INDEX OF NILPOTENCY, MULTIPLICITIES AND BLOWING-UP BY SPECIALIZATIONS

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Abstract. Let $x = (x_1, \dots, x_n)$ be indeterminates and u an indeterminate, which is considered as a parameter. The specialization of an ideal I of $R = k(u)[x]$ with respect to the substitution $u \to \alpha$ was defined as the ideal I_α, which is generated by the set $\{f(\alpha, x) \mid f(u, x) \in I \cap k[u, x]\}$. It was showed that the ideal I_α inherits most of basic properties of I and it was used to prove many important results in algebra and in algebraic geometry. In this paper we want to prove the preservation of index of nilpotency, multiplicities, Forster-Swan number and to study some properties of blowing-up of factor ring by specializations.

Introduction

The purpose of this paper is to prove the preservation of index of nilpotency, multiplicities, Forster-Swan number and to study some properties of blowing-up of factor ring by specializations. Here we want to fix some of notations that will be used throughout this paper. The groundfield k is always assumed to be infinite and perfect. A field extension of k will be denoted by K. Aggregates such as $x_1, \dots, x_n$ or $\alpha_1, \dots, \alpha_m$, where $\forall \alpha_i \in K$, will often be written x or α. Accordingly, the ring or field extensions $k[x_1, \dots, x_n]$ or $k(\alpha_1, \dots, \alpha_m)$ will be written by $k[x]$ or $k(\alpha)$, with evident variants of these designations. Denote the ring $k(u)[x]$ by R, and the ring $k(\alpha)[x]$ by R_α.

The theory of specialization of ideals was introduced by W. Krull [3]. Krull defined the specialization of an ideal I of $R = k(u)[x]$ with respect to the substitution $u \to \alpha$ as the ideal I_α, which is generated by the set $\{f(\alpha, x) \mid f(u, x) \in I \cap k[u, x]\}$, where $\alpha \in k^m$. The ideal I_α inherits most of basic properties of I.

The notion of specialization of an ideal also played an important role in the study of normal varieties by A. Seidenberg who proved that almost all hyperplane sections of a normal variety are normal again under certain conditions [10]. Following these works, N. V. Trung studied the preservation of properties from the quotient ring R/I to R_α/I_α [12]. Using specializations of finitely generated free modules and homomorphisms between them we defined in [4] the specialization of finitely generated module. We showed that basic properties and operations on modules are preserved by specializations. In [5],[6] we followed the same approach to introduce and to study specializations of finitely generated modules over a local ring and of graded modules. Present paper is a continuation of an earlier one [5]. The aim of this paper is to study the preservation of some invariants of modules and some properties of blowing-up of factor ring through a specialization.

The paper is divided in three sections. Section 1 is devoted to the discussion of index of nilpotency and multiplicities of modules. We shall see that these invariants are unchanged through a specialization. In Section 2, we shall see that the preservation of rank and Forster-Swan number of modules, and of dimension of symmetric algebras by specializations are proved. While some properties of the blowing-up of factor ring by specializations will be studied in section 3.

In this paper, we shall say that a property holds for almost all α if it holds for all α except perhaps those lying on a proper algebraic subvariety of K^m. For unexplained notations we refer the reader to [2], [11]. For convenience we will *often skip the phrase "for almost all α" when we are working with specializations.*

1. Index of nilpotency and multiplicities of modules by specialization

We start by recalling the definition of a specialization of a finitely generated S-module. Let P be an arbitrary prime ideal of R. The first obstacle in defining the specialization of R_P is that the specialization P_α of P in R_α need not to be a prime ideal. The natural candidate for the specialization of R_P is the local ring $(R_\alpha)_\wp$, where $\wp$ is an arbitrary associated prime ideal of P_α. Such a local ring was already considered with regard to specializations of points in [12]. For short we put $S = R_P$ and $S_\alpha = (R_\alpha)_\wp$. The notion S_α is not unique. However, all local rings S_α have the same dimension as S. Here the maximal ideals PS of S and $\wp S_\alpha$ of S_α will be denoted by $\mathfrak{m}$ and $\mathfrak{n}$, respectively.

Now we will recall a specialization of any element of the ring S. Note that an arbitrary element $f \in R$ may be written in the form

$$f = \frac{p(u,x)}{q(u)},\ p(u,x) \in k[u,x],\ q(u) \in k[u] \setminus \{0\}.$$

In [6], for any α such that $q(\alpha) \neq 0$ we define $f_\alpha := p(\alpha,x)/q(\alpha)$. For every element

$$a = \frac{f}{g} \in S,\ f,g \in R,\ g \notin P,$$

there is $g_\alpha \notin P_\alpha$ and we define $a_\alpha := f_\alpha/g_\alpha$. Then a_α is uniquely determined and belongs to S_α for almost all α, (see [5]).

Let M be a finitely generated S-module. Assume that the following exact sequence

$$S^q \xrightarrow{\phi} S^p \longrightarrow M \longrightarrow 0,$$

where the matrix of ϕ is $A = (a_{ij})$ with all $a_{ij} \in S$, is a finite free presentation of M. We now obtain a homomorphism $\phi_\alpha : S^q_\alpha \longrightarrow S^p_\alpha$ given by the matrix $((a_{ij})_\alpha)$. As the definition of a specialization of module, we obtain a finite free presentation

$$S^q_\alpha \xrightarrow{\phi_\alpha} S^p_\alpha \longrightarrow M_\alpha \longrightarrow 0,$$

where $M_\alpha = \operatorname{Coker} \phi_\alpha$, see [5]. The S_α-module M_α is called a specialization of M. In particular, we note that $\mathfrak{m}_\alpha = \mathfrak{n}$. Denote the dimension of M by d. The following result shows that the dimension of module is preserved by a specialization.

