NON-LINEAR AND LINEAR ANALYSIS OF STIFFENED LAMINATED PLATES

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Abstract The non-linear displacement formulation of laminated composite plates subjected to perpendicular loads by Ritz and Finite element method (FEM), are presented. Cases of stiffened and unstiffened laminated plates are considered.

Introduction

Analysis of laminated plates has been studied by many authors [1, 2, 4]. In this paper we deal with the non-linear static analysis of stiffened and unstiffened laminated plates by Ritz's method and FEM in correctizied formulation.

1. Linear and non-linear analysis of laminated plates

1.1. Laminated plates constitutive equation

The stress-strain relation for the k-layer can be expressed as follows [1]

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \\ \sigma_4 \\ \sigma_5 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{26} & 0 & 0 \\ Q_{16} & Q_{26} & C_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} \\ 0 & 0 & 0 & Q_{45} & Q_{55} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \varepsilon_4 \\ \varepsilon_5 \end{bmatrix} \tag{1}$$

The relation between internal force, moments and deformations for laminated plates are of the form [2]

$$\{\Sigma\} = [D]\{\varepsilon\}, \tag{2}$$

where

$$\{\Sigma\} = [N_x \quad N_y \quad N_{xy} \quad M_x \quad M_y \quad M_{xy} \quad Q_y \quad Q_x]^T,$$

$$\{\varepsilon\} = [\varepsilon^0_{xx} \quad \varepsilon^0_{yy} \quad \gamma^0_{xy} \quad \chi_x \quad \chi_y \quad \chi_{xy} \quad \gamma^0_{yz} \quad \gamma^0_{xz}]^T,$$

$$[D] = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & 0 & 0 \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & 0 & 0 \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & 0 & 0 \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & 0 & 0 \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & 0 & 0 \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{44} & A_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{45} & A_{55} \end{bmatrix}$$

The variation of potential energy U and work done by external force acting on the plate can be written

$$\delta U = \iint\limits_S \Sigma\, \delta\varepsilon\, dxdy = \iint\limits_S \{\delta\varepsilon\}^T [D]\{\varepsilon\}\, dxdy\,, \tag{3}$$

$$\delta A = \iint\limits_S F\, \delta u\, dxdy = \iint\limits_S \{\delta u\}^T \{F\}\, dxdy\,, \tag{4}$$

where $\{F\}$ is a matrix of external force, $\{u\}$- displacement matrix of a point of the middle surface. $\{u\} = [\,u \quad v \quad w \quad \psi_x \quad \psi_y\,]^T$

Boundary conditions

a) Simply-supported edges

$$u = w = 0 \quad \text{at} \quad x = 0; x = a \qquad v = w = 0 \quad \text{at} \quad y = 0; y = b;$$
$$\psi_x = 0 \quad \text{at} \quad y = 0; y = b; \qquad \psi_y = 0 \quad \text{at} \quad x = 0; x = a$$

b) Clamped edges

$$u = v = w = \psi_x = \psi_y = 0 \qquad \text{at } x = 0; x = a; y = 0; y = b$$

c) Mixed conditions. Clamped-suported edges

$$u = w = \psi_y = 0 \quad \text{at} \quad x = 0; x = a; y = 0; y = b$$
$$v = \psi_x = 0 \quad \text{at} \quad y = 0; y = b$$

1.2. Stiffener constitutive equation

Stiffeners are related with plate. Stiffener directions are placed along rectangular lines. Stiffener displacement components are deflection and rotation along stiffener directions. For x-stiffener we have relation between the deflection and the rotation $\psi_x = dw/dx$. The deformation along x-axis can be written:

$$\varepsilon_x = \frac{z}{\rho} = z\frac{d\psi_x}{dx} = z\frac{d^2w}{dx^2}.$$

The stiffener potential energy along x-axis is calculated as follows

$$U_{sx} = \frac{1}{2}\iiint\limits_V \varepsilon_x \cdot \sigma_x\, dV = \frac{1}{2}EJ_z \int\limits_x \left(\frac{d^2w}{dx^2}\right)^2 dx\,, \tag{5}$$

where E - elascity modulus and J_z- inertial moment for z-axis of stiffener. Similarly, we get the stiffener potential energy form along y-axis

$$U_{sy} = \frac{1}{2}\iiint\limits_V \varepsilon_y \cdot \sigma_y\, dV = \frac{1}{2}EJ_z \int\limits_y \left(\frac{d^2w}{dy^2}\right)^2 dy \tag{6}$$

