ON THE LATERAL OSCILLATION PROBLEM OF BEAMS SUBJECTED TO AXIAL LOAD

Dao Huy Bich

Vietnam National University

Nguyen Dang Bich

Institute for Building Science and Technology - Ministry of Construction

Abstract: This paper approaches the problem of lateral oscillation of beams subjected to axial load by means of seeking the exact solution of the linear Matheu equation with the periodic function h(t) having a determined form

$$\ddot{u} + h(t)u = 0 \,. \tag{1}$$

However, when $h(t) = k + a_1\cos\omega t$, the equation (1) does not posses an exact solution, but with the values of parameters k, a_1, ω satisfying some determined conditions, we can seek an approximated solution. Obtained results are summarized as follows:

The general exact solution for the equation (1) with h(t) having a determined form can be expressed in the form of known mathematical functions. It illustrates the "butterfly" effect of the Chaos phenomenon.

The condition and algorithm for finding the approximated solution of the equaton (1) with

$$h(t) = \omega^2(k + a_1\cos\omega t)\,. \tag{2}$$

From obtained results we can discuss about the oscillation of beams.

1. Lateral oscillation of beams subjected to axial load

Fig. 1 shows the oscillation of a beam having constant cross section subjected to axial load P(t). EI, EA, μ, ℓ and β represents the bending and axial rigidity, mass density, length and external damping coefficient of the beam respectively.

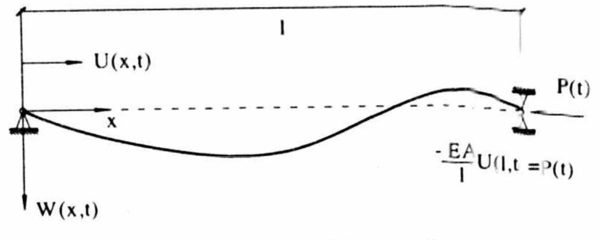


Fig.1. A beam subjected to axial load

The oscillation of the beam can be described as follows [2]

$$EIw^{IV} + \beta\dot{w} + \mu\ddot{w} - \frac{EA}{\ell}\left[U(\ell,t) + \frac{1}{2}\int_0^{\ell}(w^{II})^2 dx\right]w^{II} = 0\,, \tag{1.1}$$

w^{IV}, w^{II} - the 4th and 2nd order derivatives of w with respect to x,

$\ddot{w}$, $\dot{w}$ - the 2nd and 1st order derivatives of w with respect to t.

The boundary conditions for displacement are written as

$$w(0) = w(\ell) = w^{II}(0) = w^{II}(\ell) = 0. \tag{1.2}$$

I is assumed that the axial wave is negligible and $U(\ell, t)$ is the displacement at the right end of the beam. The boundary conditions (1.2) can be satisfied when $w(x, t)$ is set as

$$w(x,t) = u(t)\sin\frac{\pi x}{\ell}. \tag{1.3}$$

Substitute (1.3) into (1.1) it yields

$$\ddot{u} + 2D\omega_1\dot{u} + \omega_1^2[1 + q(t)]u + \gamma u^3 = 0, \tag{1.4}$$

where

$$q(t) = U(\ell,t)\frac{A\ell}{\pi^2 I};\ \omega_1 = \frac{\pi^4 EI}{\mu\ell^4};\ \gamma = \frac{1}{4}\frac{\pi^2 EA}{\mu\ell^6};\ 2D\omega_1 = \frac{\beta}{\mu}.$$

In order to investigate the phenomenon in the oscillation of beams subjected to axial loads, at first the following equation should be examined

$$\ddot{u} + \omega_1^2[1 + q(t)]u = 0. \tag{1.5}$$

If $q(t) = a\cos\omega t$, the equation (1.5) leads to the classical Mathieu's equation

$$\ddot{u} + \omega^2(k + a_1\cos\omega t)u = 0, \tag{1.6}$$

where

$$k = \frac{\omega_1^2}{\omega^2},\quad a_1 = \frac{\omega_1^2 a}{\omega^2}. \tag{1.7}$$

2. Supplementary equation

Consider the following set of equations [3]

$$\dot{v} + v^2 = \lambda(u - \alpha) - \omega^2, \tag{2.1}$$

$$v = -\frac{\dot{u}}{u - \alpha}, \tag{2.2}$$

in which v, u are functions of t; λ, α, ω are parameters.