Lemma 1.1. *[5, Theorem 2.6] Let M be a finitely generated S-module. Then, for almost all α, we have*
(i) $\operatorname{Ann} M_\alpha = (\operatorname{Ann} M)_\alpha$,
(ii) $\dim M_\alpha = \dim M$.

With the following lemma we see that specializations commute with the Tor and the Ext functors.

Lemma 1.2. *[5, proposition 3.3] Let L and M be finitely generated S-modules. For almost all α,*

$$\operatorname{Ext}^i_{S_\alpha}(L_\alpha, M_\alpha) \cong \operatorname{Ext}^i_S(L, M)_\alpha,\ i \geq 0,$$
$$\operatorname{Tor}_i^{S_\alpha}(L_\alpha, M_\alpha) \cong \operatorname{Tor}_i^S(L, M)_\alpha,\ i \geq 0.$$

Now we shall prove the preservation of the index of nilpotency of an ideal through a specialization. We first recall the definition of index of nilpotency. The *index of nilpotency* of an ideal I of a commutative ring, which is denoted by $\operatorname{nil}(I)$, is the smallest integer s such that $(\sqrt{I})^s \subset I$. It is to provide a path to the estimation of the exponent in the Nullstellensatz. This is the classical notion of multiplicity of a primary ideal regarding the Loewy length of a module.

Let M be a finitely generated S-module. When M has a finite length, the *Loewy length* of M is the smallest integer s such that $\mathfrak{m}^s M = 0$. It is denoted by $\ell\ell(M)$. Let $\mathfrak{q}$ be a prime ideal of S. Set

$$\Gamma_\mathfrak{q}(M_\mathfrak{q}) := \bigcup_{m \geq 0} (0_{M_\mathfrak{q}} : \mathfrak{q}^m).$$

The *Loewy multiplicity* of M at $\mathfrak{q}$ is $\operatorname{Lmult}_M(\mathfrak{q}) = \ell\ell(\Gamma_\mathfrak{q}(M_\mathfrak{q}))$. These notions were used in [13] in order to estimate the index of nilpotency for $\mathfrak{m}$-primary ideals. When I is a P-primary ideal, $\operatorname{nil}(I) = \operatorname{nil}(I_P) = \ell\ell(R_P/I_P)$ and it will be called the *nilpotency degree* of S/IS.

Suppose that $I \subseteq P$ is a prime ideal of R such that $\mathfrak{q} = IS$. By [5, Corollary 1.6], we set $\mathfrak{q}_\alpha = I_\alpha S_\alpha$. It is known that I_α is a radical ideal of R_α. Thus, $\mathfrak{q}_\alpha$ is a radical ideal. Suppose that Q is an arbitrary associated prime ideal of I_α such that $Q \subseteq \wp$. Then we have the following theorem.

Theorem 1.3. *Let M be a finitely generated S-module of finite length and $\mathfrak{q}$ a prime ideal of S. Then, for almost all α, we have*

$$\ell\ell(M_\alpha) = \ell\ell(M),$$
$$\operatorname{Lmult}_{M_\alpha}(QS_\alpha) = \operatorname{Lmult}_M(\mathfrak{q}).$$

Proof. We first prove the equality $\ell\ell(M_\alpha) = \ell\ell(M)$. By [5, Proposition 2.8], M_α is a S_α-module of finite length. Put $\ell\ell(M) = s$. Then $\mathfrak{m}^s M = 0$, and $\mathfrak{m}^{s-1} M \neq 0$. By Lemma 1.1, we get $\mathfrak{m}_\alpha^s M_\alpha = 0$. Since $\mathfrak{m}^{s-1} M \neq 0$, $\mathfrak{a} = \operatorname{Ann}(\mathfrak{m}^{s-1} M) \neq S$. There is an ideal I of R such that $I \subseteq P$ and $\mathfrak{a} = IS$. By [5, Corollary 1.6], we get

$$\mathfrak{a}_\alpha = I_\alpha S_\alpha \subseteq P_\alpha S_\alpha = \mathfrak{m}_\alpha.$$

Hence

$$\mathrm{Ann}(\mathfrak{m}_\alpha^{s-1}M_\alpha) = (\mathrm{Ann}(\mathfrak{m}^{s-1}M))_\alpha \subseteq \mathfrak{m}_\alpha = \mathfrak{n}.$$

Then $\mathfrak{m}_\alpha^{s-1}M_\alpha \neq 0$. Hence $\ell\ell(M_\alpha) = s = \ell\ell(M)$ for almost all α.
Next we will prove that $\mathrm{Lmult}_{M_\alpha}(QS_\alpha) = \mathrm{Lmult}_M(\mathfrak{q})$. Suppose that

$$S^h \xrightarrow{\phi} S^t \longrightarrow M \longrightarrow 0$$

is a finite free presentation of M. By localization at the prime ideal $\mathfrak{q}$, the exact sequence

$$S_\mathfrak{q}^h \xrightarrow{\phi_\mathfrak{q}} S_\mathfrak{q}^t \longrightarrow M_\mathfrak{q} \longrightarrow 0$$

is a finite free presentation of the $S_\mathfrak{q}$-module $M_\mathfrak{q}$. Since $S_\mathfrak{q} = (R_P)_{IR_P} = R_I$, we may consider $M_\mathfrak{q}$ as a finitely generated R_I-module with a finite free presentation

$$R_I^h \longrightarrow R_I^t \longrightarrow M_\mathfrak{q} \longrightarrow 0.$$

For the specialization of R_I is the local ring $(R_\alpha)_Q$. By the definition of specialization of $M_\mathfrak{q}$, we have the finite free presentation of $(M_\mathfrak{q})_\alpha$:

$$(R_\alpha)_Q^h \longrightarrow (R_\alpha)_Q^t \longrightarrow (M_\mathfrak{q})_\alpha \longrightarrow 0.$$

