THE CHOQUET WEAK TOPOLOGY ON THE SPACE OF PROBABILITY CAPACITIES IN $\mathbb{R}^d$*

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Abstract. It is shown that the space of probability capacities in $\mathbb{R}^d$ equipped with the Choquet weak topology is separable and metrizable, and contains $\mathbb{R}^d$ topologically.

1. Introduction

Let $\mathcal{K}(\mathbb{R}^d)$, $\mathcal{F}(\mathbb{R}^d)$, $\mathcal{G}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$ denote the family of all *compact sets, closed sets, open sets* and *Borel sets* in $\mathbb{R}^d$, respectively. By a *capacity* in $\mathbb{R}^d$ we mean a set function $T:\ \mathbb{R}^d \to \mathbb{R}^+ = [0, +\infty)$ satisfying the following conditions:

(i) $T(\emptyset) = 0$;

(ii) T is alternating of infinite order: For any Borel sets A_i, $i = 1, 2, \dots, n$; $n \geq 2$, we have

$$T(\bigcap_{i=1}^{n} A_i) \leq \sum_{I \in \mathcal{I}(n)} (-1)^{\# I+1} T(\bigcup_{i \in I} A_i),$$

where $\mathcal{I}(n) = \{I \subset \{1, \dots, n\},\ I \neq \emptyset\}$ and $\# I$ denotes the cardinality of I;

(iii) $T(A) = \sup\{T(C):\ C \in \mathcal{K}(\mathbb{R}^d),\ C \subset A\}$ for any Borel set $A \in \mathcal{B}(\mathbb{R}^d)$;

(iv) $T(C) = \inf\{T(G):\ G \in \mathcal{G}(\mathbb{R}^d),\ C \subset G\}$, for any compact set $C \in \mathcal{K}(\mathbb{R}^d)$.

It was shown in [8] that a capacity in $\mathbb{R}^d$ is, in fact, a generalization of a measure in $\mathbb{R}^d$. Clearly any capacity is a non-decreasing set function on Borel sets of $\mathbb{R}^d$.

By *support* of a capacity T we mean the smallest closed set $S \subset \mathbb{R}^d$ such that $T(\mathbb{R}^d \backslash S) = 0$. The support of a capacity T is denoted by supp T. We say that T is a *probability capacity* in $\mathbb{R}^d$ if T has a compact support and $T(\text{supp } T) = 1$. By $\tilde{\mathcal{C}}$ we denote the family of all probability capacities in $\mathbb{R}^d$. Let $C_b(\mathbb{R}^d)$ be the space of bounded continuous real-valued functions on $\mathbb{R}^d$.

Let T be a capacity in $\mathbb{R}^d$. Then for any bounded continuous real-valued function $f : \mathbb{R}^d \to \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R}^d)$, the function $f_A : \mathbb{R} \to \mathbb{R}$ defined by

$$f_A(t) = T(\{x \in A:\ f(x) \geq t\}) \text{ for } t \in \mathbb{R}$$

is a non-increasing function in t. Therefore we can define the *Choquet integral* $\int_A f dT$ of f with respect to T by

$$\begin{aligned}\int_A f dT &= \int_0^{\infty} f_A(t)dt + \int_{-\infty}^{0} (f_A(t) - T(A))dt \\ &= \int_0^{\infty} T(\{x \in A:\ f(x) \geq t\})dt + \int_{-\infty}^{0} [T(\{x \in A:\ f(x) \geq t\}) - T(A)]dt.\end{aligned}$$

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If $\int_A f dT < \infty$, we say that f is *integrable*. In particular for $A = \mathbb{R}^d$ we write

$$\int_{\mathbb{R}^d} f dT = \int f dT.$$

The present paper aims at studying the Choquet weak topology on the space of probability capacities in $\mathbb{R}^d$ and gives some properties of Choquet weak topology on this space. The main result of this paper is to show that the space of probability capacities equipped with the Choquet weak topology is separable and metrizable, and contains $\mathbb{R}^d$ topologically. This result is similar to one for the weak topology on the space of probability capacities in $\mathbb{R}^d$ ([8]).