2. Methods of calculating.

2.1. Ritz's method[2]

Based on Lagrange's minimum principle of the complete potential energy $(U - A)$ we have $\delta(U - A) = 0$

The potential energy U of stiffener laminated plates is equal to the total stiffener potential energy U_b and the plates potential energy U_s: $U = U_b + U_s$

We put $J = U - A$, which reduces:

$$J = \frac{1}{2}\iint\limits_S \{\varepsilon\}^T[D]\{\varepsilon\}\,dxdy + \frac{1}{2}EJ_z\int\limits_x \Big(\frac{d^2w}{dx^2}\Big)^2 dx + \frac{1}{2}EJ_z\int\limits_y \Big(\frac{d^2w}{dy^2}\Big)^2 dy - \iint\limits_S \{u\}^T[F]\,dxdy\,, \tag{7}$$

where $\{u\}^T = [u, v, w, \psi_x, \psi_y] = [u_1, u_2, u_3, u_4, u_5]$

Displacement components can be approximated by $u_i = \sum\limits_{\alpha=1}^{n} a_{i\alpha}\,\varphi_{i\alpha}(x,y)$, where functions $\varphi_{i\alpha}$ are linearly independent, and must be chosen such that the boundary conditions are satisfied.

We can write them in matrix form $\{u\}_{5\times 1} = [\Phi]_{5\times 5n} \cdot \{a\}_{5n\times 1}$

From here the deformation can be caculated by

$$\{\varepsilon\}_{8\times 1} = [B(a)(x,y)]_{8\times 5n} \cdot \{a\}_{5n\times 1}\,, \tag{8}$$

where $[B(a)(x,y)]$ depends on $a_{i\alpha}$ of first degree. The stiffener displacement along x-axis is approximated as follows

$$w = b_1 + b_2x + b_3x^2 + b_4x^3$$
$$\psi_x = \frac{dw}{dx} = b_2 + 2b_3x + 3b_4x^2\,,$$

or in matrix form

$$[w] = [\,1 \quad x \quad x^2 \quad x^3\,]\,[\,b_1 \quad b_2 \quad b_3 \quad b_4\,]^T = [F(x)] \cdot [b]. \tag{9}$$

The coefficients $b_i, (i = \overline{1,4})$ are calculated by deflection and rotation value of two boundary points of stiffener

$$[\,b_1 \quad b_2 \quad b_3 \quad b_4\,]^T = [H_x]_{4\times 5n} \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{5n} \end{bmatrix}_{5n\times 1} = [H_x] \cdot \{a\}. \tag{10}$$

From (9), (10) we have

$$\Big[\frac{d^2w}{dx^2}\Big] = \frac{d^2}{dx^2}([F(x)][H_x]\{a\}) = [G_x]_{1\times 5n} \cdot \{a\}_{5n\times 1}. \tag{11}$$

Similarly, for y-stiffener we get

$$\left[\frac{d^2 w}{dy^2}\right] = [G_y]_{1\times 5n}\cdot\{a\}_{5n\times 1} \tag{12}$$

From (7) ÷ (12) we obtain

$$J = \frac{1}{2}\iint\limits_S \{a\}^T[B]^T D][B]\{a\}\,dxdy + \frac{1}{2}EJ_z\int\limits_x \{a\}^T[G_x]^T[G_x]\{a\}\,dx +$$
$$+\frac{1}{2}EJ_z\int\limits_y \{a\}^T[G_y]^T[G_y]\{a\}\,dy - \iint\limits_S \{F\}^T[\Phi]\{a\}\,dxdy. \tag{13}$$