Since the sequence $S_\alpha^h \longrightarrow S_\alpha^t \longrightarrow M_\alpha \longrightarrow 0$ is exact, we get the exact sequence

$$(S_\alpha)_{QS_\alpha}^h \longrightarrow (S_\alpha)_{QS_\alpha}^t \longrightarrow (M_\alpha)_{QS_\alpha} \longrightarrow 0.$$

Since $(S_\alpha)_{QS_\alpha} = ((R_\alpha)_\wp)_{Q(R_\alpha)_\wp} = (R_\alpha)_Q$, we obtain the following commutative diagram

$$\begin{array}{ccccccc}
(R_\alpha)_Q^h & \longrightarrow & (R_\alpha)_Q^t & \longrightarrow & (M_\mathfrak{q})_\alpha & \longrightarrow & 0 \\
\downarrow \mathrm{id} & & \downarrow \mathrm{id} & & & & \\
(R_\alpha)_Q^h & \longrightarrow & (R_\alpha)_Q^t & \longrightarrow & (M_\alpha)_{QS_\alpha} & \longrightarrow & 0
\end{array}$$

and then there is an induced isomorphism $(M_\mathfrak{q})_\alpha \longrightarrow (M_\alpha)_{QS_\alpha}$ such that the diagram

$$\begin{array}{ccccccc}
(R_\alpha)_Q^h & \longrightarrow & (R_\alpha)_Q^t & \longrightarrow & (M_\mathfrak{q})_\alpha & \longrightarrow & 0 \\
\downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow & & \\
(R_\alpha)_Q^h & \longrightarrow & (R_\alpha)_Q^t & \longrightarrow & (M_\alpha)_{QS_\alpha} & \longrightarrow & 0
\end{array}$$

is commutative. Hence, we can identify the module $(M_\mathfrak{q})_\alpha$ with the module $(M_\alpha)_{QS_\alpha}$. Because $M_\mathfrak{q}$ is a Noertherian module, there is an integer t such that

$$\Gamma_\mathfrak{q}(M_\mathfrak{q}) = 0_{M_\mathfrak{q}} : \mathfrak{q}^t \text{ and } 0_{M_\mathfrak{q}} : \mathfrak{q}^t = 0_{M_\mathfrak{q}} : \mathfrak{q}^m \text{ for all } m \geq t.$$

Since $\Gamma_\mathfrak{q}(M_\mathfrak{q}) = \Gamma_{\mathfrak{q}S_\mathfrak{q}}(M_\mathfrak{q})$, we have $\Gamma_\mathfrak{q}(M_\mathfrak{q}) = \Gamma_\mathfrak{q}(M)_\mathfrak{q}$. Since

$$\begin{aligned}
(0_{M_\mathfrak{q}} : \mathfrak{q}^m)_\alpha &= ((0_M : \mathfrak{q}^m)_\mathfrak{q})_\alpha \\
&= ((0_M : \mathfrak{q}^m)_\alpha)_{QS_\alpha} \text{ by above proof} \\
&\cong (0_{M_\alpha} : \mathfrak{q}_\alpha^m)_{QS_\alpha} \text{ by [5, Lemma 2.5]},
\end{aligned}$$

we get $\Gamma_{\mathfrak{q}}(M_{\mathfrak{q}})_\alpha = (0_{M_{\mathfrak{q}}} : \mathfrak{q}^t)_\alpha \cong (0_{M_\alpha} : \mathfrak{q}_\alpha^t)_{QS_\alpha}$, and $(0_{M_\alpha} : \mathfrak{q}_\alpha^t)_{QS_\alpha} = (0_{M_\alpha} : \mathfrak{q}_\alpha^m)_{QS_\alpha}$ for all $m \geq t$. Hence $\ell\ell(\Gamma_{\mathfrak{q}}(M_{\mathfrak{q}})_\alpha) = \ell\ell(\Gamma_{\mathfrak{q}_\alpha}((M_{\mathfrak{q}})_\alpha))$, and it follows from the fact that $(M_{\mathfrak{q}})_\alpha = (M_\alpha)_{QS_\alpha}$ that

$$\ell\ell(\Gamma_{\mathfrak{q}}(M_{\mathfrak{q}})) = \ell\ell(\Gamma_{\mathfrak{q}_\alpha}((M_{\mathfrak{q}})_\alpha)) = \ell\ell(\Gamma_{\mathfrak{q}_\alpha}((M_\alpha)_{QS_\alpha}))$$

and therefore, $\mathrm{Lmult}_{M_\alpha}(QS_\alpha) = \mathrm{Lmult}_M(\mathfrak{q})$ for almost all α.

Using this result, we will show that the index of nilpotency of a P-primary ideal is preserved by specialization.

Corollary 1.4. *Let I be a P-primary ideal of R. Then, for almost all α, we have*

$$\mathrm{nil}(IS) = \mathrm{nil}(I_\alpha S_\alpha).$$

Proof. By Theorem 1.3, we get $\ell\ell((R/I)_P) = \ell\ell(((R/I)_P)_\alpha) = \ell\ell((R_\alpha/I_\alpha)_{\mathfrak{p}})$. Hence $\mathrm{nil}(I_P) = \mathrm{nil}((I_\alpha)_{\mathfrak{p}})$ for almost all α.

