}$$

Note that $\int_0^\infty bdbJ_0(Eb\theta)\left[e^{2i\delta_l} - 1\right] = \frac{1}{2\pi}\int_0^\infty d^2be^{iEb\theta}\left[e^{2i\delta_l} - 1\right]$, so the scattering amplitude is obtained in general form

$$f(s,t) = -\frac{i\sqrt{s}}{4\pi}\int_0^\infty d^2b e^{iEb\theta}\left[e^{2i\delta_l} - 1\right] \quad (3.5)$$

Substituting the shift phase in Eq.(2.21) into Eq(3.5) we obtain

$$f(s,t) = -\frac{i\sqrt{s}}{4\pi}\int_0^\infty d^2b e^{iEb\theta}\left[e^{-2iGs\left[\log l - \frac{1}{2l}\right]+...} - 1\right] = -\frac{i\sqrt{s}}{4\pi}\int_0^\infty d^2b e^{iEb\theta}\left[(l)^{-2iGs}.e^{\frac{iGs}{l}} - 1\right] \quad (3.6)$$

Expanding at large l, $l \gg 1$, we have

$$f(s,t) = -\frac{i\sqrt{s}}{4\pi}\int_0^\infty d^2b e^{iEb\theta}.(l)^{-2iGs}\left[1 + \frac{iGs}{l} + \frac{(iGs)^2}{l^2} + ...\right]. \quad (3.7)$$

In Eq.(3.7), if l is large, we substitute $l \approx bE = \frac{b\sqrt{s}}{2}$ into Eq.(3.7) and obtain finally expression

$$f(s,t) = -\frac{i\sqrt{s}}{4\pi}\int_0^\infty d^2b e^{iEb\theta}.\left(\frac{b\sqrt{s}}{2}\right)^{-2iGs}\left[1 + \frac{2iGs}{b\sqrt{s}} + \left(\frac{2iGs}{b\sqrt{s}}\right)^2 + ...\right] \quad (3.8)$$

The factor precedes the parenthesess in Eq.(3.8) is the leading eikonal term, the other terms in the parenthesess are the correction terms of scattering amplitude. Phases of them increase in proportion to the square root of energy $\sqrt{s}$.

In our recent paper [7, 11], we obtained exact expressions for the scattering amplitude of two particles in quantum gravity by using functional integration method. In cms system at Planck energy and small momentum transfer, we used eikonal approximation to calculate the integrals and obtained the leading term and the first order correction term for the leading term.

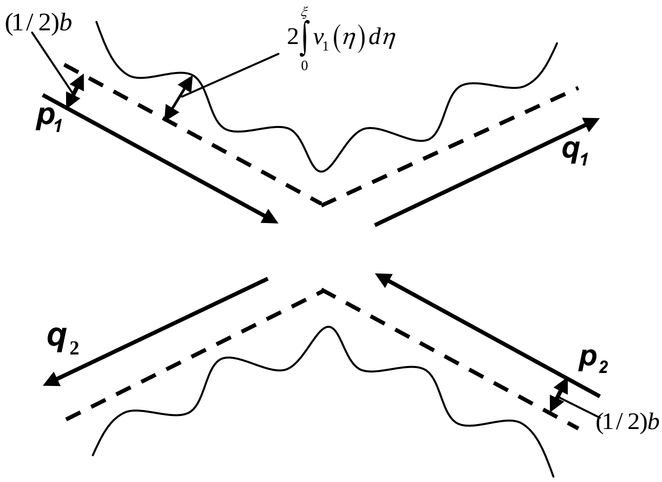

Fig. 1. The accuracy of the eikonal approximation $k_i k_j = 0, (i \neq j)$ at high energy and small momentum transfer was confirmed in the frame of perturbation theory.

In the high-energy limit $s >> M_{PL}^2 >> t$, the phase function of the scattering amplitude has the eikonal form and increases with energy [7].

If mass of the changed particle is small, we will obtain the following expression for the scattering amplitude in quantum gravity in the zeroth order of mass μ

$$f(s,t) = -2is\int_0^{\infty} d^2 b e^{i\Delta_\perp b}\left(\frac{\mu|b|}{2}\right)^{\frac{i\kappa^2 s}{2\pi}}\left[1+\left(\frac{\gamma\kappa^2 s}{2\pi}\right)+\left(\frac{\gamma\kappa^2 s}{2\pi}\right)^2+...\right] \tag{3.9}$$

with $\gamma = 0.5772...$ is the Euler Mascheroni constant.

The factor precedes the parenthesess in Eq.(3.9) is the leading eikonal term, the other terms in the parenthesess are the correction terms. Phases of them are proportional to *s*.

4. Conclusion

From eq.(3.8) and eq.(3.9) give us that the dependence of the correction terms on energy are qualitatively not the same. To explane this problem is that eq.(3.8) was obtained basing on regular perturbation theory. It has not been proven in quantum gravity because of alternating signs of correction terms [4]. Eq. (3.9) was found in the frame of functional integration method in quantum gravity, it is not related to perturbation theory. Expression (3.9) is the same as the expression obtained by using quasi-potential equation [11, 12]. Note that eikonal representation for scattering amplitude of particles at high energy in quantum field theory was first found by using quasi-potential equation [11].

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