Denote that

$$\iint\limits_S [B]^T[D][B]\,dxdy = [\mathbb{B}(a)]_{5n\times 5n},\quad EJ_z\int\limits_x [G_x]^T[G_x]\,dx = [\mathbb{G}_x]_{5n\times 5n},$$
$$EJ_z\int\limits_y [G_y]^T[G_y]\,dy = [\mathbb{G}_y]_{5n\times 5n},\quad \iint\limits_S \{F\}^T_{1\times 5}[\Phi]_{5\times 5n}\,dxdy = \{\mathbb{F}\}^T_{1\times 5n}, \tag{14}$$

where $[\mathbb{B}(a)]$ depends on $\{a_{i\alpha}\}$ of second degree and J becomes a function of multi-variable $a_{i\alpha}$

$$J = \frac{1}{2}\{a\}^T_{1\times 5n}\big([\mathbb{B}(a)] + [\mathbb{G}_x] + [\mathbb{G}_y]\big)_{5n\times 5n}\{a\}_{5n\times 1} - \{\mathbb{F}\}^T_{1\times 5n}\{a\}_{5n\times 1}, \tag{15}$$

where

$$\{a\}^T = [a_{11}, a_{12}, \cdots a_{1n}, a_{21}, a_{22}, \cdots a_{2n}, \cdots a_{51}, a_{52}, \cdots a_{5n}] = [a_1, a_2, \cdots a_{5n}].$$

Minimization of J

$$\delta J = 0 \quad \textit{reduces} \quad \frac{\partial J}{\partial a_i} = 0,\ \forall i = \overline{1, 5n}.$$

We get a system of $(5n)$ algebraic equations in matrix form for finding a_i.

$$[K(a)]_{5n\times 5n}\{a\}_{5n\times 1} = \{\mathbb{F}\}_{5n\times 1}, \tag{16}$$

where $[K(a)]$ depends on coefficients a_i of second degree.

The system (16) can be solved by an iterative method

$$[K(a)^{(k-1)}]\{a^{(k)}\} = \{\mathbb{F}\}.$$

For a plate with simply -supported edges, displacement components are chosen

$$u = a_{11}\sin(\frac{\pi x}{a}) + a_{12}\sin(\frac{\pi x}{a})\sin(\frac{\pi y}{b}),$$
$$v = a_{21}\sin(\frac{\pi y}{b}) + a_{22}\sin(\frac{\pi x}{a})\sin(\frac{\pi y}{b}),$$
$$w = a_{31}\sin(\frac{\pi x}{a})\sin(\frac{\pi y}{b}) + a_{32}\sin(\frac{3\pi x}{a})\sin(\frac{3\pi y}{b}),$$
$$\psi_x = a_{41}\sin(\frac{\pi y}{b}) + a_{42}\sin(\frac{\pi x}{a})\sin(\frac{\pi y}{b}),$$
$$\psi_y = a_{51}\sin(\frac{\pi x}{a}) + a_{52}\sin(\frac{\pi x}{a})\sin(\frac{\pi y}{b}).$$

2.2. Finite element method[2 , 3].

The plate is devided into 16 small rectangular elements with the size $(a/4) \times (b/4)$.

The element (e) having nodes (i, j, k, l) is studied. At a point $M(x, y)$ in the element (e) we choose

$$\begin{aligned} u &= a_1 + a_2\, x + a_3\, y + a_4\, x\, y\,, \\ v &= a_5 + a_6\, x + a_7\, y + a_8\, x\, y\,, \\ w &= a_9 + a_{10}\, x + a_{11}\, y + a_{12}\, x\, y\,, \\ \psi_x &= a_{13} + a_{14}\, x + a_{15}\, y + a_{16}\, x\, y\,, \\ \psi_y &= a_{17} + a_{18}\, x + a_{19}\, y + a_{20}\, x\, y\,, \end{aligned} \tag{18}$$

and in matrix form (18) can be written

$$\{u\}_{5\times 1} = [F(x, y)]_{5\times 20} \cdot \{a\}_{20\times 1}\,. \tag{19}$$

In the 4 nodes (i, j, k, l) we have

$$\{q\}^e = \begin{bmatrix} q_1{}^e \\ q_2{}^e \\ \vdots \\ q_{19}{}^e \\ q_{20}{}^e \end{bmatrix} = \begin{bmatrix} \{u\}^i \\ \{u\}^j \\ \{u\}^k \\ \{u\}^l \end{bmatrix}_{20\times 1} =$$

$$= \begin{bmatrix} 1 & x_i & y_i & x_i\, y_i & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & x_i & y_i & x_i\, y_i \\ \vdots & & & & & & & & \\ 1 & x_l & y_l & x_l\, y_l & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & x_l & y_l & x_l\, y_l \end{bmatrix}_{20\times 20} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{19} \\ a_{20} \end{bmatrix}_{20\times 1} \tag{20}$$

where $x_i = 0;\quad y_i = 0;\quad x_j = a/4;\quad y_j = 0\,,$

$x_k = a/4;\quad y_k = b/4;\quad x_l = 0;\quad y_l = b/4.$

Then the equation (20) has the form $\{q\}^e{}_{20\times 1} = [A]_{20\times 20} \cdot \{a\}_{20\times 1}$.