2. The Choquet Weak Topology on the Space of Probabilitiy Capacities in $\mathbb{R}^d$

Let $\mathbf{B}$ be a family of sets of the form

$$\mathbf{B} = \{\mathcal{U}(T; f_1, \dots, f_k; \epsilon) : T \in \tilde{\mathcal{C}}, \ \epsilon > 0, \ f_i \in C_b(\mathbb{R}^d), \ i = 1, \dots, k\}, \tag{2.1}$$

where

$$\begin{aligned}\mathcal{U}(T; f_1, \dots, f_k; \epsilon) &= \{S \in \tilde{\mathcal{C}} : |\int f_i dT - \int f_i dS| < \epsilon, \ i = 1, \dots, k\} \\ &= \bigcap_{i=1}^{k} \mathcal{U}(T; f_i; \epsilon).\end{aligned} \tag{2.2}$$

Obviously the family $\mathbf{B}$ is a base of a topology on $\tilde{\mathcal{C}}$. This topology is called the *Choquet weak topology* on $\tilde{\mathcal{C}}$ and denoted by τ_C.

For any point $x \in \mathbb{R}^d$ let $T_x = \delta_x$ be the set function defined by

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{2.3}$$

for $A \in \mathcal{B}(\mathbb{R}^d)$. Clearly that T_x is a probability capacity in $\mathbb{R}^d$. Then we have the following lemma.

2.1. Lemma. *For $x \in \mathbb{R}^d$ we take $T_x = \delta_x$ with δ_x defined by (2.3). Then for any $f \in C_b(\mathbb{R}^d)$ we have*

$$\int f dT_x = f(x) \text{ for every } x \in \mathbb{R}^d. \tag{2.4}$$

Proof. If $f(x) \geq 0$ then

$$\begin{aligned}\int f dT_x &= \int_0^{+\infty} T_x(\{f \geq t\}) dt + \int_{-\infty}^{0} [T_x(\{f \geq t\}) - 1] dt \\ &= \int_0^{f(x)} T_x(\{f \geq t\}) dt = \int_0^{f(x)} dt = f(x).\end{aligned}$$

If $f(x) < 0$,

$$\int f dT_x = \int_0^{+\infty} T_x(\{f \geq t\})dt + \int_{-\infty}^{0} [T_x(\{f \geq t\}) - 1]dt$$
$$= \int_{f(x)}^{0} [T_x(\{f \geq t\}) - 1]dt = \int_{f(x)}^{0} -dt = f(x).$$

Let $\tilde{C}$ denotes the space of all probability capacities in $\mathbb{R}^d$ equipped with the Choquet weak topology . In this section we show that

2.2. Theorem. *$\tilde{C}$ is separable and metrizable.*

The Theorem will be proved by Propositions 2.3 and 2.7 below.

2.3. Proposition. *$\tilde{C}$ is a regular space.*

Proof. Assume that $\mathcal{A}$ is a closed set in $\tilde{C}$, $T \in \tilde{C}$ and $T \notin \mathcal{A}$. We will show that there are neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of T and $\mathcal{A}$ respectively, such that $\mathcal{U} \cap \mathcal{V} = \emptyset$. Since $\mathcal{A}$ is closed and $T \notin \mathcal{A}$, there exist $f_i \in C_b(\mathbb{R}^d)$, $i = 1, \ldots, k$, and $\epsilon > 0$ such that

$$\mathcal{U}(T; f_1, \ldots, f_k; \epsilon) \cap \mathcal{A} = \emptyset. \tag{2.5}$$

For each $i = 1, \ldots, k$, we define

$$\mathcal{A}_i = \{S \in \mathcal{A} : S \notin \mathcal{U}(T; f_i; \epsilon)\}. \tag{2.6}$$

From (2.5) and (2.6) we get

$$\mathcal{A} = \bigcup_{i=1}^{k} \mathcal{A}_i. \tag{2.7}$$

We put

$$\mathcal{V}_i = \bigcup_{S \in \mathcal{A}_i} \mathcal{U}(S; f_i; \epsilon/3). \tag{2.8}$$

For any $S' \in \mathcal{V}_i$ and for any $T' \in \mathcal{U}(T; f_i; \epsilon/3)$ from (2.6) and (2.8) we have

$$|\int f_i dT' - \int f_i dS'| \geq |\int f_i dT - \int f_i dS| - |\int f_i dT - \int f_i dT'|$$
$$- |\int f_i dS - \int f_i dS'|$$
$$> \epsilon - \frac{\epsilon}{3} - \frac{\epsilon}{3} = \frac{\epsilon}{3} > 0$$

for some $S \in \mathcal{A}_i$. That means

$$\mathcal{U}(T; f_i; \epsilon/3) \cap \mathcal{V}_i = \emptyset \text{ for } i = 1, \ldots, k.$$

