PROBABILITY MEASURE FUNCTORS PRESERVING THE REGULAR PROPERTY

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Abstract Let X be a topological Hausdorff space. For each $k \in \mathbb{N}$, by $P_k(X)$ we denote the set of all probability measures on X, whose supports of no more than k points. Then probability measure functor P_k preserve the regular property.

1. Probability measure with finite supports

Let X be a topological Hausdorff space. A *probability measure with finite supports* on X is a function $\mu : X \to [0,1]$ satisfying the condition

$$\text{supp}\mu = \{x \in X : \mu(x) > 0\} \quad \text{is finite} \tag{a}$$

$$\sum_{x \in \text{supp}\mu} \mu(x) = 1. \tag{b}$$

For each $k \in \mathbb{N}$, let $P_k(X)$ denote the set of all probability measure on X, whose supports of no more than k points. Then every $\mu \in P_k(X)$ can be written in the form

$$\mu = \sum_{i=1}^{q} m_i \delta_{x_i}, \quad q \leq k$$

where δ_x is Dirac function, that is

$$\delta_x(y) = \begin{cases} 0 & \text{if} \quad y \neq x \\ 1 & \text{if} \quad y = x \end{cases}.$$

and

$$m_i = \mu(x_i) > 0, \qquad \sum_{i=1}^{q} m_i = 1.$$

Then m_i is called the mass of μ at x_i.

Fedorchuk [Fe] introduced a topology on $P_k(X)$ as follows:

Each point

$$\mu_0 = \sum_{i=1}^{q} m_i^0 \delta_{x_i^0} \in P_k(X)$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX has a neighborhood of the form $O(\mu_0, U_1, U_2, ..., U_q, \epsilon)$, where $\epsilon > 0$; $U_1, U_2, ..., U_q$ are disjoint neighborhood of $x_1^0, x_2^0, ..., x_q^0$ respectively (note that U_i can be taken from a fixed basis of topology of X).

$$O(\mu_0, U_1, U_2, ..., U_q, \epsilon) = \{\mu \in P_k(X) : \mu = \sum_{i=1}^{q+1} \mu_i, \text{supp}\mu_i \in U_i, |m_i^0 - \|\mu_i\|| < \epsilon$$

$$i = 1, 2, ..., q+1; U_{q+1} = X \setminus \bigcup_{i=1}^{q} U_i, \quad m_{q+1}^0 = 0\}.$$

It is easy to see that the family $O(\mu_0, U_1, U_2, ..., U_q, \epsilon)$ forms a basis of a topology of $P_k(X)$. This topology is called *Fedorchuk topology.*

2. The Results

In this section we shall prove that the functor P_k preserve the regular property.

Theorem 2.1. *If X metrizable, then so is $P_k(X)$, for any $k \in \mathbb{N}$.*

The proof of theorem 2.1 is based on the following fact due to Frink [Fr].

Theorem 2.2. [Fr]. *A T_1-space X is metrizable if and only if the following condition holds:*

(Fr) For each $x \in X$ there exists a neighborhood basis $\{U_n(x)\}_{n=1}^{\infty}$ satisfying the following condition: if $U_n(x)$ is given there exists an $m = m(x, n)$ such that $U_m(y) \cap U_m(x) \neq \emptyset$ implies $U_m(y) \subset U_n(x)$.

Proof. Obviously $P_k(X)$ is a T_1-space. Thus, by Theorem 2.2 it suffices to verify the condition (Fr).

For each

$$\mu = \sum_{i=1}^{q} m_i \delta_{x_i} \in P_k(X), \quad q \leq k,$$

we define a neighborhood basis $\{O_n(\mu)\}_{n=1}^{\infty}$ satisfying the condition (Fr).

For each $i = 1, ..., q$ we take $\{U^n(x_i)\}_{n=1}^{\infty}$ such that

$$\text{diam}U^n(x_i) < \frac{1}{4} \min\{2^{-n}, \text{dist}(U^n(x_i), U^n(x_j)); i \neq j\}. \tag{1}$$

$$\{U^n(x_1)\}_{n=1}^{\infty} \text{ satisfies the condition (Fr).} \tag{2}$$

We put

$$O_n(\mu_0, U_1^n, U_2^n, ..., U_q^n, \epsilon_n(\mu)),$$

where $U_i^n = U^n(x_i), i = 1, ..., q$ and

$$\epsilon_n(\mu)) < \min\{2^{-n}, m_i, i = 1, 2, ..., q\}.$$

Let us show that $\{O_n(\mu)\}_{n=1}^{\infty}$ satisfies (Fr).