Instead of finding $\{a_i\}$ we find displacement components $\{q\}^e$

$$\{a\}_{20\times 1} = [A]^{-1}_{20\times 20} \cdot \{q\}^e{}_{20\times 1}$$

The displacement in a point M(x, y) is calculated through displacement of nodes (i, j, k, l).

$$\{u\}_{5\times 1} = [F(x, y)]_{5\times 20} \cdot [A]^{-1}_{20\times 20} \cdot \{q\}^e{}_{20\times 1} = [N(x, y)]_{5\times 20}\{q\}^e{}_{20\times 1}\,, \tag{21}$$

where $[N(x, y)]_{5\times 20} = [F(x, y)]_{5\times 20} \cdot [A]^{-1}_{20\times 20}$

From (21) we obtain

$$\{\varepsilon\}^e_{8\times 1} = [\varepsilon^0_{xx} \quad \varepsilon^0_{yy} \quad \gamma^0_{xy} \quad \chi_x \quad \chi_y \quad \chi_{xy} \quad \gamma^0_{yz} \quad \gamma^0_{xz} \quad]^T = [B(q)^e]_{8\times 20}\{q\}^e_{20\times 1} \tag{22}$$

and matrix$[B(q)^e]$ depends on $\{q\}^e$ of first degree.

In [2] , $[B^e] = [B^e]_{NL} + [B^e]_L$, reduces $[\delta B^e]\{q\}^e = [B^e]_{NL}\{\delta q^e\}$, we have the relation

$$\{\delta\varepsilon^e\} = ([B^e]_L + 2[B^e]_{NL})\{\delta q^e\} = [B^e_*]\{\delta q^e\}\,, \tag{23}$$

where $[B^e_*]_{8\times 20} = [B^e]_L + 2[B^e]_{NL}$.

From here we get the variation of potential energy of a rectangular element (e)

$$\{\delta U^e_b\} = \iint\limits_{S_e} \{\delta\varepsilon^e\}^T_{1\times 8}[D]_{8\times 8}\{\varepsilon\}^e_{8\times 1}\, dxdy = \{\delta q^e\}^T(\iint\limits_{S_e} [B^e_*]^T[D][B^e]\, dxdy)\,\{q\}^e. \tag{24}$$

Put

$$[K^e]_{20\times 20} = \iint\limits_{S_e} [B^e_*]^T[D][B^e]\, dxdy\,,$$

the relation (24) can be written as the following

$$\{\delta U^e_b\} = \{\delta q^e\}^T[K^e]\{q\}^e. \tag{25}$$

A stiffener is discretized into beams in element (e) of plates. For x-stiffener we have a relation between the deflection w and nodal displacements

$$w = [N_1(x)]_{1\times 4}\{q\}^{xe}_{4\times 1}\,,$$

that reduces

$$\varepsilon_x = z\frac{d^2w}{dx^2} = z[B^e_1(x)]_x\{q\}^{xe}\,, \tag{26}$$

where

$$[B^e_1(x)]_x = \frac{d^2[N_1(x)]}{dx^2}. \tag{27}$$

The potential energy of x-stiffener is calculated as follows

$$U^e_{sx} = \frac{1}{2}\iiint\limits_V [\varepsilon_x]^T[\sigma_x]dV = \frac{1}{2}EJ_z\{q^{xe}\}^T\Big(\int\limits_x [B^e_1]^T_x[B^e_1]_x dx\Big)\{q\}^{xe}. \tag{28}$$

Similarly, we get the potential energy of y-stiffener

$$U^e_{sy} = \frac{1}{2}EJ_z\{q^{ye}\}^T\Big(\int\limits_y [B^e_1]^T_y[B^e_1]_y dy\Big)\{q\}^{ye}. \tag{29}$$