Hence

$$\mathcal{U}(T; f_1, \ldots, f_k; \epsilon/3) \cap (\bigcup_{i=1}^{k} \mathcal{V}_i) = \emptyset.$$

Consequently, take $\mathcal{U} = \mathcal{U}(T; f_1, \dots, f_k; \epsilon/3)$ and $\mathcal{V} = \bigcup_{i=1}^{k} \mathcal{V}_i$ to complete the proof of the proposition. □

To prove Prosition 2.7 we need lemmas below.

2.4. Lemma. *Let $T \in \tilde{\mathcal{C}}$ and a be a positive real-valued number. Then we have*

$$\int (f+a)dT = \int f dT + a, \text{ for every } f \in C_b(\mathbb{R}^d).$$

Proof. For $f \in C_b(\mathbb{R}^d)$ let

$$\alpha = \sup\{f(x) : x \in \mathbb{R}^d\},\ \beta = \inf\{f(x) : x \in \mathbb{R}^d\}.$$

We consider the following cases.

Case1. $\beta \geq 0$,

$$\begin{aligned}\int (f+a)dT &= \int_0^{\alpha+a} T(\{f+a \geq t\})dt \\ &= \int_a^{\alpha+a} T(\{f+a \geq t\})dt + \int_0^a T(\{f+a \geq t\})dt \\ &= \int_0^{\alpha} T(\{f \geq t\})dt + a = \int f dT + a.\end{aligned}$$

Case 2. $\beta < 0$ and $\alpha \geq 0$.

a) If $a + \beta \geq 0$,

$$\begin{aligned}\int (f+a)dT &= \int_0^{\alpha+a} T(\{f+a \geq t\})dt = \int_{-a}^{\alpha} T(\{f \geq t\})dt \\ &= \int_0^{\alpha} T(\{f \geq t\}dt + \int_{-a}^{0} T(\{f \geq t\})dt \\ &= \int_0^{\alpha} T(\{f \geq t\})dt + \int_{\beta}^{0} T(\{f \geq t\})dt + \int_{-a}^{\beta} T(\{f \geq t\})dt \\ &= \int_0^{\alpha} T(\{f \geq t\})dt + \int_{\beta}^{0} [T(\{f \geq t\}) - 1]dt - \beta + \beta + a = \int f dT + a.\end{aligned}$$

b) If $a + \beta < 0$, then

$$\begin{aligned}\int (f+a)dT &= \int_0^{\alpha+a} T(\{f+a \geq t\})dt + \int_{\beta+a}^{0} [T(\{f+a \geq t\}) - 1]dt \\ &= \int_{-a}^{\alpha} T(\{f \geq t\})dt + \int_{\beta}^{-a} [T(\{f \geq t\}) - 1]dt \\ &= \int_0^{\alpha} T(\{f \geq t\})dt + \int_{-a}^{0} T(\{f \geq t\})dt + \int_{\beta}^{0} [T(\{f \geq t\}) - 1]dt \\ &- \int_{-a}^{0} [T(\{f \geq t\}) - 1]dt = \int f dT + \int_{-a}^{0} dt = \int f dT + a.\end{aligned}$$

Case 3. $\alpha < 0$ and $\beta < 0$.

a) If $a + \alpha \le 0$, then

$$\begin{aligned}\int (f+a)dT &= \int_{\beta+a}^{0} [T(\{f+a \ge t\}) - 1]dt = \int_{\beta}^{-a} [T(\{f \ge t\}) - 1]dt \\ &= \int_{\beta}^{0} [T(\{f \ge t\}) - 1]dt - \int_{-a}^{0} [T(\{f \ge t\}) - 1]dt \\ &= \int f dT + \int_{-a}^{0} dt = \int f dT + a.\end{aligned}$$

b) The case $a + \alpha > 0$ and $a + \beta < 0$ is similar to the case 2.b).