Lemma 1.5. *Let I be an ideal of R. Then, for almost all α, we have*

$$(\sqrt{I})_\alpha = \sqrt{I_\alpha}.$$

Proof. Assume that $I = I_1 \cap \cdots \cap I_s$ is a minimal primary decomposition of I, where $P_i = \sqrt{I_i}, i = 1, \ldots, s$. By [3, Satz 3], $I_\alpha = (I_1)_\alpha \cap \cdots \cap (I_s)_\alpha$. Suppose that

$$(I_i)_\alpha = H_{i1} \cap \cdots \cap H_{it_i}, i = 1, \ldots, s$$

is a minimal primary decomposition. Let

$$P_{il} = \sqrt{H_{il}}, i = 1, \ldots, s, l = 1, \ldots, t_i.$$

Since k is perfect, $(P_i)_\alpha = P_{i1} \cap \ldots \cap P_{it_i}, i = 1, \ldots, s$. Hence

$$(\sqrt{I})_\alpha = \bigcap_{i=1}^{s}(P_{i1} \cap \ldots \cap P_{it_i}).$$

By [3, Satz 8 and Satz 9], the minimal primary decomposition of I_α is

$$I_\alpha = \bigcap_{i=1}^{s}(H_{i1} \cap \cdots \cap H_{it_i}).$$

Hence $\sqrt{I_\alpha} = \bigcap_{i=1}^{s}(P_{i1} \cap \ldots \cap P_{it_i}) = (\sqrt{I})_\alpha$.

Proposition 1.6. *Let $\mathfrak{a}$ be an ideal of S. Then, for almost all α, we have*

$$\mathrm{nil}(\mathfrak{a}) = \mathrm{nil}(\mathfrak{a}_\alpha).$$

Proof. Assume that $\mathfrak{a} = IS$, where $I \subseteq P$ is an ideal of R. There is $\mathfrak{a}_\alpha = I_\alpha S_\alpha$. Since $\sqrt{\mathfrak{a}} = \sqrt{IS}$ and $\sqrt{\mathfrak{a}_\alpha} = \sqrt{I_\alpha S_\alpha}$, therefore

$$\begin{aligned}(\sqrt{\mathfrak{a}})_\alpha &= (\sqrt{IS})_\alpha = (\sqrt{I})_\alpha S_\alpha \text{ by [5, Corollary 1.6]} \\ &= \sqrt{I_\alpha S_\alpha} \text{ by Lemma 1.5} = \sqrt{\mathfrak{a}_\alpha}.\end{aligned}$$

Put $\mathrm{nil}(\mathfrak{a}) = s$. Since $(\sqrt{\mathfrak{a}})^s \subseteq \mathfrak{a}$, by an easy computation it follows that

$$(\sqrt{\mathfrak{a}_\alpha})^s = (\sqrt{\mathfrak{a}})^s_\alpha \subseteq \mathfrak{a}_\alpha.$$

Hence $\mathrm{nil}(\mathfrak{a}_\alpha) \leq s$. We need still show that $\mathrm{nil}(\mathfrak{a}_\alpha) \geq s$. Indeed, by definition of the index of nilpotency, there is $(\sqrt{\mathfrak{a}})^{s-1} \nsubseteq \mathfrak{a}$. Then $(\sqrt{\mathfrak{a}_\alpha})^{s-1} \nsubseteq \mathfrak{a}_\alpha$. This proves $\mathrm{nil}(\mathfrak{a}_\alpha) \geq s$, and therefore $\mathrm{nil}(\mathfrak{a}_\alpha) = s = \mathrm{nil}(\mathfrak{a})$ for almost all α.

We continue with the preservation of multiplicites of modules through a specialization. Before stating the theorem we first recall some notations and definitions.

Let M be a finitely generated S-module. A sequence of elements $\underline{a} = a_1, \ldots, a_p$ in $\mathfrak{m}$ is a *multiplicity system* of M if $\lambda(M/(\underline{a})M)$ is finite, where λ is the length of modules. Let $\mathfrak{q} = (a_1, \ldots, a_p)S$ an ideal of S. Now we will show that the multiplicity of M with respect to $\mathfrak{q}$ is unchanged through a specialization. By [11, Lemma 1.4] it is sufficient to treat the case in which $a_1, \ldots, a_p$ is a system of parameters for M with $p = \dim M = d$. We know that the *multiplicity* of M with respect to $\mathfrak{q}$ is defined as the number

$$e(\mathfrak{q}, M) = \lim_{h \to \infty} \frac{\lambda(M/\mathfrak{q}^h M).d!}{h^d}.$$

Note that $e(\mathfrak{q}, S)$ will be denoted by $e(\mathfrak{q})$. The following theorem shows that the multiplicity of M with respect to $\mathfrak{q}$ is unchanged by a specialization.

Theorem 1.7. *[7, Theorem 1.6] Let M be a finitely generated S-module of dimension d and let*

$$\mathfrak{q} = (a_1, \ldots, a_d)S$$

be a parameter ideal on M. Then, for almost α, we have

$$e(\mathfrak{q}_\alpha; M_\alpha) = e(\mathfrak{q}; M),$$

where $e(\mathfrak{q}_\alpha; M_\alpha)$ and $e(\mathfrak{q}; M)$ are the multiplicities of M_α and M with respect to $\mathfrak{q}_\alpha$ and $\mathfrak{q}$ respectively.