We denotes

$$[K^e_x]_{4\times 4} = \frac{1}{2}EJ_z\Big(\int\limits_x [B^e_1]^T_x[B^e_1]_x dx\Big),\ [K^e_y]_{4\times 4} = \frac{1}{2}EJ_z\Big(\int\limits_y [B^e_1]^T_y[B^e_1]_y dy\Big)\,, \tag{30}$$

and the variation of potential energy of beam can be written

$$\{\delta U^e_{sx}\} = \{\delta q^{xe}\}^T[K^e_x]\{q\}^{xe}, \quad \{\delta U^e_{sy}\} = \{\delta q^{ye}\}^T[K^e_y]\{q\}^{ye} \tag{31}$$

The variation of work done by external force is calculated as follows

$$\{\delta A^e\} = \iint_{S_e} \{\delta u^e\}^T_{1\times 5}\{F\}_{5\times 1}\, dxdy = \{\delta q^e\}^T \iint_{S_e} [N(x,y)^e]^T \{F\}\, dxdy. \tag{32}$$

The plate has 25 nodes, i.e. there are 125 nodal displacement components. Denotes the global vector of displacement $\{q\}$

$$\{q\}_{125\times 1} = [u_1 \quad v_1 \quad w_1 \quad \psi_{x1} \quad \psi_{y1} \cdots \quad u_{25} \quad v_{25} \quad w_{25} \quad \psi_{x25} \quad \psi_{y25}]^T$$

In the element (e) we have relation between nodal and global displacements

$$\{q\}^e_{20\times 1} = [L^e]_{20\times 125}\{q\}_{125\times 1} \tag{33}$$

Nodal displacements of beam $\{q\}^{xe}$ depends on global displacements $\{q\}$ as follows

$$\{q\}^{xe}_{4\times 1} = [L^e_x]_{4\times 125}.\{q\}_{125\times 1} \quad \{q\}^{ye}_{4\times 1} = [L^e_y]_{4\times 125}.\{q\}_{125\times 1}.$$

A stiffened laminated plate is discretized into L_e element (e), L_{xe} beams - along x-axis and L_{ye} beams - along y-axis.

From $(25), (31) \div (34)$, for stiffened laminated plates we have the variation of potential and work done by external forces

$$\begin{aligned}
\delta U &= \sum_{e=1}^{Le} \delta U^e_b + \sum_{e=1}^{L_{xe}} \delta U^e_{sx} + \sum_{e=1}^{L_{ye}} \delta U^e_{sy} \\
&= \{\delta q\}^T \Big(\sum_{e=1}^{Le}[L^e]^T[K^e][L^e] + \sum_{e=1}^{L_{xe}}[L^e_x]^T[K^e_x][L^e_x] + \sum_{e=1}^{L_{ye}}[L^e_y]^T[K^e_y][L^e_y]\Big)\{q\}, \\
\delta A &= \sum_{e=1}^{Le} \delta A^e = \sum_{e=1}^{Le} \iint_{S_e} \{\delta q\}^T [L^e]^T [N(x,y)^e]^T \{F\}\, dxdy \\
&= \{\delta q\}^T \Big(\sum_{e=1}^{Le}[L^e]^T \iint_{S_e} [N(x,y)^e]^T \{F\}\, dxdy\Big).
\end{aligned} \tag{35}$$

The global stiffness and the forces matrix are determined such as

$$\begin{aligned}
[K]_{125\times 125} &= \sum_{e=1}^{Le}[L^e]^T[K^e][L^e] + \sum_{e=1}^{L_{xe}}[L^e_x]^T[K^e_x][L^e_x] + \sum_{e=1}^{L_{ye}}[L^e_y]^T[K^e_y][L^e_y] \\
\{P\}_{125\times 1} &= \sum_{e=1}^{Le}[L^e]^T_{125\times 20} \iint_{S_e} [N(x,y)^e]^T_{20\times 5}\{F\}_{5\times 1}\, dxdy.
\end{aligned} \tag{36}$$

Then equations $(35), (36)$ can be rewritten

$$\delta U = \{\delta q\}^T [K]\{q\}, \quad \delta A = \{\delta q\}^T \{P\}.$$

According to $\delta U = \delta A$ and (37) we have the equation for finding global displacements in the matrix form

$$[K]_{125\times125}\{q\}_{125\times1} = \{P\}_{125\times1}.$$

Because matrix $[K]$ depends on $\{q\}$ of second degree, we can solve (38) by an iterative method $[K^{(k-1)}]\{q^{(k)}\} = \{P\}$

3. Numerical results

We consider a four layer laminated plate:$a = 400mm$; $b/a = 2$; $h = 10mm$ or $h = 20mm$;$E_1 = 280GPa$; $E_2 = E_3 = 7GPa$;$G_{12} = G_{13} = 4,2GPa$; $G_{23} = 3,5GPa$; $\nu_{12} = \nu_{13} = \nu_{23} = 0,25$.