c) If $a + \alpha > 0$ and $a + \beta \ge 0$, then

$$\begin{aligned}\int (f+a)dT &= \int_{0}^{a+\alpha} T(\{f+a \ge t\})dt = \int_{-a}^{\alpha} T(\{f \ge t\})dt \\ &= \int_{\beta}^{\alpha} T(\{f \ge t\})dt + \int_{-a}^{\beta} T(\{f \ge t\})dt \\ &= \int_{\beta}^{0} [T(\{f \ge t\}) - 1]dt - \int_{0}^{\alpha} T(\{f \ge t\})dt - \beta + a + \beta = \int f dT + a.\end{aligned}$$

The lemma is proved. □

Let T be a probability capacity in $\mathbb{R}^d$ and let $f : \mathbb{R}^d \to \mathbb{R}^+$ be a bounded continuous function and let $A = \{x_i : i = 1, \dots, k\} \subset \operatorname{supp} f$ be a finite set in supp f which we may assume that

$$0 < f(x_1) < f(x_2) < \cdots < f(x_k).$$

We define

$$T_f^A = \sum_{i=1}^{k-1} (t_i - t_{i+1})\delta_{x_i} + t_k \delta_{x_k} + (1 - t_1)\delta_{x_1},$$

where $t_i = T(\{x \in \mathbb{R}^d : f(x) > f(x_i)\})$ for $i = 1, \dots, k$ and δ_x is defined by (2.3).

Observe that $T_f^A \in \tilde{C}$ and

$$\begin{aligned}\int f dT_f^A &= \sum_{i=1}^{k-1} (t_i - t_{i+1}) f(x_i) + t_k f(x_k) + (1 - t_1) f(x_1) \\ &= f(x_1) + \sum_{i=1}^{k-1} [f(x_{i+1}) - f(x_i)] t_{i+1}.\end{aligned} \tag{2.9}$$

Then we have the following lemma.

2.5. Lemma. *Let D be a countable dense set in $\mathbb{R}^d$. Then for any $T \in \tilde{C}$, for any $f \in C_b(\mathbb{R}^d)$ and for any $\epsilon > 0$ there exists a finite set $A = \{x_i : i = 1, \dots, k-1\} \subset D$ such that $T_f^A \in \mathcal{U}(T; f; \epsilon)$.*

Proof. Let
$$\alpha = \sup\{f(x) : x \in \mathbb{R}^d\};\ \beta = \inf\{f(x) : x \in \mathbb{R}^d\}.$$

By Lemma 2.4 we may assume that $\beta > 0$. Since $f \in C_b(\mathbb{R}^d)$ and D is dense in $\mathbb{R}^d$, $D \cap (\text{supp } f)$ is dense in supp f. Therefore, for any $\epsilon > 0$ we can choose $x_i \in D \cap (\text{supp } f)$, $i = 1, \dots, k-1$ with

$$\alpha_0 = \beta \le \alpha_1 = f(x_1) < \dots < \alpha_{k-1} = f(x_{k-1}) \le \alpha_k = \alpha \tag{2.10}$$

such that
$$0 \le \alpha_{i+1} - \alpha_i < \epsilon/3 \text{ for every } i = 0, \dots, k-1. \tag{2.11}$$

For every $i = 0, \dots, k$ let

$$t_i = T(\{x \in \mathbb{R}^d :\ f(x) > \alpha_i\}). \tag{2.12}$$

Then for $t \in (\alpha_i, \alpha_{i+1}]$ we have

$$t_{i+1} \le\ T(\{x \in \mathbb{R}^d :\ f(x) \ge t\}) \le\ t_i,\ i = 0, \dots, k-1.$$

Hence, by virtue of (2.11) and with noting that $t_0 = 1$ and $t_k = 0$ (see (2.12)) we have

$$\begin{aligned}
0 &\le \int f dT - \sum_{i=0}^{k-1} (\alpha_{i+1} - \alpha_i) t_{i+1} - \beta \\
&\le \sum_{i=0}^{k-1} [(\alpha_{i+1} - \alpha_i) t_i - (\alpha_{i+1} - \alpha_i) t_{i+1}] \\
&= \sum_{i=0}^{k-1} (\alpha_{i+1} - \alpha_i)(t_i - t_{i+1}) < \epsilon/3 \sum_{i=0}^{k-1} (t_i - t_{i+1}) = \epsilon/3.
\end{aligned}$$

