ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF NEUTRAL DELAY DIFFERENCE EQUATIONS

Nguyen Van Minh, Nguyen Minh Man

Department of Mathematics, College of Science, VNU

Abstract. For a natural number r we denote the Banach space $C := \{\phi : \{-r, -r+1, \cdots, 0\} \to \mathbb{X}\}$ with norm $\|\phi\| := \sup_{-r\le n\le 0} \|\phi(n)\|$. If $x : \mathbb{Z} \to \mathbb{X}$, $x_n \in C$ stands for the function $x_n(\theta) = x(n+\theta)$, $\forall n \in \mathbb{Z}, -r \le \theta \le 0$. We consider conditions for the existence of bounded and almost periodic solutions of neutral delay difference equations of the form

$$\Delta(Dx_n) = Lx_n + f(n), \quad n \in \mathbb{Z},$$

where $f \in l_\infty(\mathbb{X})$, $D, L \in L(C, \mathbb{X})$ and $\Delta(Dx_n) : Dx_{n+1} - Dx_n$. Conditions for the existence of bounded and almost periodic solutions of difference equations have obtained which extend and complement several recent results.

1. Introduction and preliminaries

1.1. Notations

In this paper $\mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ denote the sets of integer, real and complex numbers, respectively. Γ stands for the unit circle in $\mathbb{C}$. Given complex Banach spaces $\mathbb{X}$, $\mathbb{Y}$. $L(\mathbb{X}, \mathbb{Y})$ stands for the space of all bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$. For a linear operator A on $\mathbb{X}$ we denote by $\sigma(A), \rho(A)$ its spectrum and resolvent set. The space of all bounded sequences $\{x(n)\}_{n\in\mathbb{Z}} \subset \mathbb{X}$ is denoted by $l_\infty(\mathbb{X})$. For a natural number r we denote the Banach space $C := \{\phi : \{-r, -r+1, \cdots, 0\} \to \mathbb{X}\}$ with norm $\|\phi\| := \sup_{-r\le n\le 0} \|\phi(n)\|$. If $x : \mathbb{Z} \to \mathbb{X}$, $x_n \in C$ stands for the function $x_n(\theta) = x(n+\theta)$, $\forall n \in \mathbb{Z}, -r \le \theta \le 0$.

1.2. Problem Setting

With the above notations we consider conditions for the existence of bounded and almost periodic solutions of neutral delay difference equations of the form

$$\Delta(Dx_n) = Lx_n + f(n), \quad n \in \mathbb{Z}, \tag{1}$$

where $f \in l_\infty(\mathbb{X})$, $D, L \in L(C, \mathbb{X})$ and $\Delta(Dx_n) : Dx_{n+1} - Dx_n$.

Conditions for the existence of bounded and almost periodic solutions of difference equations have been studied for a long time. There are a lot of works with various methods of study dedicated to this topic some of which can be found in our list of references and the references therein (see e.g. [5, 10, 11, 12, 14, 23, 27, 31, 32]). Actually, it goes back to the Poincare method of studying periodic solutions of ordinary differential equations. We refer the reader to [10, 11, 14, 23] with further introduction to the relations between solutions of

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difference equations and those of differential equations. As shown in recent works (see e.g. [5, 11, 23] and the references therein) the spectral theory of sequences is a powerful tool for studying the asymptotic behavior solutions of difference equations. On the other hand, in our recent paper [19], the method of commutative operators seems to be a natural way to approach various problems on the asymptotic behavior solutions of differential equations. Although we are trying this method for neutral functional differential equations, there are still several abstract obstacles to be overcome. However, for neutral delay difference equations, there turn out to be no such obstacles.

As a continuation of some results in [11, 23, 27], in this paper we combine the method which makes use of the spectral theory of sequences used in [23] with the one of commutative operators used in [19] to study conditions for the existence of bounded and almost periodic sloutions of Eq. (1). The main results of this paper are Theorems 2.2, 2.8 and 2.10, which extend and complement several ones in [8, 18, 11, 17, 21, 23, 27] to neutral delay difference equations Eq. (1).

1.3. Preliminaries

Spectral theory of sequences

In this paper we use the following definition of spectrum of a sequence $\{x(n)\}_{n\in\mathbb{Z}}$ which corresponds to the notion of spectrum of the sequence $\{\overline{x}(n) := S(n)x\}_{n\in\mathbb{Z}} \subset l_\infty(\mathbb{X})$ of [5], where $S(n)$ denotes the translation $x(\cdot) \in l_\infty(\mathbb{X}) \mapsto x(n+\cdot) \in l_\infty(\mathbb{X})$.