After v being eliminated from (2.1), (2.2) it yields

$$-\frac{\ddot{u}}{u-\alpha} + \frac{2\dot{u}^2}{(u-\alpha)^2} = \lambda(u - \alpha) - \omega^2. \tag{2-3}$$

Instead of the function v, function y is used, with the following alteration:

$$v = \frac{\dot{y}}{y}. \tag{2.4}$$

From (2.2), (2.4) y can be calculated

$$y = \frac{1}{u - \alpha}. \tag{2.5}$$

Using (2.4), (2.5), the equation (2.1) can be written as

$$\ddot{y} + \omega^2 y = \lambda . \tag{2.6}$$

The solution of (2.6) is

$$y = \frac{\lambda}{\omega^2} + A\cos(\omega t + \varphi), \tag{2.7}$$

where φ, A – integral constants.

Equation (2.3) can be alternated into the form:

$$\frac{d}{du}\left(\dot{u}^2\right) - \frac{4}{u - \alpha}\dot{u}^2 = -2\lambda(u - \alpha)^2 + 2\omega^2(u - \alpha), \tag{2.8}$$

that yields

$$\dot{u}^2 = -\gamma(u - \alpha)^2\left[(u - \alpha)^2 - \frac{2\lambda}{\gamma}(u - \alpha) + \frac{\omega^2}{\gamma}\right], \tag{2.9}$$

γ - integral constant.

After solving (2.9) it can be found that

$$u - \alpha = \frac{\omega^2}{\lambda + \beta\cos(\omega t + \psi)}, \tag{2.10}$$

where ψ - integral constant,

$$\beta^2 = \lambda^2 - \omega^2\gamma > 0. \tag{2.11}$$

From (2.5), (2.7), (2.10) it can be inferred that:

$$y = \frac{1}{\omega^2}[\lambda + \beta\cos(\omega t + \psi)], \tag{2.12}$$

$$A = \frac{\beta}{\omega^2}, \quad \varphi = \psi .$$

Our aim is to find any supplementary Mathieu's equation which has an exact solution. Differentiating the equation (2.9) with respect to t we obtain

$$\ddot{u} = -(u - \alpha)\left[2\gamma(u - \alpha)^2 - 3\lambda(u - \alpha) + \omega^2\right]. \tag{2.13}$$

and from (2.5) we have

$$\frac{yu}{\alpha y + 1} = 1, \quad \frac{\alpha yu}{\alpha y + 1} = \alpha . \tag{2.14}$$

Based on (2.14), equation (2.13) can be written in the form

$$\ddot{u} = -u\left[2\gamma(u - \alpha)^2 - 3\lambda(u - \alpha) + \omega^2\right]\frac{1}{\alpha y + 1} . \tag{2.15}$$

Substitute u - α calculated in (2.10), y calculated in (2.12) with ψ = 0 into (2.15) it yields

$$\ddot{u} + \omega^2\left[\frac{2\gamma\omega^2}{(\lambda + \beta\cos\omega t)^2} - \frac{2\alpha\gamma + 3\lambda}{\lambda + \beta\cos\omega t} + \frac{\omega^2 + 3\alpha\lambda + 2\gamma\alpha^2}{\omega^2 + \alpha\lambda + \alpha\beta\cos\omega t}\right]u = 0. \quad (2.16)$$

3. Solution and characteristic of the solution

Equation (2.16) has the following particular periodic solution

$$u_1 = \frac{\omega^2 + \alpha\lambda + \alpha\beta\cos\omega t}{\lambda + \beta\cos\omega t} = \alpha + \frac{\omega^2}{\lambda + \beta\cos\omega t}. \quad (3.1)$$

When the particular solution (3.1) is found, the general solution for equation (2.16) can be estimated as follows

$$u = \frac{\omega^2 + \alpha\lambda + \alpha\beta\cos\omega t}{\lambda + \beta\cos\omega t}\left[C_1 + C_2\int_0^t \frac{(\lambda + \beta\cos\omega x)^2 dx}{(\omega^2 + \alpha\lambda + \alpha\beta\cos\omega x)^2}\right], \quad (3.2)$$

in which C_1, C_2 – integral constants.

From (3.2), the velocity $\dot{u}$ and acceleration $\ddot{u}$ can be calculated:

$$\dot{u} = \frac{\beta\omega^3\sin\omega t}{(\lambda + \beta\cos\omega t)^2}\left[C_1 + C_2\int_0^t \frac{(\lambda + \beta\cos\omega x)^2 dx}{(\omega^2 + \alpha\lambda + \alpha\beta\cos\omega x)^2}\right] + C_2\frac{\lambda + \beta\cos\omega t}{\omega^2 + \alpha\lambda + \alpha\beta\cos\omega t}, \quad (3.3)$$

$$\ddot{u} = \omega^2\left[-\frac{2\omega^4\gamma}{(\lambda + \beta\cos\omega t)^3} + \frac{3\lambda\omega^2}{(\lambda + \beta\cos\omega t)^2} - \frac{\omega^2}{\lambda + \beta\cos\omega t}\right]\left[C_1 + C_2\int_0^t \frac{(\lambda + \beta\cos\omega x)^2 dx}{(\omega^2 + \alpha\lambda + \alpha\beta\cos\omega x)^2}\right]. \quad (3.4$$