That means
$$\left| \int f dT - \sum_{i=0}^{k-1} (\alpha_{i+1} - \alpha_i) t_{i+1} \right| < \epsilon/3. \tag{2.13}$$

Note that for $A = \{x_i :\ i = 1, \dots, k-1\}$ from (2.9) we have

$$\int f dT_f^A = \alpha_1 + \sum_{i=0}^{k-2} (\alpha_{i+1} - \alpha_i) t_{i+1}. \tag{2.14}$$

Thus, from (2.13) and (2.14) we get

$$\begin{aligned}
\left| \int f dT - \int f dT_f^A \right| &< \epsilon/3 + (\alpha_1 - \alpha_0)(1 - t_1) \\
&< \epsilon/3 + \epsilon/3 < \epsilon.
\end{aligned}$$

Therefore $T_f^A \in \mathcal{U}(T; f; \epsilon)$. The lemma is proved. □

Note that the set function T_f^A defined in the proof of Lemma 2.5 is a probability capacity with finite support. Hence, Lemma 2.5 immediately implies the following corollary.

2.6. Corollary. *The probability capacities with finite support are Choquet weakly dense in the space $\tilde{\mathcal{C}}$.*

Let C and Q denote a countable dense set of $C_b(\mathbb{R}^d)$ and $(0,1)$, respectively. Denote

$$\mathbf{G} = \{\bigcap_{i=1}^{k} \mathcal{U}(T_{g_i}^{A_i}; g_i; \delta_i) : \ A_i \in \mathcal{F}(D),\ g_i \in C,\ \delta_i \in Q,\ i = 1, \dots, k\},$$

where $\mathcal{F}(D)$ is the family of all finite sets of D. Using Lemmas 2.4 and 2.5 we will show that

2.7. Proposition. *$\mathbf{G}$ is a countable base of the Choquet weak topology τ_C in $\tilde{\mathcal{C}}$.*
Proof. Clearly $\mathbf{G}$ is countable. We prove that $\mathbf{G}$ is a base of the Choquet weak topology in $\tilde{\mathcal{C}}$.

Given $\mathcal{U}(T; f_1, \dots, f_k; \epsilon) \in \mathbf{B}$. Since C is dense in $C_b(\mathbb{R}^d)$, for each $i = 1, \dots, k$ there exists $g_i \in C$ such that

$$|f_i(x) - g_i(x)| \ < \delta_i \text{ for all } \ x \in \mathbb{R}^d, \tag{2.15}$$

where $\delta_i \in Q, \delta_i \ < \epsilon/4$.

From the definition of Choquet integral it is easy to see that if $f(x) \le g(x)$ for all $x \in \mathbb{R}^d$, then $\int f dT \le \int g dT$ for any $T \in \tilde{\mathcal{C}}$. Hence, by Lemma 2.4 and (2.15),

$$|\int f_i dT - \int g_i dT| \ \le \delta_i \text{ for every } \ T \in \tilde{\mathcal{C}}, i = 1, \dots, k. \tag{2.16}$$

On the other hand, by Lemma 2.5 for each $i = 1, \dots, k$ we can choose

$$A_i = \{x_j^i : \ j = 1, \dots, n_i\} \ \in \mathcal{F}(D)$$

such that

$$|\int g_i dT - \int g_i dT_{g_i}^{A_i}| < \delta_i. \tag{2.17}$$

Therefore for every $S \in \mathcal{U}(T_{g_i}^A; g_i; \delta_i)$, from (2.16) and (2.17) we have

$$\begin{aligned} |\int f_i dT - \int f_i dS| &\le |\int f_i dT - \int g_i dT| + |\int g_i dT - \int g_i dT_{g_i}^{A_i}| \\ &+ |\int g_i dT_{g_i}^{A_i} - \int g_i dS| + |\int g_i dS - \int f_i dS| \\ &< \delta_i + \delta_i + \delta_i + \delta_i = 4\delta_i < \epsilon \end{aligned}$$

for every $i = 1, \dots, k$.

Thus

$$S \in \mathcal{U}(T; f_i; \epsilon) \text{ for every } i = 1, \dots, k.$$

Consequently

$$\bigcap_{i=1}^{k} \mathcal{U}(T_{g_i}^{A_i}; g_i; \delta_i) \subset \mathcal{U}(T; f_1, \dots, f_k; \epsilon),$$

and the proposition is proved. □

The proof of the theorem is finished. Thus, since $\tilde{C}$ equipped with the Choquet weak topology is a metric space, we can define the notion of weak convergence of a sequence in $\tilde{C}$ as follows.