With stiffeners placed along x-axis and y-axis : $E = 200GPa$; $b_x = 10mm$ or $b_x = 20mm$; $b_y = 10mm$ or $b_y = 20mm$;$h_x = 2b_x$ v $h_y = 2b_y$.

The plates is acted on by perpendicular extenal force $p = 25N/mm^2$;

Boundary conditions : 4-simply- supported edges (SS);

2-simply- supported and 2-clamped edges (CS); 4-clamped edges (CC);

The first case: Laminated plate $0^o/90^o/90^o/0^o$;

The second case: Laminated plate $45^o/-45^o/-45^o/45^o$;

For illustration in the table 1-2 numerical calculation of deflection w_{max} at the center of plate is presented for the unstiffened plate and stiffened plate.

Table 1. Plate $0^o/90^o/90^o/0^o$. SS.

FEM: Unstiffened plate $w_{\max} = 0.0100\,m$ (L), $w_{\max} = 0.0091\,m$ (NL)

Ritz's: Unstiffened plate $w_{\max} = 0.0103\,m$ (L), $w_{\max} = 0.0091\,m$ (NL)

Stiffener size (m)	Quantity of stiffener	w_{max}. FEM, (m)		w_{max}. Ritz's, (m)	
		Linear	Non-linear	Linear	Non-linear
$b_y = 0.01$	$1D_y$	0.0100	0.0091	0.0101	0.0089
$h_y = 0.02$	$3D_y$	0.0099	0.0091	0.0100	0.0089
$b_y = 0.02$	$1D_y$	0.0094	0.0088	0.0092	0.0083
$h_y = 0.04$	$3D_y$	0.0087	0.0082	0.0085	0.0078
$b_x = 0.01$	$1D_x$	0.0080	0.0075	0.0081	0.0076
$h_x = 0.02$	$3D_x$	0.0068	0.0066	0.0071	0.0068
$b_x = 0.02$	$1D_x$	0.0045	0.0044	0.0047	0.0046
$h_x = 0.04$	$3D_x$	0.0032	0.0032	0.0033	0.0033
$b_x = b_y = 0.01$	$1D_x$, $1D_y$	0.0081	0.0076	0.0081	0.0075
$h_x = h_y = 0.02$	$3D_x$, $3D_y$	0.0069	0.0066	0.0070	0.0067
$b_x = 0.01$, $b_y = 0.02$	$1D_x$, $1D_y$	0.0079	0.0075	0.0075	0.0071
$h_x = 0.02$, $h_y = 0.04$	$3D_x$, $3D_y$	0.0064	0.0062	0.0062	0.0060
$b_x = b_y = 0.02$	$1D_x$, $1D_y$	0.0049	0.0048	0.0044	0.0044
$h_x = h_y = 0.04$	$3D_x$, $3D_y$	0.0032	0.0032	0.0031	0.0031

Table 2. Plate $45^o/-45^o/-45^o/45^o$. SS.
FEM: Unstiffened plate $w_{\max} = 0.0133\,m$ (L), $w_{\max} = 0.0119\,m$ (NL)
Ritz's: Unstiffened plate $w_{\max} = 0.0128\,m$ (L), $w_{\max} = 0.0111\,m$ (NL)

