ON WEAK CONVERGENCE IN THE SPACE OF PROBABILITY CAPACITIES IN $\mathbb{R}^d$*

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Abstract. In this paper we are going to extend the weak convergence in the space of probability measures to the one in the space of probability capacities and investigate the relation on weak convergence between probability capacities and their associated probability measures.

1. Introduction

As well known that integration theory could be founded on order and monotonicity. Indeed it turned out that many aspects of integration theory are sustained if additivity is replaced by order and monotonicity. It is due to Choquet [2] who was lead to the problem from his research in electrostatics and potential theory. Then Choquet's results had been applied to several research areas, including Artificial Intelligence, Mathematical Economics and Bayesian statistics, particularly to the areas of upper and lower probabilities, (see, e.g., [14] and [15] for an introduction to their use in these areas). In a such way, non-additive set functions, known as capacities, have become uncertainty measures in situations where probability measures do not seem to be approriate. In the last twenty five years the study of non-additive set functions is useful in interval computations where interval probabilities present uncertainty, and has been carried out by several authors.

In [10] we have introduced the notion of capacites in $\mathbb{R}^d$, where $\mathbb{R}^d$ denotes d-dimensional Euclidean space with the ordinary metric $\rho(x,y) = [\sum_{i=1}^{d}(x_i - y_i)^2]^{\frac{1}{2}}$, and constructed the Choquet integral for these capacities. Then the weak topology on the space of probability capacities in $\mathbb{R}^d$ was investigated in [11]. We first recall some definitions and facts used in this paper, the details can be found in [10] and [11].

Let $\mathcal{K}(\mathbb{R}^d)$, $\mathcal{F}(\mathbb{R}^d)$, $\mathcal{G}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$ denote the family of all *compact sets, closed sets, open sets* and *Borel sets* in $\mathbb{R}^d$, respectively. By a *capacity* in $\mathbb{R}^d$ we mean a set function $T:\ \mathbb{R}^d \to \mathbb{R}^+ = [0,\ +\infty)$ satisfying the following conditions:

(i) $T(\emptyset) = 0$;

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(ii) T is alternating of infinite order: For any Borel sets A_i, $i = 1, 2, \dots, n$; $n \geq 2$, we have

$$T(\bigcap_{i=1}^{n} A_i) \leq \sum_{I \in \mathcal{I}(n)} (-1)^{\#I+1} T(\bigcup_{i \in I} A_i),$$

where $\mathcal{I}(n) = \{I \subset \{1, \dots, n\},\ I \neq \emptyset\}$ and $\#I$ denotes the cardinality of I;

(iii) $T(A) = \sup\{T(C) :\ C \in \mathcal{K}(\mathbb{R}^d),\ C \subset A\}$ for any Borel set $A \in \mathcal{B}(\mathbb{R}^d)$;

(iv) $T(C) = \inf\{T(G) :\ G \in \mathcal{G}(\mathbb{R}^d),\ C \subset G\}$, for any compact set $C \in \mathcal{K}(\mathbb{R}^d)$.

A capacity in $\mathbb{R}^d$ is, in fact, a generalization of a measure in $\mathbb{R}^d$. Clearly any capacity is a non-decreasing set function on Borel sets of $\mathbb{R}^d$.

By *support* of a capacity T we mean the smallest closed set $S \subset \mathbb{R}^d$ such that $T(\mathbb{R}^d \backslash S) = 0$. The support of a capacity T is denoted by supp T. We say that T is a *probability capacity* in $\mathbb{R}^d$ if T has a compact support and $T(\text{supp } T) = 1$. By $\tilde{\mathcal{C}}$ we denote the family of all probability capacities in $\mathbb{R}^d$.