Definition 1.1. *The spectrum $\sigma(g)$ of a given bounded sequence $g \in l_\infty(\mathbb{X})$ is defined to be the set of all complex numbers λ_0 in the unit circle Γ, at which the function $\hat{g}(\lambda)$ defined by*

$$\hat{g}(\lambda) := \begin{cases} \sum_{n=0}^{\infty} \lambda^{-n-1} S(n)g, & \forall\, |\lambda| > 1 \\ -\sum_{n=1}^{\infty} \lambda^{n-1} S(-n)g, & \forall\, |\lambda| < 1. \end{cases}$$

has no holomorphic continuation at any neighborhood of λ_0.

We list below some properties of the spectrum of $\{g_n\}$.

Proposition1.2. *Let $g := \{g_n\}$ be a two-sided bounded sequence in $\mathbb{X}$. Then the following assertions hold:*

i. *$\sigma(g)$ is closed.*

ii. *If g^n is a sequence in $l_\infty(\mathbb{X})$.converging to g such that $\sigma(g^n) \subset \Lambda$ for all $n \in \mathbb{N}$, where Λ is a closed subset of the unit circle, then $\sigma(g) \subset \Lambda$.*

iii. *If $g \in l_\infty(\mathbb{X})$ and A is a bounded linear operator on the Banach space $\mathbb{X}$, then $\sigma(Ag) \subset \sigma(g)$, where $Ag \in l_\infty(\mathbb{X})$ is given by $(Ag)_n := Ag_n, \forall n \in \mathbb{Z}$.*

iv *Let the space $\mathbb{X}$ not contain any subspace which is isomorphic to c_0 (the Banach space of numerical sequences which converge to 0) and $x \in l_\infty(\mathbb{X})$ be a sequence such that $\sigma(x)$ is countable. Then x is almost periodic.*

Proof. For the proof we refer the reader to [23].

As an immediate consequence of the above proposition is the following:

Corollary 1.3. *Let $\mathbb{X}$ be a Banach space and let $\Lambda \subset \Gamma$ be closed. Then the subspace*

$$\Lambda(\mathbb{X}) := \{x \in l_\infty(\mathbb{X}) : \sigma(x) \subset \Lambda\}$$

is a closed subspace of $l_\infty(\mathbb{X})$.

In what follows we use the notation $\mathcal{M}_x := \overline{span\{S(n)x, n \in \mathbb{Z}\}}$.

Lemma 1.4. *Assume that $x \in l_\infty(\mathbb{X})$. Then we have the following assertion:*

$$\sigma(x) = \sigma\left(S|_{\mathcal{M}_x}\right) \tag{2}$$

where $S = S(1)$.

Proof. For the proof see [23].

We denote by $APZ(\mathbb{X})$ the space of all almost periodic two-sided sequences in a complex Banach space $\mathbb{X}$ in the sense of Bohr. Note that by the Approximation Theorem for Almost Periodic Sequences, $APZ(\mathbb{X})$ is the closure of the linear space spanned over all trigonometric polynomial sequences, i.e., sequences $\{x(n)\}$ of the form

$$x(n) = \sum_{k=1}^{N} a_k q_k^n, \ \forall n \in \mathbb{Z},$$

where $a_k \in \mathbb{X}, q_k \in \Gamma$.

Commutative Operators and Their Spectra

In what follows, we list some spectral properties of commutative bounded linear operators.

Lemma 1.5. *Let $\mathbb{X}$ be a complex Banach space and let U and V be bounded linear operators acting on $\mathbb{X}$ such that $UV = VU$ (commutativeness). Then, the following assertions hold:*

$$\sigma(UV) \subset \sigma(U) + \sigma(V) \tag{3}$$
$$\sigma(UV) \subset \sigma(U)\sigma(V). \tag{4}$$

Proof. For the proof see [28, Theorem 11.23].

Next, we will show how to apply this result to study the existence of bounded solutions of difference equations. Let A be a bounded linear operator on $\mathbb{X}$ and $\Lambda \subseteq \Gamma$ be a closed subset of the unit circle. Then we can define the operator of multiplication by A on $\Lambda(\mathbb{X})$ by the formulas

$$(\mathcal{A}_\Lambda x)(n) = Ax_n = Ax(n+1), \ \forall x \in \Lambda(\mathbb{X}), \forall n \in \mathbb{Z}$$
$$(S_\Lambda x)(n) = x(n+1), \ \forall x \in \Lambda(\mathbb{X}), \forall n \in \mathbb{Z}.$$

The following lemma can be easily proved.

