SOME REMARKS AND EXAMPLES ON DOMAIN OF SUMS OF SERIES IN BANACH SPACES

Dau The Cap, Le Xuan Truong
Ho chi Minh city University of Pedagogy

Abstract. In this note we present some remarks on the domains of sums of series and some examples on domains of sums of series in infinite - dimensional Banach spaces.

1. Introduction

Suppose that $\sum_{k=1}^{\infty} x_k$ is a series in Banach space X. The domain of sums of series is defined to be the set DS($\sum x_k$) of x such that the $\sum_{k=1}^{\infty} x_{\sigma(k)}$ converges to x for some permutation $\sigma : \mathbb{N} \to \mathbb{N}$. Let $\Re_m$ denote the set of all series in X with domain of sums is an m-dimensional linear set; Ω_m is the set of all series in $\Re_m$ with terms in some finite demensional linear subspace of X. and

$$\Re = \bigcup_{m=0}^{\infty} \Re_m, \quad \Omega = \bigcup_{m=0}^{\infty} \Omega_m.$$

Suppose that $\sum_{k=1}^{\infty} x_k$ is a convergence series with sum s in a Banach space X. A linear functional $f \in X^*$ is called a convergent functional for the series $\sum_{k=1}^{\infty} x_k$ if $\sum_{k=1}^{\infty} |f(x_k)| < \infty$. The set of all convergent functionals of series $\sum_{k=1}^{\infty} x_k$ is a linear subspace $\Gamma \subset X^*$. By $\Gamma_0 \subset X$ we denote the annihilator of the convergence set, i.e

$$\Gamma_0 = \{\, x \in X : f(x) = 0, \forall f \in \Gamma\}.$$

A series $\sum_{k=1}^{\infty} x_k$ is said to be *unconditionally convergent* if it converges for any rearrangement for its terms.

A series is said to be *conditionally convergent* if it converges, but not unconditionally.

We recall the following well-known result.

Riemann theorem. *If $\sum_{k=1}^{\infty} x_k$ is conditionally convergent series of real numbers, then DS($\sum x_k$) = $\mathbb{R}$.*

The following is a perfect extension of Riemann theorem over the finite-dimensional spaces.

Steinitz theorem [2]. *Let $\sum_{k=1}^{\infty} x_k$ be a convergent series in an m-dimensional space X and let $\sum_{k=1}^{\infty} x_k = s$. Then the domain of sums of the series $\sum_{k=1}^{\infty} x_k$ is the linear set $s + \Gamma_0$, where Γ_0 is the annihilator of the set of convergent functionals.*

From Steinitz theorem, it follows that any series in finite-dimensional space having domain of sums is a linear set. It is known that in any finite -dimensional Banach space

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-$\TeX$

there exist series with domains of sums that is not linear set. In this note we construct some series in infinite-dimensionaly spaces with domains of sums which are linear sets.

2. Some properties the set $\Re$

Theorem. *Let $\sum_{k=1}^{\infty} x_k$ be a unconditionally convergent series in a Banach space X then*

$$\sum_{k=1}^{\infty} x_k \in \Re_0 \quad and \quad \Gamma_0 = \{\ 0\}.$$

Proof. By theorem 1.3.1 in [2],$\sum_{k=1}^{\infty} x_k \in \Re_0$. For every $f \in X^*$, the series of real numbers$\sum_{k=1}^{\infty} f(x_k)$ is unconditionally convergent, hence $\sum_{k=1}^{\infty} |f(x_k)| < \infty$. Therefore X = Γ. On the other hand any $x \in X, x \neq 0$, from Hahn-Banach's theorem there exists $f \in X^*$ such that $f(x) \neq 0$. Hence, $\Gamma_0 = \{\ 0\}$. The theorem is proved.