Let T be a capacity in $\mathbb{R}^d$. Then for any measurable function $f : \mathbb{R}^d \to \mathbb{R}^+$ and $A \in \mathcal{B}(\mathbb{R}^d)$, the function $f_A : \mathbb{R} \to \mathbb{R}$ defined by

$$f_A(t) = T(\{x \in A :\ f(x) \geq t\}) \text{ for } t \in \mathbb{R}$$

is a non-increasing function in t. Therefore we can define the *Choquet integral* $\int_A f dT$ of f with respect to T by

$$\int_A f dT = \int_0^\infty f_A(t) dt = \int_0^\infty T(\{x \in A :\ f(x) \geq t\}) dt.$$

If $\int_A f dT < \infty$, we say that f is *integrable*. In particular for $A = \mathbb{R}^d$, we write

$$\int_{\mathbb{R}^d} f dT = \int f dT.$$

Observe that if f is bounded, then

$$\int_A f dT = \int_0^\alpha T(\{x \in A :\ f(x) \geq t\} dt,$$

where $\alpha = \sup\{f(x) :\ x \in A\}$.

In the general case if $f :\ \mathbb{R}^d \to \mathbb{R}$ is a measurable function, then we define

$$\int_A f dT = \int_A f^+ dT - \int_A f^- dT,$$

where $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$.

In this note we are going to extend the weak convergence in the space of probability measures to the one in the space of probability capacities and study the relation on the weak convergence between probability capacities and probability measures. The paper is organized as follows. In section 2 we give a version of the Portmanteau Theorem, in which the space of all bounded continuous real valued functions is replayed by the space of all continuous non-negative real valued functions with compact support in $\mathbb{R}^d$. In Section 3, we prove a generalization of Portmanteau Theorem in the case of space of probability capacities in $\mathbb{R}^d$. The main result of this paper is presented in Section 4, where the relation on weak convergence between probability capacities and their associated measures is investigated.

2. A version of portmanteau theorem

Let P be a measure on $\mathcal{B}(\mathbb{R}^d)$. Then the measure P is called the *probability measure* if $P(\mathbb{R}^d) = 1$. By slight modification of Billingsley's definition [1] of weak convergence, we say that a sequence of probability measures P_n *converges weakly* to a probability measure P if $\int f dP_n \to \int f dP$ for every $f \in C_0^+(\mathbb{R}^d)$, where $C_0^+(\mathbb{R}^d)$ denotes all non-negative real valued continuous functions with compact support in $\mathbb{R}^d$. From the definition of a weakly convergent sequence and the property of a convergent sequence, it follows immediately the following proposition.

2.1. Proposition. *P_n convergences weakly to P if and only if each subsequence $\{P_{n'}\}$ contains a further subsequence $\{P_{n''}\}$ such that $P_{n''}$ converges weakly to P.*

Following [1], we say that a set $A \subset \mathbb{R}^d$ is called T-*continuity set* if its boundary ∂A satisfies $T(\partial A) = 0$. In this section we prove the following theorem, which is a version of the Portmanteau Theorem in situation where every bounded continuous real valued function is replayed by non-negative real valued continuous function with compact support and provides useful conditions equivalent to weak convergence for probability measures.

2.2. Theorem. *Let P, P_n be probability measures on $\mathcal{B}(\mathbb{R}^d)$. Then the following statements are equivalent*

(i) P_n converges weakly to P;

(ii) $\limsup_n P_n(F) \leq P(F)$ *for all closed F;*

(iii) $\liminf_n P_n(G) \geq P(G)$ *for all open G;*

(iv) $\lim_{n\to\infty} P_n(A) = P(A)$ *for all P-continuity set A.*

We prove Theorem 2.1 by establishing the implications in the following diagram

$$(i) \leftrightarrow (ii) \leftrightarrow (iii) \text{ and } (ii) \leftrightarrow (iv).$$