Lemma 1.6. *Let $\Lambda \subset \Gamma$ be closed. Then*

i) S_Λ *commutes with* $\mathcal{A}_\Lambda$,

ii)

$$\sigma(\mathcal{A}_\Lambda) = \sigma(A) \tag{5}$$

Similar assertions hold for the restrictions of the above operators to $\Lambda_{AP}(\mathbb{X}) := \Lambda(\mathbb{X}) \cap APZ(\mathbb{X})$.

On the other hand, we have

Lemma 1.7. *For any closed $\Lambda \subset \Gamma$ we have*

$$\sigma(S_\Lambda) = \Lambda. \tag{6}$$

Proof. For the proof see [23].

2. Main results

From the definitions of the operators D and L in Eq. (1) it is easily seen that they can be represented in the forms:

$$D\phi = \sum_{k=-r}^{0} A_k \phi(k), \quad \forall \phi \in C \tag{7}$$

$$L\phi = \sum_{k=-r}^{0} B_k \phi(k), \quad \forall \phi \in C, \tag{8}$$

where $A_k, B_k \in L(\mathbb{X})$, $\forall k = -r, \cdots 0$. Hence, the operator $\Delta(Dx_n)$ in Eq.(1) can be re-written in the form:

$$\begin{aligned}
\Delta(Dx_n) &= Dx_{n+1} - Dx_n \\
&= \sum_{k=-r}^{0} A_k x_{n+1}(k) - \sum_{k=-r}^{0} A_k x_n(k) \\
&= \sum_{k=-r}^{0} A_k x(n+1+k) - \sum_{k=-r}^{0} A_k x(n+k) \\
&= A_0 x(n+1) + \sum_{k=-r+1}^{0} (A_{k-1} - A_k) x(n+k) - A_{-r} x(n-r).
\end{aligned}$$

Combined with (8) the Eq. (1) can be re-written as follows:

$$A_0 x(n+1) = \sum_{k=-r}^{0} C_k x(n+k) + f(n), \tag{9}$$

where

$$\begin{aligned}
C_k &= A_k - A_{k-1} + B_k, \quad \forall k = -r+1, \cdots, 0 \\
C_{-r} &= A_{-r} + B_{-r}.
\end{aligned}$$

Definition2.1. *Eq. (1) is said to be atomic at zero if A_0 is invertible.*

Note that this notion of atomicness is an analog of what has been known in the theory of functional differential equations (see e.g. [13]).

We now transform Eq.(9) into a first order equation by setting

$$\begin{aligned} y^0(n) &= x(n-r) \\ y^1(n) &= x(n-r+1) \\ &\cdots\cdots \\ y^r(n) &= x(n). \end{aligned}$$

Finally, we re-write Eq. (1) in the form

$$Ay(n+1) = By(n) + F(n), \tag{10}$$

where

$$\begin{aligned} y(n) &:= (y^0(n), y^1(n), \cdots, y^r(n))^T \in \mathbb{Y} := \mathbb{X}^{r+1}, \forall n \in \mathbb{Z}, \\ F(n) &= (0, \cdots, 0, f(n))^T, \quad \forall n \in \mathbb{Z}, \end{aligned}$$

and

$$A := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 1 \\ C_{-r} & C_{-r+1} & \dots & C_0 \end{pmatrix}.$$

From now on we denote

$$\Sigma_0 := \{z \in \mathbb{C} : \nexists (z^{r+1}I - \sum_{j=-r}^{0} z^{r-j}C_j)^{-1}\} \tag{11}$$

We are now in a position to formulate a main result of this paper.

Theorem 2.2. *Let A_0 commute with C_k, $k = -r, \cdots 0$ and let $\Lambda\sigma(A_0) \cap \Sigma_0 = \emptyset$. Then, for every $f \in \Lambda(\mathbb{X})$ Eq. (1) has a unique bounded solution x_f such that $\sigma(x_f) \subset \sigma(\Lambda)$. Moreover, if f is almost periodic, then so is x_f.*

Proof. By the above notations, observe that $\sigma(A_0) = \sigma(A)$, $\Sigma_0 = \sigma(B)$ and $\sigma(f) = \sigma(F)$. Consider the space $\Lambda(\mathbb{Y})$. Then, by the assumption and 1.6, 1.7 and 1.5, we have

$$\emptyset = \{\sigma(\mathcal{A}_\Lambda)\Lambda \cap \sigma(\mathcal{B}_\Lambda)\} \supset \{\sigma(\mathcal{A}_\Lambda S_\Lambda) \cap \sigma(\mathcal{B}_\Lambda)\}.$$