The contrary sentence of theorem 1 is false if X is infinite-dimensional. Indeed, in the Hilbert space l_2 for the canomical orthonormal basis $\{\ e_k\}\ _{k=1}^{\infty}$ consider series having terms e_1, $\frac{(-1)^k}{2^{n-1}} e_n$ for each $n \geq 2$, $\ 2^{n-1} - 1 \geq k \geq 0$:

$$e_1 + \frac{1}{2}e_2 - \frac{1}{2}e_2 + \frac{1}{4}e_3 - \frac{1}{4}e_3 + \frac{1}{4}e_3 - \frac{1}{4}e_3 + ...$$

This is a convergent series, and its sum is equal to e_1. The series constructed does not converge unconditionally, but domain of sums of this consists of only the one point e_1.

Theorem. *If $\sum_{k=1}^{\infty} x_k \in \Omega$ then $\Gamma_0 = DS(\sum_{k=1}^{\infty} x_k) - x_0$ for any $x_0 \in DS(\sum x_k)$.*

Proof. We denote E is the linear subspaces of X generated by terms of series $\sum_{k=1}^{\infty} x_k$. Since $dimE < \infty$ so E is a closed subset of X. Therefore DS($\sum x_k$) $\subset E$. Denote Γ^E is the set of all convergence functionals of $\sum_{k=1}^{\infty} x_k$ in E,

$$\Gamma_0^E = \{\ x \in E \quad : \quad f(x) = 0, \quad \forall f \in \Gamma^E\}$$

From Steinitz theorem we have

$$\Gamma_0^E = DS(\sum x_k) - x_0, \quad x_0 \in DS(\sum x_k).$$

On the other hand, from Hahn-Banach's theorem, any $f \in \Gamma^E$ there exists $f_0 \in \Gamma$ such that $f_{o|E} = f$. Hence $\Gamma_0^E = \Gamma_0$. The theorem is proved.

3. Series with domains of sums are finite dimensional linear set in Banach spaces

Theorem. *Let X be a Banach space and S be an m-dimensionally linear set of X. Then there exists $\sum_{k=1}^{\infty} x_k \in \Re_m$ such that $DS(\sum x_k) = S$.*

Proof. Given fix $x_0 \in S$. Then $S - x_0$ is a linear subspace of X. Let $\{\ e_i\}\ _{i=1}^{m}$ be a basis of $S - x_0$. For any $i \in \{\ 1, 2, ..., m\}$ and $j \in \mathbb{N}$, put $t_{ij} = \frac{(-1)^j}{j} e_i$. Obviously, for any $k \in \mathbb{N}$ there exists a unique pair (i,j) for $i \in \{\ 1, 2, ..., m\}$ and $j \in \mathbb{N}$ such that

$$k = m(j-1) + i \quad (*)$$

It follows that the element $x_k = t_{ij} + \frac{1}{2^k}x_0 \in X$ is unique. We shall prove that $\mathrm{DS}(\sum x_k) = \mathrm{S}$.

Given $x \in S$. Since $x - x_0 \in S - x_0$, there exist numbers $\lambda_1, \lambda_2, ...\lambda_m$ such that $x - x_0 = \sum_{i=1}^{m} \lambda_i e_i$. For any $i \in \{ 1, 2, ..., m\}$ choose permutation $\sigma_i : \mathbb{N} \to \mathbb{N}$ such that $\sum_{j=1}^{\infty} t_{i\sigma_i(j)} = \lambda_i e_i$. By relation (*), put

$$\sigma(k) = m[\sigma_i(j) - 1] + i, \quad k \in \mathbb{N}$$

we obtain the permutation $\sigma : \mathbb{N} \to \mathbb{N}$. We have $\sum_{k=1}^{\infty} x_{\sigma(k)} = x$. Indeed, for any $\varepsilon > 0$ and $i \in \{ 1, 2, ..., m\}$ there exist n_0 and $n_i \in \mathbb{N}$ such that

$$||\sum_{j=1}^{n} t_{i\sigma_i(j)} - \lambda_i e_i|| < \frac{\varepsilon}{2m}, \quad \forall m \geq n_i,$$

$$||\sum_{k=1}^{n} \frac{1}{2^{\sigma(k)}} x_0 - x_0|| < \frac{\varepsilon}{2}, \quad \forall n \geq n_0.$$