Proof of (i)→ (ii). Let $F \in \mathcal{F}(\mathbb{R}^d)$ and $\epsilon > 0$. Then there exists $G \in \mathcal{G}(\mathbb{R}^d), G \supset F$ such that

$$P(G) < P(F) + \epsilon. \tag{2.1}$$

Suppose that $f_{F,G} : \mathbb{R}^d \to [0,1]$ is a continuos function such that

$$f_{F,G}(x) = \begin{cases} 1 & \text{if } x \in F \\ 0 & \text{if } x \notin G. \end{cases}$$

Thus function exists by Urysohn-Tietze Theorem. Then by (2.1) we get

$$P_n(F) = \int_F f_{F,G} dP_n \le \int f_{G,F} dP_n \to \int f_{F,G} dP$$
$$= \int_G f_{F,G} dP \le P(G) < P(F) + \epsilon,$$

which implies

$$\limsup_n P_n(F) \le P(F) + \epsilon.$$

Since ϵ was arbitrary, the assertion follows.

Proof of (ii)→ (i). Suppose (ii) holds. We first show that for any real-valued continuous f with compact support, we have

$$\limsup_n \int f dP_n \le \int f dP. \tag{2.2}$$

By addition in f a constant if necessary, we may assume that $f(x) \ge 0$ for all $x \in \mathbb{R}^d$. Then we have

$$0 < \alpha = \sup\{f(x) : x \in \mathbb{R}^d\} < \infty.$$

For a given $k \in \mathbb{N}$, let

$$0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = \alpha$$

with

$$\alpha_{i+1} - \alpha_i = \alpha/k \text{ for } i = 0, 1, \ldots, k-1.$$

We put

$$t_i = P(\{x \in \mathbb{R}^d : f(x) \ge \alpha_i\}) \text{ for } i = 0, 1, \ldots, k.$$

Then for $t \in [\alpha_i; \alpha_{i+1})$ we have

$$t_{i+1} \le P(\{x \in \mathbb{R}^d : f(x) \ge t\}) \le t_i \text{ for } i = 0, 1, \ldots, k-1.$$

Hence

$$\sum_{i=0}^{k-1} t_{i+1}(\alpha_{i+1} - \alpha_i) \le \int f dP \le \sum_{i=0}^{k-1} t_i(\alpha_{i+1} - \alpha_i).$$

That means

$$\frac{\alpha}{k} \sum_{i=1}^{k-1} P(F_i) \le \int f dP \le \frac{\alpha}{k} + \frac{\alpha}{k} \sum_{i=1}^{k-1} P(F_i), \tag{2.3}$$

where the set $F_i = \{x \in \mathbb{R}^d : f(x) \ge \alpha_i\}$ is closed (even compact!) for $i = 1, \ldots, k-1$.

Similarly for each $n \in \mathbb{N}$ we have

$$\frac{\alpha}{k}\sum_{i=1}^{k-1} P_n(F_i) \le \int f dP_n \le \frac{\alpha}{k} + \frac{\alpha}{k}\sum_{i=1}^{k-1} P_n(F_i). \tag{2.4}$$

If (ii) holds, then from (2.3) and (2.4) we get

$$\begin{aligned}\limsup_n \int f dP_n &\le \frac{\alpha}{k} + \frac{\alpha}{k}\sum_{i=1}^{k-1} \limsup_n P_n(F_i)\\ &\le \frac{\alpha}{k} + \frac{\alpha}{k}\sum_{i=1}^{k-1} P(F_i)\\ &\le \frac{\alpha}{k} + \int f dP.\end{aligned}$$

Letting $k \to \infty$, we obtain (2.2). Applying (2.2) to -f yiels

$$\liminf_n \int f dP_n \ge \int f dP.$$

From the latter and (2.2) it follows that

$$\lim_{n\to\infty} \int f dP_n = \int f dP$$

for all real valued continuous functions with compact support f. In particular

$$\lim_{n\to\infty} \int f dP_n = \int f dP \text{ for all } f \in C_0^+(\mathbb{R}^d),$$

i.e., P_n converges weakly to P.