Stiffener size (m)	Quantity of stiffener	w_{max}, FEM, (m)		w_{max}, Ritz's, (m)	
		Linear	Non-linear	Linear	Non-linear
$h_y = 0.01$	$1D_y$	0.0129	0.0117	0.0123	0.0108
$h_y = 0.02$	$3D_y$	0.0124	0.0113	0.0120	0.0106
$h_y = 0.02$	$1D_y$	0.0115	0.0107	0.0107	0.0097
$h_y = 0.04$	$3D_y$	0.0100	0.0095	0.0096	0.0090
$h_x = 0.01$	$1D_x$	0.0110,	0.0102	0.0108	0.0098
$h_x = 0.02$	$3D_x$	0.0096	0.0091	0.0095	0.0089
$h_x = 0.02$	$1D_x$	0.0058	0.0056	0.0060	0.0058
$h_x = 0.04$	$3D_x$	0.0039	0.0039	0.0040	0.0040
$h_x = h_y = 0.01$	$1D_x, 1D_y$	0.0108	0.0101	0.0104	0.0095
$h_x = h_y = 0.02$	$3D_x, 3D_y$	0.0091	0.0088	0.0091	0.0085
$h_x = 0.01$, $h_y = 0.02$	$1D_x, 1D_y$	0.0100	0.0095	0.0092	0.0086
$h_x = 0.02$, $h_y = 0.04$	$3D_x, 3D_y$	0.0078	0.0076	0.0076	0.0073
$h_x = h_y = 0.02$	$1D_x, 1D_y$	0.0058	0.0057	0.0055	0.0054
$h_x = h_y = 0.04$	$3D_x, 3D_y$	0.0037	0.0037	0.0036	0.0036

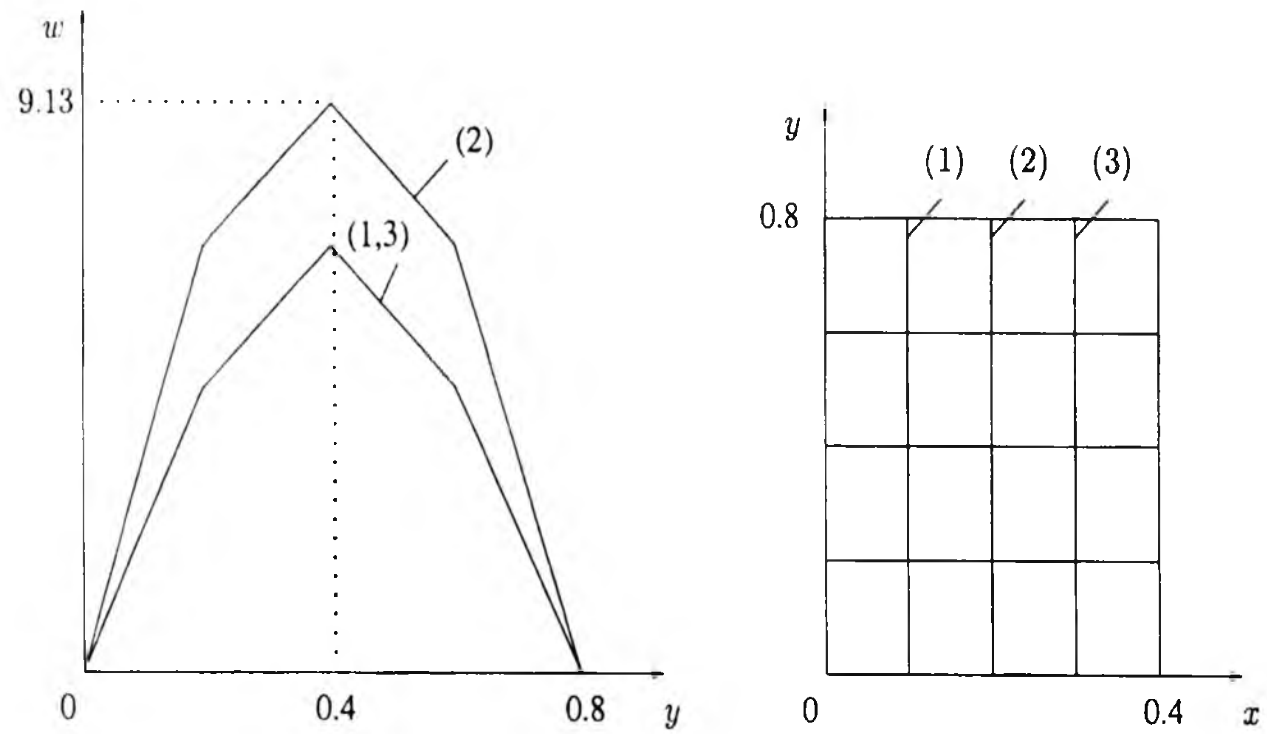

Fig 1. Deflection w along vertical cuts (1), (2), (3) of unstiffened plate $0^o/90^o/90^o/0^o$, FEM, non-linear problem, SS, $p = 25N/mm^2$. $w - (10^{-3}m)$, $,y - (m)$.