Let $H^i_{\mathfrak{m}}(M)$, $i \in \mathbb{Z}$, denote the ith local cohomology modules of M. In [1], the ith *pseudo-support* $\mathrm{Psupp}^i(M)$ and ith *pseudo-dimension* $\mathrm{psd}^i(M)$ of M are defined as follows

$$\begin{aligned}\mathrm{Psupp}^i(M) &= \{\mathfrak{p} \in \mathrm{Spec}(S) \mid H^{i-\dim S/\mathfrak{p}}_{\mathfrak{p}S_\mathfrak{p}}(M_\mathfrak{p}) \neq 0\}, \\ \mathrm{psd}^i(M) &= \sup\{\dim S/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Psupp}^i(M)\}.\end{aligned}$$

Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of S. The multiplicity of local cohomology modules is defined by

$$e'(\mathfrak{q}, H^i_{\mathfrak{m}}(M)) = \sum_{\substack{\mathfrak{p} \in \operatorname{Psupp}^i(M) \\ \dim S/\mathfrak{p} = \operatorname{psd}^i(M)}} \ell_{S_{\mathfrak{p}}}(H^{i-\dim S/\mathfrak{p}}_{\mathfrak{p}S_{\mathfrak{p}}}(M_{\mathfrak{p}}))e(\mathfrak{q}, S/\mathfrak{p}).$$

To prove the preservation of multiplicity of local cohomology modules by a specialization we will study the specializations of modules

$$K^i_M = \operatorname{Hom}_S(H^i_{\mathfrak{m}}(M), E(S/\mathfrak{m})) \text{ for } i = 0, \dots, d,$$

where $E(S/\mathfrak{m})$ denotes the injective hull of the residue field $S/\mathfrak{m}$. The modules K^i_M are again the finitely generated S-modules. Clearly $K^i_M = 0$ for all $i < 0$ and $i > \dim M$. We often write K_M instead of K^d_M. This module K_M is called the *canonical module* of M, see [2], [8].

Lemma 1.8. *Let M be a finitely generated S-module. Then, for almost α, we have*

$$(K^i_M)_\alpha \cong K^i_{M_\alpha}, i = 0, \dots, d.$$

Proof. By Lemma 1.1, $\dim M_\alpha = \dim M = d$. Let $r = \dim S = \dim S_\alpha$. Since S and S_α are regular rings, they are Gorenstein rings. Therefore $K^i_M = \operatorname{Ext}^{r-i}_S(M, S)$ and $K^i_{M_\alpha} = \operatorname{Ext}^{r-i}_{S_\alpha}(M_\alpha, S_\alpha)$ by Matlis duality. Since $\operatorname{Ext}^{r-i}_S(M, S)_\alpha \cong \operatorname{Ext}^{r-i}_{S_\alpha}(M_\alpha, S_\alpha)$, $i = 0, \dots, d$, by Lemma 1.2, we have $(K^i_M)_\alpha \cong K^i_{M_\alpha}$ for almost α.

Proposition 1.9. *Let M be a finitely generated S-module and $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal of S. Then, for almost α, we have*
(i) *an ideal $\mathfrak{b}$ of S with $\operatorname{Psupp}^i(M) = V(\mathfrak{b})$ such that $\operatorname{Psupp}^i(M_\alpha) = V(\mathfrak{b}_\alpha)$,*
(ii) $e'(\mathfrak{q}_\alpha, H^i_{\mathfrak{n}}(M_\alpha)) = e'(\mathfrak{q}, H^i_{\mathfrak{m}}(M))$,
(iii) $\operatorname{psd}^i(M_\alpha) = \operatorname{psd}^i(M)$.

Proof. (i) By [1, Proposition 1.2 (iii)], $\operatorname{Psupp}^i(M) = \operatorname{Supp}(K^i_M) = V(\mathfrak{b})$, where $\mathfrak{b} = \operatorname{Ann} K^i_M$. Since $(K^i_M)_\alpha \cong K^i_{M_\alpha}$ by Lemma 1.8 and $\mathfrak{b}_\alpha = (\operatorname{Ann} K^i_M)_\alpha = \operatorname{Ann} K^i_{M_\alpha}$ by Lemma 1.1, $\operatorname{Psupp}^i(M_\alpha) = \operatorname{Supp}(K^i_{M_\alpha}) = V(\mathfrak{b}_\alpha)$.
(ii) By [1, Proposition 1.2 (ii)], we know that

$$e'(\mathfrak{q}, H^i_{\mathfrak{m}}(M)) = e(\mathfrak{q}, K^i_M) \text{ and } e'(\mathfrak{q}_\alpha, H^i_{\mathfrak{n}}(M_\alpha)) = e(\mathfrak{q}_\alpha, K^i_{M_\alpha}).$$

Since $(K^i_M)_\alpha \cong K^i_{M_\alpha}$ by Lemma 1.8 and $e(\mathfrak{q}_\alpha, (K^i_M)_\alpha) = e(\mathfrak{q}, K^i_M)$ by Theorem 1.7, we have $e(\mathfrak{q}_\alpha, K^i_{M_\alpha}) = e(\mathfrak{q}, K^i_M)$. Hence $e'(\mathfrak{q}_\alpha, H^i_{\mathfrak{n}}(M_\alpha)) = e'(\mathfrak{q}, H^i_{\mathfrak{m}}(M))$.
(iii) By [1, Theorem 2.4], we know that $\operatorname{psd}^i(M)$ is equal to the dimension of $H^i_{\mathfrak{m}}(M)$. Since $\operatorname{Ann} H^i_{\mathfrak{n}}(M_\alpha) = \operatorname{Ann} H^i_{\mathfrak{m}}(M)_\alpha$ by [5, Lemma 3.5], upon simple computation, we get $\dim H^i_{\mathfrak{n}}(M_\alpha) = \dim H^i_{\mathfrak{m}}(M)$ from [5, Lemm 1.1]. Hence $\operatorname{psd}^i(M_\alpha) = \operatorname{psd}^i(M)$ for almost α.

2. Preservation of Forster-Swan number by specializations.

In this section, the problem of concern is the preservation of Forster-Swan number and of dimension of symmetric algebra of a module by specializations. First we want to study the preservation of Forster-Swan number and of dimension of symmetric algebra of a module by specializations.