It is assumed that when $t = 0$

$$u(0) = u_0, \dot{u}(0) = \dot{u}_0. \quad (3.5)$$

Based on the initial condition (3.5), from (3.2) and (3.3), the integral constants C_1, C_2 can be found:

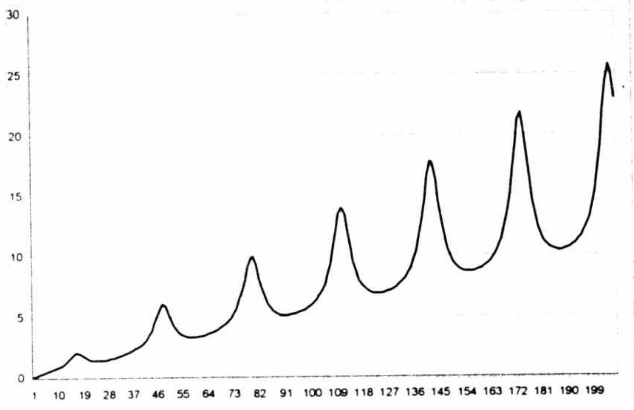
$$C_1 = \frac{(\lambda + \beta)u_0}{\omega^2 + \alpha\lambda + \alpha\beta}, C_2 = \frac{(\omega^2 + \alpha\lambda + \alpha\beta)\dot{u}_0}{\lambda + \beta}. \quad (3.6)$$

Hence, it can be concluded that (3.2) is the general solution for (2.16).

Based on (3.2), (3.3) the graphs of the functions $u(t)$ and $\dot{u}(u)$ with different parameters can be plotted as shown in Figs 2-5.

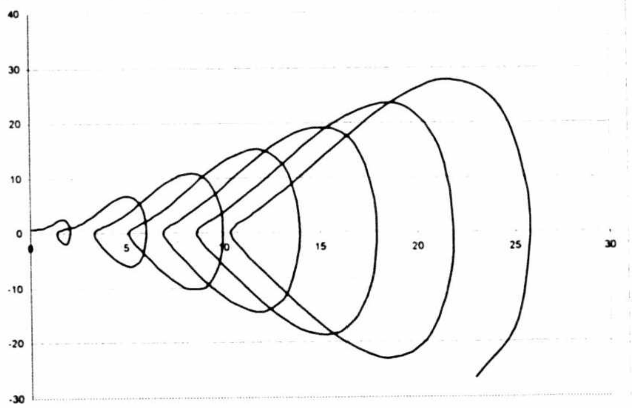
There exist constant maxima and minima of the function under the integral in (3.2). Therefore, it can be proved that this integral be generalized diverse when $t \to \infty$. From Figs. 2-5, it can be observed that the solution $u(t)$ expressed in (3.2) have the characteristic of

- Diffusively variable $\lim_{t\to\infty} u(t) = \infty$.
- The solution (3.2) depends sensitively on the initial boundary condition, when $\dot{y}_0 = 0$ it is periodic, when $\dot{y}_0 \neq 0$ it has the exceptional characteristic of the effect named "butterfly" as seen in the "chaos" phenomenon.



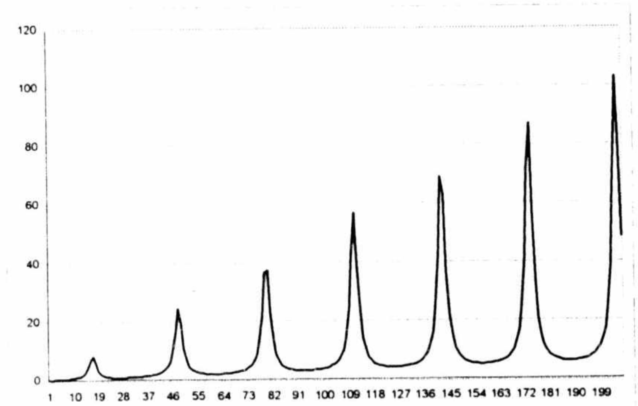
ω	αβ	αλ
2	7.48	9.69
U_o	$\dot{U}_0$	$\alpha\gamma^{1/2}$
0	0.813	3,46

Fig.2. Graph of function u(t)