Proof of (ii)↔(iii). Suppose that (ii) holds and $G \in \mathcal{G}(\mathbb{R}^d)$. Then $F = \mathbb{R}^d \backslash G$ is a closed set. Hence

$$\begin{aligned}\liminf_n P_n(G) &= \liminf_n (1 - P_n(F))\\ &= 1 - \limsup_n P_n(F)\\ &\ge 1 - P(F) = P(G).\end{aligned}$$

That means (*iii*) holds.

Conversely, if (*iii*) holds and $F \in \mathcal{F}(\mathbb{R}^d)$, then $G = \mathbb{R}^d \backslash F \in \mathcal{G}(\mathbb{R}^d)$. Hence

$$\begin{aligned}\limsup_n P_n(F) &= \limsup_n (1 - P_n(G))\\ &= 1 - \liminf_n P_n(G)\\ &\le 1 - P(G) = P(F).\end{aligned}$$

Consequently (*ii*) ↔ (*iii*).

Proof of (ii)→ (iv). Suppose that (ii) holds and A is a P-continuity set. Let intA and $\overline{A}$ denote the interior and closure of A, respectively. Note that if (ii) holds, then so does (iii), and hence

$$\begin{aligned} P(\overline{A}) &\geq \limsup_n P_n(\overline{A}) \geq \limsup_n P_n(A) \\ &\geq \liminf_n P_n(A) \geq \liminf_n P_n(\text{int}A) \\ &\geq P(\text{int}A). \end{aligned} \tag{2.5}$$

Since A is a P-continuity set, $P(\overline{A}) = P(\text{int}A \bigcup \partial A) = P(\text{int}A)$. Therefore, from (2.5) we get

$$\lim_{n\to\infty} P_n(A) = P(A).$$

Proof of (iv) → (ii). Asumme that F is a closed set in $\mathbb{R}^d$. Since $\partial\{x : \rho(x,F) \leq \delta\} \subset \{x : \rho(x,F) = \delta\}$, we have

$$\partial F_\delta \bigcap \partial F_\gamma = \emptyset \text{ for } \delta \neq \gamma \text{ and } \delta, \gamma > 0,$$

where $F_\delta = \{x : \rho(x,F) \leq \delta\}$. Hence, there are at most countably many of the sets $\{F_\delta : \delta > 0\}$, which can have positive P-measure. Therefore, there exixts a positive sequence δ_k descreasing to 0, such that

$$F_{\delta_k} = \{x : \rho(x,F) \leq \delta_k\} \text{ are } P-\text{continuity sets for every k} \in \mathbb{N}.$$

If (iv) holds, then

$$\limsup_n P_n(F) \leq \lim_{n\to\infty} P_n(F_{\delta_k}) = P(F_{\delta_k}) \text{ for every } k. \tag{2.6}$$

Since F is closed and $F_{\delta_k} \downarrow F$, we have

$$P(F_{\delta_k}) \to P(F) \text{ as } k \to \infty.$$

From the latter and (2.6), (ii) follows.

3. A generalization of portmanteau theorem for capacities

We topologize $\tilde{\mathcal{C}}$ as follows. The *weak topology* on $\tilde{\mathcal{C}}$ is the topology with the base

$$\{\mathcal{U}(T; f_1, \ldots, f_k; \epsilon) : T \in \tilde{\mathcal{C}},\ f_i \in C_0^+(\mathbb{R}^d),\ \epsilon > 0,\ i = 1, \ldots, k\},$$

where

$$\begin{aligned} \mathcal{U}(T; f_1, \ldots, f_k; \epsilon) &= \{S \in \tilde{\mathcal{C}} : |\int f_i dT - \int f_i dS| < \epsilon,\ i = 1, \ldots, k\} \\ &= \bigcap_{i=1}^{k} \mathcal{U}(T; f_i; \epsilon). \end{aligned}$$

It is shown in [11] that $\tilde{C}$ equipped with the weak topology is metrizable and separable. Therefore, we can define the weak convergence of a sequence of probability capacities $\{T_n\}$ to a capacity T as the convergence of the sequence $\{\int fdT_n\}$ to $\int fdT$ for every $f \in C_0^+(\mathbb{R}^d)$.