Take $N = max\{ mn_1, mn_2, \ldots, mn_m, n_0\}$. For each $n \geq N$ we have numbers $k_i \geq n_i$ such that

$$\begin{aligned}
||\sum_{k=1}^{n} x_{\sigma(k)} - x|| &= ||\sum_{i=1}^{m}(\sum_{j=1}^{k_i} t_{i\sigma_i(j)}) + \sum_{k=1}^{n} \frac{1}{2^{\sigma(k)}} x_0 - x|| \\
&= ||\sum_{i=1}^{m}(\sum_{j=1}^{k_i} t_{i\sigma_i(j)} - \lambda_i e_i) + \sum_{k=1}^{n} \frac{1}{2^{\sigma(k)}} x_0|| \\
&\leqslant \sum_{i=1}^{m} ||\sum_{j=1}^{k_i} t_{i\sigma_i(j)} - \lambda_i e_i|| + ||\sum_{k=1}^{n} \frac{1}{2^{\sigma(k)}} x_0|| \\
&< m.\frac{\varepsilon}{2m} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore $\sum_{k=1}^{\infty} x_{\sigma(k)} = x$. The theorem is proved.

3. Series with domains of sums are linear subsets in the hilbert spaces

Theorem. *Let X be a separable Hilbert space and A be a closed subspace of X. Then there exists a convergent series* $\sum_{k=1}^{\infty} x_k$ *with* $DS(\sum x_k) = A$.

Proof. From theorem 3 we can assume $dimA = \infty$. Given a orthonormal basis $\{ e_n\}_{n=1}^{\infty}$ such that $\{ e_{n_j}\}_{j=1}^{\infty}$ is an orthonormal basis of A. For any $i, j \in \mathbb{N}$, we put

$$t_{ij} = \frac{(-1)^j}{j} e_{n_i}$$

We shall prove that series $\sum_{k=1}^{\infty} x_k$ have terms t_{ij}, $i, j \in \mathbb{N}$ satisfying

$$DS(\sum x_k) = A.$$

For any $x \in A$, we write $x = \sum_{i=1}^{\infty} \lambda_i e_{n_i}$. Take permutaion $\sigma_i : \mathbb{N} \to \mathbb{N}$ such that

$$\sum_{j=1}^{\infty} t_{i\sigma_i(j)} = \sum_{j=1}^{\infty} \frac{(-1)^{\sigma_i(j)}}{\sigma_i(j)} e_{n_i} = \lambda_i e_{n_i}.$$

Let $\varphi : \mathbb{N}^2 \to \mathbb{N}$ denote the map defined by

$$\varphi(i,j) = \frac{1}{2}(i+j-1)(i+j-2) + i.$$

Then φ is a surjection. Let $x_{\sigma(k)} = t_{\varphi^{-1}(k)}$. It is easy to show that $\sum_{k=1}^{\infty} x_{\sigma(k)} = x$. Since A is closed of X, $DS(\sum x_k) = A$. The Theorem is proved.

References

1. J. Bone, A.Defant, The Levy-Steinitz rearrangement theory for duals of metrizable spaces, *Israel Journal of mathematics*, **117**(2000), 131-156.
2. V.M.Kadets, M.I.Kadets, Rearrangements of series in Banach spaces, Translations of mathematical Monographs, *American Mathematical Society* Vol **86**(1991).
3. V.M.Kadets, M.I.Kadets,*Series in Banach spaces*, Operator theory, Advances an Applications 94, Birkhauser (1997).
4. L.A.Lyusternik, V.I.Sobolev,*Elements of functional analysis*, Hindustan, Delhi, Wiley, NewYork(1974).
5. M.I.Ostrovskii, Domains of sums of conditionally convergent series in Banach spaces, *Teor Fucktsional. Anal. i Prilozhen.vyp.* **46**(1986), 77-85.