Thus

$$\sigma(\mathcal{A}_\Lambda S_\Lambda) \cap \sigma(\mathcal{B}_\Lambda) = \emptyset.$$

This yields

$$0 \notin \sigma(\mathcal{A}_\Lambda S_\Lambda) + \sigma(-\mathcal{B}_\Lambda), \tag{12}$$

i.e., the operator $(\mathcal{A}_\Lambda S_\Lambda - \mathcal{B}_\Lambda)^{-1}$ exists on $\Lambda(\mathbb{Y})$. Hence $y_f = (\mathcal{A}_\Lambda S_\Lambda - \mathcal{B}_\Lambda)^{-1}$ exists on $\Lambda(\mathbb{Y})$, that is y_f is a unique solution of Eq. (10) in $\Lambda(\mathbb{Y})$. This yields that $x(n) := y^0(n)$ is a unique solution of Eq. (1) in $\mathbb{X}$.

By the remark following Lemma 1.6, if f is almost periodic our argument works well with the operator theoretic setting on $\Lambda(\mathbb{X}) \cap APZ(\mathbb{X})$. Thus, the last assertion follows.

2.3. *In general, without assumption on the commutativeness of A and B we do not know if the assertions of the above theorem is true. We have the following which extends [23, Theorem 3.1].*

Corollary 2.4. *Let A_0 commute with C_k, $k = -r, \cdots, 0$ and let $f \in l_\infty(\mathbb{X})$. Then Eq. (1) has a unique solution $x_f \in l_\infty(\mathbb{X})$ provided that $\sigma(f)\sigma(A_0) \cap \Sigma_0 = \emptyset$. Morover, if f is almost periodic, then so is x_f.*

Proof. We can set $\Lambda := \sigma(f)$ and apply Theorem 2.2.

Corollary 2.5. *Let the operator D in Eq. (1) be atomic at zero, i.e., A_0 is invertible, and let Λ be a closed subset of the unit circle such that $\sigma(f) \cap \Sigma_1 = \emptyset$. Then, there exists a unique solution $x_f \in \Lambda(\mathbb{X})$ of Eq. (1).*

Proof. Re-write the equation in the form (9). Then, since A_0 is invertible, we have

$$x(n+1) = \sum_{k=-r}^{0} A_0^{-1} C_k x(n+k) + A^{-1} f(n),$$

Observe that $\sigma(A_0^{-1} f(\cdot)) \subset \sigma(f)$. Thus, $A_0^{-1} f(\cdot) \in \Lambda(\mathbb{X})$. By Corollary 2.4 we have the assertion of the corollary.

To consider the converse of Theorem 2.2 we need several results.

Definition 2.6. *Let A and $B_j, j = 1, 2, \cdots n$ be bounded linear operators acting on a complex Banach space $\mathbb{X}$. Then, we define*

$$\sigma_A(B) := \{\lambda \in \mathbb{C} : \nexists (\lambda^{n+1} A - \sum_{j=1}^{n} \lambda^j B)^{-1} \in L(\mathbb{X})\}. \tag{13}$$

Proposition 2.7. *Let A and $B_j, j = 1, 2, \cdots, n$, be bounded linear operators acting on a complex Banach space $\mathbb{X}$. Then the following assertions hold true:*

i) The $\sigma_A(B)$ is a closed subset of $\mathbb{C}$;

ii) The function $\rho_A(B) := \mathbb{C} \backslash \sigma_A(B) \ni \lambda \mapsto (\lambda^{n+1} A - \sum_{j=1}^{n} \lambda^j B)^{-1} \in L(\mathbb{X})$ is analytic.

Proof. (i) It suffices to show that $\mathbb{C} \backslash \sigma_A(B) =: \rho_A(B)$ is open. In fact, take $\lambda_0 \in \rho_A(B) := \{\lambda \in \mathbb{C} : \ \exists (\lambda^{n+1} A - \sum_{j=1}^{n} \lambda^j B)^{-1} \in L(\mathbb{X})\}$. Then $(\lambda^{n+1} A - \sum_{j=1}^{n} \lambda^j B)^{-1}$ exists. And thus, by the Open Mapping Theorem, the operator $(\lambda^{n+1} A - \sum_{j=1}^{n} \lambda^j B)^{-1} : \mathbb{X} \to \mathbb{X}$ is bounded.