In the sequel we need the following lemma, which was proved in [10].

3.1. Lemma. *Let T be a capacity in $\mathbb{R}^d$. If $A \in \mathcal{B}(\mathbb{R}^d)$ with $T(A) = 0$, then*

$$T(B) = T(A \cup B) \textit{ for } B \in \mathcal{B}(\mathbb{R}^d).$$

In this section, we prove the following theorem, which is a generalization of Portmanteau Theorem for capacities.

3.2. Theorem. *Let T_n, T be probability capacities in $\mathbb{R}^d$ and suppose T_n converges werakly to T. Then*

(i) $\limsup_n T_n(K) \le T(K)$ *for all compact* $K \in \mathcal{K}(\mathbb{R}^d)$;

(ii) $\liminf_n T_n(G) \ge T(G)$ *for all open* $G \in \mathcal{G}(\mathbb{R}^d)$;

(iii) $\lim_{n\to\infty} T_n(A) = T(A)$ *for all bounded T-continuity set* $A \subset \mathbb{R}^d$.

Proof. *(i)* Let $K \in \mathcal{K}(\mathbb{R}^d)$. By definition

$$T(K) = \inf\{T(G) : \ G \in \mathcal{G}(\mathbb{R}^d), \ G \supset K\},$$

for $\epsilon > 0$ there exists $G_0 \in \mathcal{G}(\mathbb{R}^d), G_0 \supset K$ such that

$$T(G_0) < T(K) + \epsilon.$$

We put

$$f_K(x) = \begin{cases} 1 & if\ x \in K \\ 0 & if\ x \notin K \end{cases}$$

and suppose that $f_{K,G_0} : \mathbb{R}^d \to [0,1]$ is a continuos function such that

$$f_{K,G_0}(x) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if } x \notin G_0. \end{cases}$$

Thus function f_{K,G_0} exists by the Urysohn-Tietze Theorem. Clearly $f_{K,G_0} \in C_0^+(\mathbb{R}^d)$. We have

$$T_n(K) = \int f_K dT_n \le \int f_{K,G_0} dT_n \to \int f_{K,G_0} dT \le \int_0^1 T(G_0)dt = T(G_0).$$

Hence

$$\limsup_n T_n(K) \le \lim_{n\to\infty} \int f_{K,G_0} dT_n = \int f_{K,G_0} dT \le T(G_0) < T(K) + \epsilon.$$

Since ϵ was arbitrary, we get

$$\limsup_n T_n(K) \leq T(K).$$

(*ii*) Suppose that T_n converges weakly to T and $G \in \mathcal{G}(\mathbb{R}^d)$. By definition

$$T(G) = \sup\{T(K) : \ K \in \mathcal{K}(\mathbb{R}^d), \ K \subset G\},$$

for $\epsilon > 0$ there exists $K_0 \in \mathcal{K}(\mathbb{R}^d), K_0 \subset G$ such that

$$T(K_0) > T(G) - \epsilon.$$

Let $f_{K_0,G} : \mathbb{R}^d \to [0,1]$ be a continuos function such that

$$f_{K_0,G}(x) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if } x \notin G_0. \end{cases}$$

Then $f_{K_0,G} \in C_0^+(\mathbb{R})$. We have

$$T_n(G) \geq \int f_{K_0,G} dT_n \to \int f_{F_0,G} dT.$$

Hence

$$\liminf_n T_n(G) \geq \lim_{n\to\infty} \int f_{K_0,G} dT_n = \int f_{K_0,G} dT \geq T(K_0) > T(G) - \epsilon.$$