Let M be a finitely generated S-module. In this section we want to study the preservation of Forster-Swan number and dimension of symmetric algebra of M by specializations. Recall that the *Forster-Swan number* of M is defined to be

$$b(M) = \sup_{\mathfrak{q}\in \operatorname{Spec}(S)} \{\dim S/\mathfrak{q} + \mu(M_{\mathfrak{q}})\},$$

where $\mu(M_{\mathfrak{q}})$ is the minimal number of generators of $M_{\mathfrak{q}}$. Note that $b(M)$ is always bounded below by $b_0(M) = \dim S + \operatorname{rank} M$. The *tensor algebra* $T_S(M)$ of M is defined the graded, non-commutative algebra

$$T_S(M) = S \oplus M \oplus (M \otimes_S M) \oplus \cdots,$$

where the product of $y_1 \otimes \cdots \otimes y_s$ and $z_1 \otimes \cdots \otimes z_h$ is $y_1 \otimes \cdots \otimes y_s \otimes z_1 \otimes \cdots \otimes z_h$. The *symmetric algebra* $S(M)$ of M obtained from $T_S(M)$ by factoring out the two-sided ideal generated by relations $y \otimes z - z \otimes y$ for all $y, z \in M$.

Denote the total ring of fractions of S by H. Then M has *rank* r, which is denoted by $\operatorname{rank} M$, if $M \otimes H$ is a free H-module of rank r. If $\psi : M \longrightarrow N$ is a homomorphism of S-modulesm then the number $\operatorname{rank}(\psi)$ is defined to be the number $\operatorname{rank}\operatorname{Im}(\psi)$. We shall see that these numbers are unchanged through a specialization.

Lemma 2.1. *Let M be a finitely generated S-module with a finite free presentation $S^q \xrightarrow{\phi} S^p \longrightarrow M \longrightarrow 0$. Then $\operatorname{rank} M_\alpha = \operatorname{rank} M$ and $\operatorname{rank}(\phi_\alpha) = \operatorname{rank}(\phi)$ for almost all α.*

Proof. Suppose that $\mathbf{F}_\bullet : 0 \longrightarrow S^{p_\ell} \xrightarrow{\phi_\ell} S^{p_{\ell-1}} \longrightarrow \cdots \longrightarrow S^{p_1} \xrightarrow{\phi} S^{p_0} \longrightarrow M \longrightarrow 0$ is a finite free resolution of M. Then, the following complex

$$(\mathbf{F}_\bullet)_\alpha : 0 \longrightarrow S_\alpha^{p_\ell} \xrightarrow{(\phi_\ell)_\alpha} S_\alpha^{p_{\ell-1}} \longrightarrow \cdots \longrightarrow S_\alpha^{p_1} \xrightarrow{\phi_\alpha} S_\alpha^{p_0} \longrightarrow M_\alpha \longrightarrow 0$$

is a finite free resolution of M_α by [5, Theorem 3.1]. By [2, Corollary 1.4.5], we get

$$\operatorname{rank} M_\alpha = \sum_{j=0}^{\ell} (-1)^j p_j = \operatorname{rank} M.$$

By [2, Proposition 1.4.3] we get $\operatorname{rank}(\phi_\alpha) = \operatorname{rank}(\phi)$ for almost all α.

Let $I_t(\phi)$ be the ideal generated by $t \times t$ minors of the matrix A of ϕ. For an integer h, the module M satisfies the condition F_h if

$$\operatorname{ht} I_t(\phi) \geq \operatorname{rank}(\phi) - t + 1 + h,\ 1 \leq t \leq \operatorname{rank}(\phi),$$

see [14, Definition 1.3.1]. We have the following lemma.

Lemma 2.2. *Let M be a finitely generated S-module. If M satisfies the condition F_h, then M_α also satisfies the condition F_h for almost all α.*

Proof. Let $a_1, \dots, a_s$ be the $d \times d$-minors of $A, d = \operatorname{rank} \phi$. We may write $a_j = f_j/g_j$ with $f_j, g_j \in R$, $g_j \notin P$. Then

$$I((\phi)_\alpha) = ((a_1)_\alpha, \dots, (a_s)_\alpha) = ((f_1)_\alpha, \dots, (f_s)_\alpha)S_\alpha.$$

Let I be the ideal of R generated by the elements $f_1, \dots, f_s$. Then $I_\alpha = ((f_1)_\alpha, \dots, (f_s)_\alpha)$ by [4, Corollary 3.3]. Therefore, $I((\phi_j)_\alpha) = I_\alpha S_\alpha$. Since S and S_α are regular rings and $\dim S = \dim S_\alpha$, $\operatorname{ht} I_t(\phi_\alpha) = \operatorname{ht} I_t(\phi)$ for all t by Lemma 1.1. Since $\operatorname{rank}(\phi_\alpha) = \operatorname{rank}(\phi)$ by Lemma 2.1, we get

$$\operatorname{ht} I_t(\phi_\alpha) \geq \operatorname{rank}(\phi_\alpha) - t + 1 + h,\ 1 \leq t \leq \operatorname{rank}(\phi_\alpha).$$

Hence, M_α satisfies the condition F_h for almost all α.

We are now in a position to prove the following proposition about the preservation of the Forster-Swan number and the dimension of $S(M)$ of M through a specialization.

Define the following integral valued function on $[1, \operatorname{rank}(\phi)]$:

$$d(t) = \begin{cases} \operatorname{rank}(\phi) - t + 1 - \operatorname{ht} I_t(\phi) \text{ if } F_0 \text{ is violated at } t, \\ 0 \quad \text{otherwise.} \end{cases}$$

If we put $d(M) = \sup_t\{d(t)\}$, we have the dimension formula: $b(M) = b_0(M) + d(M)$ by [14, Theorem 1.3.5].

Theorem 2.3. *Let M be a finitely generated S-module. Then, for almost all α, we have*
(i) $b(M_\alpha) = b(M)$,
(ii) $\dim S(M_\alpha) = \dim S(M)$.