2.8. Definition. A sequence of capacities $\{T_n\}_{n=1}^{\infty} \subset \tilde{C}$ is said to be *Choquet weakly convergent* to the capacity $T \in \tilde{C}$ if and only if $\int f dT_n \to \int f dT$ for every $f \in C_b(\mathbb{R}^d)$.

The following proposition shows that the convergence on compact sets is stronger than the Choquet weak convergence.

3. Topological embedding $\mathbb{R}^d$ into $\tilde{C}$

Note that the corresponding $x \to T_x = \delta_x$, with δ_x defined by (2.3), is one-to-one between $\mathbb{R}^d$ and the set of the probability capacities $\{T_x : x \in \mathbb{R}^d\} \subset \tilde{C}$. Therefore, in some sense the class of capacities in $\mathbb{R}^d$ also cotains $\mathbb{R}^d$. In a such way, let $V : \mathbb{R}^d \to \tilde{C}$ be a transform defined by

$$V(x) = T_x \text{ for every } x \in \mathbb{R}^d. \tag{3.1}$$

We now show that

3.1. Theorem. *The map $V : \mathbb{R}^d \to \tilde{C}$ is a topological embedding, i.e, $\mathbb{R}^d$ is homeomorphic to $V(\mathbb{R}^d)$, which is the closed subset of $\tilde{C}$.*

Proof. Clearly $V(x) \neq V(y)$ for $x \neq y$. Moreover, if $x_n \to x$ then for any $f \in C_b(\mathbb{R}^d)$, from (2.4) we have

$$\int f dT_{x_n} = f(x_n) \to f(x) = \int f dT_x.$$

Therefore $V(x_n) \to V(x)$, and so V is continuous in the Choquet weak topology.

Conversely, assume that $T_{x_n} \to T \in \tilde{C}$ in the Choquet weak topology. We claim that $x_n \to x$ and $T = T_x$ for some $x \in \mathbb{R}^d$. In fact, $T_{x_n} \to T$ implies $\int f dT_{x_n} = f(x_n) \to \int f dT$ for every $f \in C_b(\mathbb{R}^d)$.

Since T is a probability capacity, by the Urysohn-Tietze Theorem we can find a continuous function $f \in C_0^+(\mathbb{R}^d)$ such that

$$\gamma = \int f dT > 0,$$

where $C_0^+(\mathbb{R}^d)$ denotes all continuous non-negative real-valued functions with compact support in $\mathbb{R}^d$. Since $f(x_n) \to \gamma > 0$, for $0 < \delta < \gamma$ there is $n_0 \in \mathbb{N}$ such that

$$|f(x_n) - \gamma| < \delta \text{ for all } n \geq n_0.$$

That means

$$x_n \in \text{supp } f \text{ for all } n \geq n_0.$$

By the compactness of supp f there is a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x \in \mathbb{R}^d$.

If $x_n \nrightarrow x$, then there exists a subsequence $\{x'_{n_k}\} \subset \{x_n\}$ such that

$$|x'_{n_k} - x| \geq \beta > 0 \text{ for all } k \in \mathbb{N} \text{ and for some } \beta > 0. \tag{3.2}$$

Again, by the compactness of supp f, there is a subsequence $\{x''_{n_k}\} \subset \{x'_{n_k}\}$ such that $x''_{n_k} \to x' \in \mathbb{R}^d$. Then we have

$$T_{x''_{n_k}} \to T_{x'} \text{ and } T_{x_{n_k}} \to T_x.$$

Since $\{T_{x''_{n_k}}\}, \{T_{x_{n_k}}\} \subset \{T_{x_n}\}$and $T_{x_n} \to T$, we get $T_{x'} = T_x = T$, and so $x = x'$. From (3.2) we obtain a contradiction. Hence $x_n \to x$. Consequently V^{-1} is continuous. □

3.2. Remark. By Theorem 3.1 we can identify $\mathbb{R}^d$ with the closed subset $V(\mathbb{R}^d)$ of $\tilde{\mathcal{C}}$. Therefore, the space $\tilde{\mathcal{C}}$ contains $\mathbb{R}^d$ topologically.

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