Given $O_n(\mu)$. Since $\epsilon_n(\gamma) < 2^{-n}$ for every $\gamma \in P_k(X)$ there exists an $m \in \mathbb{N}$ such that

$$\epsilon_m(\gamma) < \frac{1}{4k} \min\{\epsilon_n(\mu), m_i, i = 1, ..., q\} \tag{3}$$

for every $\gamma \in P_k(X)$. We shall prove that $m(\mu, n) = \max\{m, m(x_i, n), i = 1, ..., q\}$ satisfies the desired property of (Fr).

Assume that $O_m(\gamma) = O_m(\gamma, V_1^m, V_2^m, ..., V_q^m, \epsilon_m(\gamma))$ with $O_m(\gamma) \cap O_m(\mu) \neq \emptyset$. Take $\theta \in O_m(\gamma) \cap O_m(\mu)$ and write $\theta_i = \theta|_{U_i^m}, i = 1, ..., q$ and let

$$\theta_{q+1} = \theta|_{X \setminus \bigcup_{i=1}^q U_i^m}, A_i = \mathrm{supp}\theta_i, i = 1, 2, ..., q+1.$$

Since

$$||\theta_i|| \geq m_1 - \epsilon_m(\mu) > m_i - \frac{1}{4}m_i = \frac{3}{4}m_i > \epsilon_m(\gamma), i = 1, 2, .., q,$$

we infer that for every $i \leq q$ there exists at least $j \in \{1, ..., r\}$ such that $A_i \cap A_j \neq \emptyset$. Let

$$G_i = \bigcup\{V_j : V_j \cap A_i \neq \emptyset\}, i = 1, ..., q; G_{q+1} = \bigcup\{V_j : V_j \subset X \setminus \bigcup_{i=1}^q A_i\}.$$

Since $A_i \subset U_i^m$ from (2) it follows that

$$G_i \subset U_i^m \text{ for every } i = 1, ..., q. \tag{4}$$

We shall show that $O_m(\gamma) \subset O_n(\mu)$. For every $w \in O_m(\gamma)$ we denote $w_i = w|_{G_i}$ for $i = 1, 2, ..., q+1$; $w_{ij} = w_i|_{V_j}$ for $V_j \subset G_i$; $\theta_{ij} = \theta_i|_{V_j}$ for $V_j \subset G_i$. Since $w, \theta \in O_m(\gamma)$ it follows

$$\Big|||w_{ij}|| - ||\theta_{ij}||\Big| < 2\epsilon_m(\gamma).$$

Note that $k \geq r \geq \mathrm{Card}\{j : V_j \subset G_i\}$. From (3) we obtain

$$\Big|||w_i|| - ||\theta_i||\Big| \leq \sum_{V_j \subset G_i} \Big|||w_{ij}|| - ||\theta_{ij}||\Big| < 2k\epsilon_m(\gamma) < \frac{1}{2}\epsilon_n(\mu) \tag{5}$$

for every $i = 1, ..., q+1$. Hence

$$\Big|||w_i|| - m_i\Big| \leq \Big|||w_i|| - ||\theta_i||\Big| + ||\theta_i - m_i|| < \frac{1}{2}\epsilon_n(\mu) + \epsilon_m(\mu) < \epsilon_n(\mu)$$

for every $i = 1, ..., q$ and by (5) we have

$$||w_{q+1}|| \leq ||\theta_{q+1}|| + \frac{1}{2}\epsilon_n(\mu) \leq \epsilon_m(\mu) + \frac{1}{2}\epsilon_n(\mu) < \epsilon_n(\mu).$$

Consequencetly from (4) we infer that

$$w \in O_n(\mu).$$

This completes the proof of theorem 2.1.

Theorem 2.3. *If topological space X is T_1 and regular, and the topology has a $\sigma-$locally finite base, then so is $P_k(X)$ for any $k \in \mathbb{N}$.*

Proof. Since X is T_1 and regular and topology has $\sigma-$lacally finite base, then X metrizable. Thus by theorem 2.1 it follows that $P_k(X)$ is metrizable and satisfies condition T_1-space, and regular, and topology has a $\sigma-$locally finite base.
This completes the proof of theorem 2.3.

References

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