Proof. (i) Since $\dim S_\alpha = \dim S$ and $\operatorname{rank}(M_\alpha) = \operatorname{rank}(M)$ by Lemma 2.1, $b_0(M_\alpha) = b_0(M)$. From the definition of $d(t)$ it follows that $d(M_\alpha) = d(M)$. Since $b(M_\alpha) = b_0(M_\alpha) + d(M_\alpha)$ and $b(M) = b_0(M) + d(M)$, we have $b(M_\alpha) = b(M)$.
(ii) By the theorem of Hunecke-Rossi, $\dim S(M_\alpha) = b(M_\alpha)$ and $\dim S(M) = b(M)$, see [14, Theorem 1.2.1]. By (i), $\dim S(M_\alpha) = \dim S(M)$ for almost all α.

3. Blowing-up of factor ring by specialization

The present section will be devoted to study the preservation of Blowing-up of factor ring by specializations. We now assume that $\mathfrak{J}$ is an ideal of S. We consider the local ring $A = S/\mathfrak{J}$ and $\mathfrak{b} = \mathfrak{m}/\mathfrak{J}$. A and $\mathfrak{b}$ may be considered as finitely generated S-modules, we have $A_\alpha = S_\alpha/\mathfrak{J}_\alpha$, and $\mathfrak{b}_\alpha = \mathfrak{m}_\alpha/\mathfrak{J}_\alpha$. Each ideal of A may be written in the form $\mathfrak{q}A$, where $\mathfrak{q}$ is an ideal of S. If L' is an A-module, then we can regard L' as a S-module and there is a S-isomorphism $\phi : H^i_{\mathfrak{q}A}(L') \xrightarrow{\simeq} H^i_{\mathfrak{q}}(L')$, so that, it doesn't matter whether we calculate these local cohomology modules over S or A.

Let $\mathfrak{q}$ be a parameter ideal of A. Denote by $\operatorname{Proj} R_{\mathfrak{q}}(A)$ the set of homogeneous prime ideals of $R_{\mathfrak{q}}(A)$ not containing the ideal

$$R_{\mathfrak{q}}(A)_+ = \bigoplus_{h>0} \mathfrak{q}^h.$$

$\operatorname{Proj} R_{\mathfrak{q}}(A)$ is called the *blowing-up* of A with centre of $\mathfrak{q}$. We shall begin with recalling the following lemma.

Lemma 3.1. [5, Theorem 3.7] *If $\mathfrak{q}$ is a standard parameter ideal of A, then $\mathfrak{q}_\alpha$ is again a standard parameter ideal of A_α for almost α.*

Lemma 3.2. *We have $H^0_{\mathfrak{b}}(A)_\alpha \cong H^0_{\mathfrak{b}_\alpha}(A_\alpha)$ for almost all α.*

Proof. There is an integer t such that $H^0_{\mathfrak{b}}(A) \cong 0_A : \mathfrak{b}^t$ and $0_A : \mathfrak{b}^t = 0_A : \mathfrak{b}^m$ for all $m \geq t$. By [5, Lemma 2.5], $(0_A : \mathfrak{b}^m)_\alpha \cong 0_{A_\alpha} : \mathfrak{b}^m_\alpha$. Therefore, $H^0_{\mathfrak{b}}(A)_\alpha \cong (0_A : \mathfrak{b}^t)_\alpha = 0_{A_\alpha} : \mathfrak{b}^t_\alpha$ and $0_{A_\alpha} : \mathfrak{b}^t_\alpha = 0_{A_\alpha} : \mathfrak{b}^m_\alpha$ for $m \geq t$. Hence $H^0_{\mathfrak{b}}(A)_\alpha \cong H^0_{\mathfrak{b}_\alpha}(A_\alpha)$ for almost all α.

Proposition 3.3. *Let $\mathfrak{q} = (a_1, \ldots, a_d)A$ be a parameter ideal of A, where $d = \dim A$. If $R_{(a_1^{t_1}, \ldots, a_d^{t_d})}(A)$ is a Cohen-Macaulay scheme for all d-tupels $(t_1, \ldots, t_d)$ of positive integers, then, for almost α, $R_{((a_1)_\alpha^{t_1}, \ldots, (a_d)_\alpha^{t_d})}(A_\alpha)$ is again a Cohen-Macaulay scheme for all d-tupels $(t_1, \ldots, t_d)$ of positive integers.*

Proof. By [9, Theorem 5.1], it is well-known that $R_{(a_1^{t_1}, \ldots, a_d^{t_d})}(A)$ is a Cohen-Macaulay scheme for all d-tupels $(t_1, \ldots, t_d)$ of positive integers if and only if the images of $(a_1, \ldots, a_d)$ form a standard system of parameters in $A/H^0_{\mathfrak{b}}(A)$. By Lemma 1.1, $\dim A_\alpha = d$. Since

$$\dim A_\alpha/((a_1)_\alpha, \ldots, (a_d)_\alpha)A_\alpha = \dim(A/(a_1, \ldots, a_d)A)_\alpha = \dim A/(a_1, \ldots, a_d)A = 0$$

by Lemma 1.1, $(a_1)_\alpha, \ldots, (a_d)_\alpha$ is a system of parameters on A_α. Using the S-isomorphism ϕ and by Lemma 3.2 and [5, Lemma 2.3], we have

$$A_\alpha/H^0_{\mathfrak{b}_\alpha}(A_\alpha) \cong (A/H^0_{\mathfrak{b}}(A))_\alpha.$$

Therefore, the images of $((a_1)_\alpha, \ldots, (a_d)_\alpha)$ form a standard system of parameters in $A_\alpha/H^0_{\mathfrak{b}_\alpha}(A_\alpha)$ by Lemma 3.1. Hence $R_{((a_1)_\alpha^{t_1}, \ldots, (a_d)_\alpha^{t_d})}(A_\alpha)$ is again a Cohen-Macaulay scheme for all d-tupels $(t_1, \ldots, t_d)$ of positive integers.