Since ϵ was arbitrary, we get

$$\liminf_n T_n(G) \geq T(G).$$

(*iii*) Suppose A is a bounded T-continuity in $\mathbb{R}^d$. Let int A and $\overline{A}$ denote the interior of A and the closure of A, respectively. Note that $\overline{A}$ is compact. By Lemma 3.1. we have

$$T(\overline{A}) = T(\text{int } A \bigcup \partial A) = T(\text{int } A). \tag{3.1}$$

If T_n converges weakly to T, then (i) and (ii) hold. Since T_n and T are increasing set functions, we have

$$\begin{aligned} T(\overline{A}) &\geq \limsup_n T_n(\overline{A}) \geq \limsup_n T_n(A) \\ &\geq \liminf_n T_n(A) \geq \liminf_n T_n(\text{int } A) \\ &\geq T(\text{int } A), \end{aligned}$$

which, together with (3.1) implies

$$\lim_{n\to\infty} T_n(A) = T(A).$$

3.3.Problem. *Does each of three statements (i), (ii), and (iii) of Theorem 3.2 imply the weak convergence of capacities?*

4. Weak convergence of capacities and of their associated measures

Let E be a locally compact separable Hausdorff space. Let $\mathcal{K}, \mathcal{F}, \mathcal{G}$ denote the classeses of all compact, closed and open subsets of E, respectively.

Following Matheron [9], we topologize $\mathcal{F}$ as follows. For every $A \subset E$ we denote

$$\mathcal{F}_A = \{F \in \mathcal{F} : F \bigcap A \neq \emptyset\} \text{ and } \mathcal{F}^A = \{F \in \mathcal{F} : F \bigcap A = \emptyset\}.$$

The *miss-and-hit* topology on $\mathcal{F}(\mathbb{R}^d)$ is the topology with the base

$$\{\mathcal{F}^K_{G_1 \dots G_n} : K \in \mathcal{K} \text{ and } G_1, \dots, G_n \in \mathcal{G}\},$$

where

$$\mathcal{F}^K_{G_1 \dots G_n} = \mathcal{F}^K \bigcap \mathcal{F}_{G_1} \bigcap \dots \bigcap \mathcal{F}_{G_n}, n \in \mathbb{N}.$$

It was shown in [9] that for a locally compact separable Hausdorff space E, the space $\mathcal{F}$ with the miss-and-hit topology is compact, Hausdorff and separable. Let $\mathcal{B}(\mathcal{F})$ denote the family of all Borel sets of $\mathcal{F}$ in the miss-and-hit topology.

By Choquet Theorem, there exists a bijection between probability measures P on $\mathcal{B}(\mathcal{F})$ and probability capacities $T : \mathcal{K} \to [0, +\infty)$ satisfying the equality

$$P(\mathcal{F}_K) = T(K) \text{ for every } K \in \mathcal{K}.$$

In this case we say that the probability measure P is *associated with* the probability capacity T.

Now we take E to be a compact set K in $\mathbb{R}^d$. For given probability capacities $T, T_n, n = 1, 2, \dots$, let $P, P_n, n = 1, 2, \dots$ denote their associated probability measures. To prove the main result in this section, we need the following Lemma.

4.1. Lemma. *Let T, S be the probability capacities in $\tilde{\mathcal{C}}$ such that*

$$\int f dT = \int f dS \text{ for all } f \in C_0^+(\mathbb{R}^d).$$

Then $S = T$ on $\mathcal{B}(\mathbb{R}^d)$, i.e., $T(A) = S(A)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$.

From this follows that $\{T_n\} \subset \tilde{\mathcal{C}}$ can not converge weakly to two different limits at the same time.