Consider $(\lambda^{n+1} A - \sum_{j=1}^{n} \lambda^j B)$, where λ is sufficiently close to λ_0. We have

$$(\lambda^{n+1} A - \sum_{j=1}^{n} \lambda^j B) = \left[(\lambda^{n+1} A - \lambda_0^{n+1} A) - \sum_{j=1}^{n} (\lambda^j - \lambda_0^j) B \right] + (\lambda_0^{n+1} A - \sum_{j=1}^{n} \lambda_0^j B)$$

Since A, B_j are bounded, if $\lambda - \lambda_0$ is sufficiently small, then so is $(\lambda^{n+1} A - \lambda_0^{n+1} A) - \sum_{j=1}^{n} (\lambda^j - \lambda_0^j) B$. Finally, since $(\lambda_0^{n+1} A - \sum_{j=1}^{n} \lambda_0^j B)^{-1}$ is bounded, if $|\lambda - \lambda_0|$ is so small that

$$\|(\lambda^{n+1} A - \lambda_0^{n+1} A) - \sum_{j=1}^{n} (\lambda^j - \lambda_0^j) B\| < \|(\lambda_0^{n+1} A - \sum_{j=1}^{n} \lambda_0^j B)^{-1}\|,$$

then $(\lambda^{n+1}A - \sum_{j=1}^{n} \lambda^j B)$ is invertible, i.e., $\lambda \in \rho_A(B)$.

(ii) Set $\mathcal{G} \subset L(\mathbb{X})$ consists of all invertible elements of $L(\mathbb{X})$. Then $\mathcal{G}$ is open in $L(\mathbb{X})$ with the usual operator topology. Moreover, if $U_0 \in \mathcal{G}$, U^{-1} exists if U is sufficiently close to U_0 and can be represented in the form

$$U^{-1} = \sum_{k=0}^{\infty}(I - U_0^{-1}U)^k U_0^{-1}.$$

Suppose that $z_0 \in \rho_A(B)$ Then for $|\lambda - \lambda_0|$ sufficiently small, by setting

$$U = U(\lambda) = (\lambda^{n+1}A - \sum_{j=1}^{n} \lambda^j B); \quad U_0 = U(\lambda_0) = (\lambda_0^{n+1}A - \sum_{j=1}^{n} \lambda_0^j B)$$

we have

$$\begin{aligned}
U^{-1} &= \sum_{k=0}^{\infty}(I - U_0^{-1}U)^k U_0^{-1} \\
&= \sum_{k=0}^{\infty}\left[I - (\lambda_0^{n+1}A - \sum_{j=1}^{n} \lambda_0^j B)^{-1}(\lambda^{n+1}A - \sum_{j=1}^{n} \lambda^j B)\right]^k (\lambda_0^{n+1}A - \sum_{j=1}^{n} \lambda_0^j B)^{-1} \\
&= \sum_{k=0}^{\infty}\left[(\lambda_0^{n+1}A - \sum_{j=1}^{n} \lambda_0^j B) - (\lambda^{n+1}A - \sum_{j=1}^{n} \lambda^j B)\right]^k (\lambda_0^{n+1}A - \sum_{j=1}^{n} \lambda_0^j B)^{-k-1} \\
&= \sum_{k=0}^{\infty}\left[(\lambda_0^{n+1} - \lambda^{n+1})A - \sum_{j=1}(\lambda_0^j - \lambda^j)B_j\right]^k (\lambda_0^{n+1}A - \sum_{j=1}^{n} \lambda_0^j B)^{-k-1} \\
&= \sum_{k=0}^{\infty}(\lambda_0 - \lambda)^k (\varphi(\lambda))^k U_0^{-k-1},
\end{aligned}$$

where $\varphi(\lambda)$ is analytic function, and thus, it is bounded around λ_0.