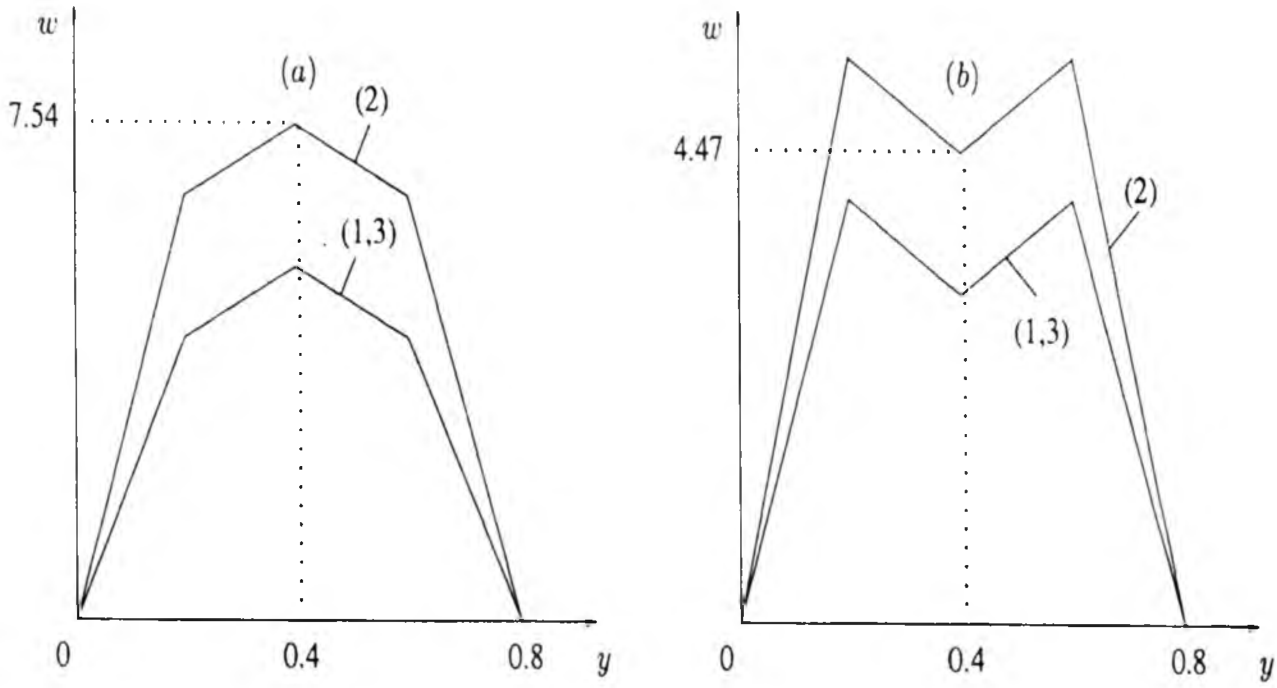

Fig 2. Deflection w along vertical cuts (1), (2), (3) of stiffened plate $0^o/90^o/90^o/0^o$, FEM, non-linear problem, SS, $p = 25N/mm^2$. $w - (10^{-3}m)$, $y - (m)$.
$(a) - 3D_x$ with $b_x = 0.01, h_x = 0.02$, $(b) - 1D_x$ with $b_x = 0.02, h_x = 0.04$,

Conclusions

- Displacement in non-linear problem is smaller than that one in linear problem. If external force is small, displacement in non-linear problem approximately equal with linear displacement. When external force increases, the difference between linear and non-linear displacement also get increased.

- The difference between result by Ritz's method and FEM in the case SS is not more than 0,8%.

- Ritz's method is suitable for cases with simply-supported edges; while FEM is used for cases with more complex boundary conditions.

- Time for solving by Ritz's method (about 5 mins) is much shorter than by FEM (about 25 mins). This publication is completed with financial support of the Council for Natural Science of Vietnam.

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