Proof. Because of the equality

$$T(A) = \sup\{T(K) : K \in \mathcal{K}(\mathbb{R}^d), K \subset A\}$$

for any Borel set $A \in \mathcal{B}(\mathbb{R}^d)$, it suffices to show that

$$T(K) = S(K) \text{ for all } K \in \mathcal{K}(\mathbb{R}^d).$$

If it is not the case, then $T(K) \neq S(K)$ for some compact set $K \in \mathcal{K}(\mathbb{R}^d)$. Assume that $T(K) - S(K) = \delta > 0$. Then there exists an open set $G \supset K$ such that

$$S(G) < S(K) + \frac{\delta}{2}. \tag{4.1}$$

Let $f_{K,G}(x) \in C_0^+(\mathbb{R}^d)$ such that

$$f_{K,G}(x) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if } x \notin G. \end{cases}$$

Then by (4.1) we have

$$\begin{aligned} \int f_{K,G} dS &\leq S(G) < S(K)) + \frac{\delta}{2} \\ &= T(K) - \frac{\delta}{2} < T(K) \\ &\leq \int f_{K,G} dT. \end{aligned}$$

This contradiction completes the proof. □

In this section we prove the folollowing theorem.

4.2. Theorem. *The sequence of probability capacities T_n converges weakly to T if and only if the associated sequence of probability measures P_n converges weakly to P.*

Proof. Assume that P_n converges weakly to P. Let $f \in C_0^+(\mathbb{R}^d)$. We put

$$\alpha = \sup\{f(x) : x \in \mathbb{R}^d\} < \infty.$$

Note that if $t_1, t_2 \in (0, \alpha)$ are distinct, then $\partial\mathcal{F}_{\{f \geq t_1\}}$ and $\partial\mathcal{F}_{\{f \geq t_2\}}$ are disjoint, and hence at most countably many of them can have positve P-measure. Therefore, $\mathcal{F}_{\{f \geq t\}}$ is a P-continuity set almost everywhere on $(0.\alpha)$, i.e.,

$$\int_0^\alpha P[\partial(\mathcal{F}_{\{f \geq t\}})] dt = 0.$$

Hence, by Theorem 2.2 and Lebesgue's bounded convergence Theorem, we have

$$\begin{aligned} \lim_{n\to\infty} \left(\int f dT_n - \int f dT\right) &= \lim_{n\to\infty} \int_0^\alpha [T_n(\{f \geq t\}) - T(\{f \geq t\})] dt \\ &= \lim_{n\to\infty} \int_0^\alpha [P_n(\mathcal{F}_{\{f \geq t\}}) - P(\mathcal{F}_{\{f \geq t\}})] dt \\ &= \int_0^\alpha \lim_{n\to\infty} [P_n(\mathcal{F}_{\{f \geq t\}}) - P(\mathcal{F}_{\{f \geq t\}})] dt = 0 \end{aligned}$$

Conversely, assume that T_n converges weakly to T. We will show that P_n converges weakly to P. Since the space $\mathcal{F}(K)$ with the miss-and-hit topology is a compact metric space, by Theorem 6.4 [13] for each subsequence $\{P_{n'}\}$ there exists a futher subsequence $\{P_{n''}\}$ such that

$$\int f dP_{n''} \to \int f dP'$$

for every $f \in C_b(\mathbb{R}^d)$ and for some probability measure P' on $\mathcal{F}(K)$, where $C_b(\mathbb{R}^d)$ denotes the space of all bounded continuos real valued functions on $\mathbb{R}^d$. In particular

$$\int f dP_{n''} \to \int f dP' \text{ for every} f \in C_0^+(\mathbb{R}^d),$$

i.e.,$P_{n''}$ converges weakly to P'. Hence $T_{n''}$ converges weakly to T'. By Lemma 4.1 we have

$$T = T'.$$

Therefore, by Choquet Theorem we get

$$P = P'.$$

That means

$$\int f dP_{n''} \to \int f dP \qquad \text{for} \quad \text{every} \quad f \in C_0^+(\mathbb{R}^d).$$

Consequently, P_n converges weakly to P by Proposition 2.1. □

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