Now we can easily see that

$$\begin{aligned}
U^{-1}(\lambda) - U^{-1}(\lambda_0) &= \sum_{k=1}^{\infty}(\lambda_0 - \lambda)^k (\varphi(\lambda))^k U_0^{-k-1} \\
&= (\lambda_0 - \lambda)\sum_{j=0}^{\infty}(\lambda_0 - \lambda)^j (\varphi(\lambda))^{j+1} U_0^{-j-2}
\end{aligned}$$

Since $\varphi(\lambda)$ is bounded in $\{|\lambda - \lambda_0| < \epsilon\}$ with small ϵ, the series

$$\sum_{j=0}^{\infty}(\lambda_0 - \lambda)^j (\varphi(\lambda))^{j+1} U_0^{-j-2} \tag{14}$$

is absolutely and uniformly convergent in $\{|\lambda - \lambda_0| < \epsilon\}$ for sufficiently small ϵ. Finally, if x^* is any bounded functional on $\mathbb{X}$

$$\frac{x^*(U^{-1}(\lambda)) - x^*(U^{-1}(\lambda_0))}{(\lambda - \lambda_0)} = -\sum_{j=0}^{\infty}(\lambda_0 - \lambda)^j x^*((\varphi(\lambda))^{j+1}U_0^{-j-2})$$

By the absolute and uniform convergence of (14), we have

$$\lim_{\lambda\to\lambda_0} \frac{x^*(U^{-1}(\lambda)) - x^*(U^{-1}(\lambda_0))}{(\lambda_0 - \lambda)} = -\sum_{j=0}^{\infty} \lim_{\lambda\to\lambda_0}(\lambda_0 - \lambda)^j x^*((\varphi(\lambda))^{j+1}U_0^{-j-2})$$
$$= x^*(\varphi(\lambda_0)U_0^{-2}).$$

This shows that for each bounded functional x^* the complex function $x^*(U^{-1}(\lambda))$ is holomorphic on $\rho_A(B)$. By the well known facts on vector-valued analytic functions, this implies in particular the analyticity of $U^{-1}(\lambda)$ as a vector-valued function of λ on $\rho_A(B)$.

To proceed we denote

$$\Sigma_1 := \{\lambda \in \mathbb{C} : \nexists((\lambda^{r+1}A_0 - \sum_{k=-r}^{1} \lambda^{r+k}C_k)^{-1}\}. \tag{15}$$

Theorem 2.8. *Let $\Lambda \subset \Gamma$ be closed and let A_0 and $C_j, j = -r, \cdots, 0$ be bounded linear operators on $\mathbb{X}$ defined as in (9). Assume further that for every $f \in \Lambda(\mathbb{X})$ Eq. (1) has a unique bounded solution $x_f \in \Lambda(\mathbb{X})$. Then, $\Sigma_1 \cap \Lambda = \emptyset$.*

Proof.

Take $\lambda \in \Lambda \subset \Gamma$. Consider the sequence $f_n = \lambda^n x$, $x \neq 0, x \in \mathbb{X}$. Then, $f = \{f_n\}_{n\in\mathbb{Z}} \in \Lambda(\mathbb{X})$ and $\sigma(f) \subset \Lambda)$. We are going to show that there exists a unique solution x_f to Eq. (1) of the form

$$(x_f)_n = \lambda^n y, \tag{16}$$

for certain $y \in \mathbb{X}$. In fact, suppose that x_f is a solution to Eq. (1). Then,

$$A_0 x(n+1) = \sum_{k=-r}^{0} C_k x(n+k) + f(n), \ \forall n \in \mathbb{Z}.$$

Denoting by $\mathcal{A}_0$, $\mathcal{C}_k$ the operators of multiplication by A_0, C_k on $\Lambda(\mathbb{X})$, respectively, we have

$$(\mathcal{A}_0 S - \sum_{k=-r}^{0} \mathcal{C}_k S(k))x_f = f.$$

Thus

$$x_f = (\mathcal{A}_0 S - \sum_{k=-r}^{0} \mathcal{C}_k S(k))^{-1} f. \tag{17}$$

Next, the linear operator $(\mathcal{A}_0 S - \sum_{k=-r}^{0} \mathcal{C}_k S(k))^{-1}$ exists and is bounded according to the Open Mapping Theorem. Set $G := (\mathcal{A}_0 S - \sum_{k=-r}^{0} \mathcal{C}_k S(k))^{-1}$. We claim that the operator

$G: \Lambda(\mathbb{X}) \to \Lambda(\mathbb{X})$, $f \longmapsto x_f$ commutes with the translation operator S. In fact, suppose that $f = \{f(n)\}_{n\in\mathbb{Z}} \in \Lambda(\mathbb{X})$. By checking directly we can see that Sx_f is a solution to Eq. (1) with the forcing term Sf. By the assumption on the uniqueness and by the definition of G we have $G(Sf) = Sx_f = S(Gf)$, i.e., G and S commutes. Now letting $f = \lambda^n x$ we have

$$\begin{aligned}(GS(\lambda^m x))(n) &= (S(Gf))(n)\\ (G\lambda^{m+1}x)(n) &= (Sx_f)(n)\\ (\lambda.G\lambda^m x)(n) &= S(x_f)(n)\\ \lambda(Gf)(n) &= (x_f)(n+1)\\ \lambda(x_f)(n) &= (x_f)(n+1).\end{aligned}$$