Proposition 3.4. *If $R_{\mathfrak{a}}(A)$, $(resp. G_{\mathfrak{a}}(A))$, is a locally Cohen-Macaulay module for all parameter ideals $\mathfrak{a}$ of A, then, for almost α, $R_{\mathfrak{c}}(A_\alpha)$, $(resp. G_{\mathfrak{c}}(A_\alpha))$, is a locally Cohen-Macaulay module for all parameter ideals $\mathfrak{c}$ of A_α.*

Proof. By [11, Chapter IV Theorem 3.2], we only need to show that if $A/H^0_{\mathfrak{b}}(A)$ is a Buchsbaum module, then $A_\alpha/H^0_{\mathfrak{b}_\alpha}(A_\alpha)$ is also a Buchsbaum module for almost all α. The proof for this claim follows immediately by virtue of [5, Corollary 3.8] and Lemma 3.2.

From Proposition 3.4 we obtain a following corollary.

Corollary 3.5. *If* $\operatorname{Proj} R_{\mathfrak{a}}(A)$ *is a Cohen-Macaulay scheme for all parameter ideals* $\mathfrak{a}$ *of* A, *then* $\operatorname{Proj} R_{\mathfrak{c}}(A_\alpha)$ *is a Cohen-Macaulay scheme for all parameter ideals* $\mathfrak{c}$ *of* A_α.

Proof. By [9, Corollary 6.1], $\operatorname{Proj} R_{\mathfrak{a}}(A)$ is a Cohen-Macaulay scheme for all parameter ideals $\mathfrak{a}$ of A if and only if $A/H^0_{\mathfrak{b}}(A)$ is a Buchsbaum ring. We consider the ring $A/H^0_{\mathfrak{b}}(A)$ as a finitely generated S-module. This module is a Buchsbaum S-module. Using the S-isomorphism ϕ and by Lemma 3.2, we obtain

$$A_\alpha/H^0_{\mathfrak{b}_\alpha}(A_\alpha) \cong (A/H^0_{\mathfrak{b}}(A))_\alpha.$$

Since $A/H^0_{\mathfrak{b}}(A)$ is a Buchsbaum S-module, $(A/H^0_{\mathfrak{b}}(A))_\alpha$ is a Buchsbaum S_α-module by [5, Corollary 3.8]. Thus, $A_\alpha/H^0_{\mathfrak{b}_\alpha}(A_\alpha)$ is a Buchsbaum S_α-module. Hence $\operatorname{Proj} R_{\mathfrak{c}}(A_\alpha)$ is a Cohen-Macaulay scheme for all parameter ideals $\mathfrak{c}$ of A_α by [9, Corollary 6.1].

Proposition 3.6. *If there is a parameter ideal* $\mathfrak{a}$ *such that* $\operatorname{Proj} G_{\mathfrak{a}}(A)$ *is a Cohen-Macaulay scheme, then there is a parameter ideal* $\mathfrak{c}$ *such that* $\operatorname{Proj} G_{\mathfrak{c}}(A_\alpha)$ *is a Cohen-Macaulay scheme.*

Proof. Put $d = \dim A$. Then $\dim A_\alpha = \dim A = d$. By [9, Proposition 4.2], it is well-known that there is a parameter ideal $\mathfrak{a}$ such that $\operatorname{Proj} G_{\mathfrak{a}}(A)$ is a Cohen-Macaulay scheme if and only if $H^i_{\mathfrak{b}}(A)$ is of finite length for all $i < \dim A$. We consider $H^i_{\mathfrak{b}}(A)$ as finitely generated S-modules. Using the S-isomorphism ϕ and Lemma 3.2, and by [5, Theorem 3.6], we obtain

$$\lambda(H^i_{\mathfrak{b}_\alpha}(A_\alpha)) = \lambda(H^i_{\mathfrak{b}}(A)) < \infty \text{ for all } i < d.$$

By [9, Proposition 4.2], there is a parameter ideal $\mathfrak{c}$ such that $\operatorname{Proj} G_{\mathfrak{c}}(A_\alpha)$ is a Cohen-Macaulay scheme.

Proposition 3.7. *If* $\operatorname{Proj} R_{\mathfrak{a}}(A)$ *is a Gorenstein scheme for every parameter ideal* $\mathfrak{a}$ *of* A, *then* $\operatorname{Proj} R_{\mathfrak{c}}(A_\alpha)$ *is a Gorenstein scheme for every parameter ideal* $\mathfrak{c}$ *of* A_α.

Proof. By [9, Theorem 5.4], $\operatorname{Proj} R_{\mathfrak{a}}(A)$ is a Gorenstein scheme for every parameter ideal $\mathfrak{a}$ of A if and only if $A/H^0_{\mathfrak{b}}(A)$ is a Gorenstein ring. We consider the ring $A/H^0_{\mathfrak{b}}(A)$ as a finitely generated S-module. This module is Gorenstein. By using the S-isomorphism ϕ and by Lemma 3.2, we obtain

$$A_\alpha/H^0_{\mathfrak{b}_\alpha}(A_\alpha) \cong (A/H^0_{\mathfrak{b}}(A))_\alpha.$$

Since $A/H^0_{\mathfrak{b}}(A)$ is Gorenstein, $(A/H^0_{\mathfrak{b}}(A))_\alpha$ is Gorenstein by [5, Proposition 4.2]. Thus, $A_\alpha/H^0_{\mathfrak{b}_\alpha}(A_\alpha)$ is Gorenstein. Hence $\operatorname{Proj} R_{\mathfrak{c}}(A_\alpha)$ is a Gorenstein scheme for evey parameter ideal $\mathfrak{c}$ of A_α by [9, Theorem 5.4].

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