Hence, $x_f = \{x_f(n)\}_{n\in\mathbb{Z}}$ is of the form $x_f(n+1) = \lambda x_f(n)$ $\forall n \in \mathbb{Z}$. Consequently, $x_f(n) = \lambda^n x_f(0)$. Set

$$y := x_f(0). \tag{18}$$

Then, $y \in \mathbb{X}$, and thus, $x_f(n) = \lambda^n.y$ $\forall n \in \mathbb{Z}$. This implies (16).

Next, we consider the operator $T : \mathbb{X} \ni x \longmapsto y \in \mathbb{X}$, where y is defined by (18), i.e.,

$$T : x \mapsto f = \{\lambda^n x\}_{n\in\mathbb{Z}} \longrightarrow x_f \longrightarrow x_f(0) = y. \tag{19}$$

This shows that the map taking x into y is a bounded linear operator. Moreover, it is easily checked that $T : x \mapsto y$ is the continuous inverse of $(\lambda^{r+1}A_0 - \sum_{k=-r}^{1} \lambda^{r+k}C_k)$, so $\lambda \notin \Sigma_1$. This shows that $\Sigma_1 \cap \Lambda = \oslash$.

Theorem 2.9. *Let x be a bounded solution of Eq. (1). Then,*

$$\sigma(x) = \Sigma_{\Gamma,1} \cup \sigma(f), \tag{20}$$

where $\Sigma_{\Gamma,1} := \Gamma \cap \Sigma_1$.

Proof. First, by definition we have

$$S\hat{x}(z) = \widehat{Sx}(z) = z\hat{x}(z) - x, \quad \forall x \in l_\infty(\mathbb{X}), \quad 0 < |z| \neq 1.$$

By induction,

$$S(n)\hat{x}(z) = S(\hat{n})x(z) = z^n\hat{x}(z) - (z^{n-1} + z^{n-2} + \cdots + 1)x, \quad \forall n = 1, 2, \cdots.$$

For negative integers k, by the above computation we can show that

$$S(k)\hat{x}(z) = S(\hat{k})x(z) = z^k\hat{x}(z) + \phi_k(z), \quad 0 < |z| \neq 1, \tag{21}$$

where ϕ_k is an analytic function in $\{0 < |z|\}$.

Re-write (1) in the form

$$\mathcal{A}_0 Sx = \left(\sum_{k=-r}^{0} C_k S(k)\right) x + f. \tag{22}$$

Taking the Carleman transform of both sides of the above equality, for $|z| > 1$, by (21) we have

$$\begin{aligned}
\mathcal{A}_0 \widehat{Sx}(z) &= \mathcal{A}_0 z.\hat{x}(z) - \mathcal{A}_0 x \\
&= \sum_{n=0}^{\infty} z^{-n-1} S(n)(\mathcal{A}_0 Sx) \\
&= \sum_{n=0}^{\infty} z^{-n-1} S(n)((\sum_{k=-r}^{0} C_k S(k))x + f) \\
&= \sum_{n=0}^{\infty} z^{-n-1} S(n)((\sum_{k=-r}^{0} C_k S(k))x) + \sum_{n=0}^{\infty} z^{-n-1} S(n) f \\
&= (\sum_{k=-r}^{0} C_k S(k)) \sum_{n=0}^{\infty} z^{-n-1} S(n) x + \hat{f}(z) \\
&= (\sum_{k=-r}^{0} C_k S(k)) \hat{x}(z) + \hat{f}(z).
\end{aligned}$$

Hence,

$$\left(\mathcal{A}_0 z - \sum_{k=-r}^{0} C_k S(k)\right) \hat{x}(z) = \mathcal{A}_0 x + \hat{f}(z). \tag{23}$$

Similarly, for $|z| < 1$ we get (23).

Now, by (21) we have

$$\left(\mathcal{A}_0 z - \sum_{k=-r}^{0} C_k S(k)\right) \hat{x}(z) = \left(\mathcal{A}_0 z - \sum_{k=-r}^{0} C_k z^k\right) \hat{x}(z) + \phi(z),$$

where $\phi(z)$ is analytic in $\{0 < |z|\}$. Hence we have

$$\left(\mathcal{A}_0 z - \sum_{k=-r}^{0} C_k z^k\right) \hat{x}(z) = \phi_1(z) + \hat{f}(z),$$

where $\phi_1(z)$ is analytic in $\{0 < |z|\}$. Consequently,

$$\left(\mathcal{A}_0 z^{n+1} - \sum_{k=-r}^{0} C_k z^{n+k}\right) \hat{x}(z) = z^n \phi_1(z) + z^n \hat{f}(z),$$

If $z_0 \in (\Gamma\backslash\Sigma_{\Gamma,1}) \cap (\mathbb{C}\backslash\sigma(f))$, for z near z_0 we have

$$\hat{x}(z) = \left(A_0 z^{n+1} - \sum_{k=-r}^{0} C_k z^{n+k}\right)^{-1} z^n \phi_1(z) + z^n \hat{f}(z).$$

By Proposition 2.7, $\left(A_0 z^{n+1} - \sum_{k=-r}^{0} C_k z^{n+k}\right)^{-1}$ is analytic in a neighborhood of z_0. On the other hand, near $z_0 \in (\mathbb{C}\backslash\sigma(f))$, $\hat{f}(z)$ has an analytic extension to a neighborhood of z_0. This shows that $\hat{x}(z)$ has an analytic extension to a neighborhood of z_0. Hence, $z_0 \notin \sigma(x)$. Finally, this yields that $\sigma(x) \subset \Sigma_1 \cup \sigma(f)$.

We are going to prove a Massera type theorem for the neutral delay difference equation (1).

Theorem 2.10. *Let $\overline{\Sigma_{\Gamma,1}\backslash\sigma(f)} \cap \sigma(f) = \oslash$ and let x be a bounded solution of Eq. (1). Then, there exists a bounded solution w to Eq. (1) such that $\sigma(w) \subset \sigma(f)$.*

Proof. By definition, $\overline{\Sigma_{\Gamma,1}\backslash\sigma(f)} \cap \sigma(f) = \oslash$. Set $\Lambda := \Sigma_{\Gamma,1} \cup \sigma(f)$, $\Lambda_1 := \overline{\Sigma_{\Gamma,1}\backslash\sigma(f)}$ and $\Lambda_2 := \sigma(f)$. Then, as in the proof of [23, Theorem 3.4]naiminminyiham, we can show that there is a splitting $\Lambda(\mathbb{X}) = \Lambda_1(\mathbb{X}) \oplus \Lambda_2(\mathbb{X})$, and the projection $P : \Lambda(\mathbb{X}) \to \Lambda_2(\mathbb{X})$ commutes with translation operator S and multiplication operators. Hence, by (22) we have

$$PA_0Sx = A_0SPx = P\left(\sum_{k=-r}^{0} C_k S(k)\right)x + Pf = \left(\sum_{k=-r}^{0} C_k S(k)\right)Px + Pf.$$

Obviously, $w := Px$ is a bounded solution of Eq. (1) satisfying $\sigma(w) \subset \sigma(f)$.

Corollary 2.11. *Let all assumptions of the above theorem be safisfied. Moreover, assume that f is almost periodic, $\sigma(f)$ is countable, and $\mathbb{X}$ does not contain any subspace isomorphic to c_0. Then, if there is a bounded solution to Eq. (1), then there is an almost periodic solution to this equation as well.*

Proof. As shown in the above theorem, if there is a bounded solution x to Eq. (1), there is a bounded solution w such that $\sigma(w) \subset \sigma(f)$. Under the additional conditions of the corollary, w is almost periodic according to [23, Theorem 5.3].

3. Discussions

In particular, if $dim\ \mathbb{X} < \infty$, then $\mathbb{X}$ does not contain any subspaces isomorphic to c_0. In this finite dimensional case, the noninvertibility of the matrix $(\lambda^{r+1}A_0 - \sum_{k=-r}^{1} \lambda^{r+k}C_k)$ is equivalent to $\det(\lambda^{r+1}A_0 - \sum_{k=-r}^{1} \lambda^{r+k}C_k) = 0$. If we assume a further condition on the nondegenerateness that $g(z) := \det(\lambda^{r+1}A_0 - \sum_{k=-r}^{1} \lambda^{r+k}C_k)$ is not identically equal to zero, then one can see that Σ_1 is always countable due to the discreteness of the set of zeros of an analytic function. In a separate paper, by using the spectral theory of sequences we have proved that in this case any bounded solution should be almost periodic. And the decomposition technique can be refined so that the countability of $\sigma(f)$ is not necessary.

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