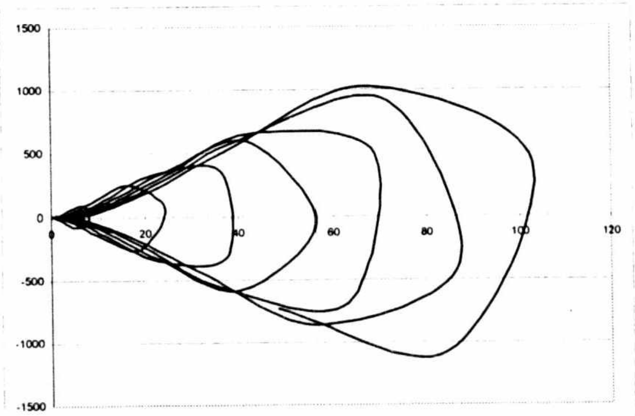
ω	αβ	αλ
2	7.48	9.69
U_o	$\dot{U}_0$	$\alpha\gamma^{1/2}$
0	0.813	3,46

Fig.3. Graph of function $\dot{u}(u)$



ω	αβ	αλ
2	-0,19	-1,25
U_o	$\dot{U}_0$	$\alpha\gamma^{1/2}$
0	1,56	0,19

Fig.4. Graph of function u(t)



ω	αβ	αλ
2	-0,19	-1,25
U_o	$\dot{U}_0$	$\alpha\gamma^{1/2}$
0	1,56	0,19

Fig.5. Graph of function $\dot{u}(u)$

4. Potential of equation (2.16)

In equation (2.16) the following function is called potential of the equation

$$h(t)=\frac{2\gamma\omega^2}{(\lambda+\beta\cos\omega t)^2}-\frac{2\alpha\gamma+3\lambda}{\lambda+\beta\cos\omega t}+\frac{\omega^2+3\alpha\lambda+2\gamma\alpha^2}{\omega^2+\alpha\lambda+\alpha\beta\cos\omega t}. \quad (4.1)$$

With the following condition

$$\lambda^2 - \beta^2 > 0;\ \ \omega^2 + 2\alpha\lambda + \gamma\alpha^2 > 0, \quad (4.2)$$

the potential function h(t) is continuous and periodic.

From (4.1) yields

$$\frac{dh}{dt}=\left[\frac{4\alpha^2\gamma\omega^2}{(\alpha\lambda+\alpha\beta\cos\omega t)^3}-\frac{2\alpha^2\gamma+3\alpha\lambda}{(\alpha\lambda+\alpha\beta\cos\omega t)^2}+\frac{\omega^2+3\alpha\lambda+2\gamma\alpha^2}{(\omega^2+\alpha\lambda+\alpha\beta\cos\omega t)^2}\right]\alpha\beta\sin\omega t. \quad (4.3)$$

that can be rearranged as

$$\frac{dh}{dt}=f(\cos\omega t)\frac{\omega^3\alpha^4\beta^4\sin\omega t}{(\alpha\lambda+\alpha\beta\cos\omega t)^3(\omega^2+\alpha\lambda+\alpha\beta\cos\omega t)^2}, \quad (4.4)$$

in which

$$f(\cos\omega t)=\cos^3\omega t-3\frac{\lambda}{\beta}\cos^2\omega t-\left(7\frac{\lambda^2}{\beta^2}+3\frac{\omega^2}{\alpha\beta}\frac{\lambda}{\beta}+2\right)\cos\omega t+$$

$$+\left(\frac{\lambda^3}{\beta^3}+\frac{\omega^2}{\alpha\beta}\frac{\lambda^2}{\beta^2}-6\frac{\lambda}{\beta}-4\frac{\omega^2}{\alpha\beta}\right). \quad (4.5)$$

Let $\frac{dh}{dt}=0$ only when $\sin\omega t=0$, from (4.4) it can be seen that

$$f(\cos\omega t)\neq 0 \quad \text{with all t such that} \quad -1\leq\cos\omega t\leq 1. \quad (4.6)$$

In order to satisfy (4.6) the following preliminary requirement can be used

$$f(-1)f(1)>0, \quad (4.7)$$

in which

$$f(-1)=\left(\frac{\lambda}{\beta}-1\right)\left[\frac{\lambda^2}{\beta^2}+\left(8+\frac{\omega^2}{\alpha\beta}\right)\frac{\lambda}{\beta}-\left(1-4\frac{\omega^2}{\alpha\beta}\right)\right], \quad (4.8)$$

$$f(1)=\left(\frac{\lambda}{\beta}+1\right)\left[\frac{\lambda^2}{\beta^2}-\left(8-\frac{\omega^2}{\alpha\beta}\right)\frac{\lambda}{\beta}-\left(1+4\frac{\omega^2}{\alpha\beta}\right)\right]. \quad (4.9)$$

From the condition (4.7) together with (4.8), (4.9) it yields

$$2\frac{\lambda}{\beta}>8+\sqrt{\frac{\omega^4}{\alpha^2\beta^2}+68}-\frac{\omega^2}{\alpha\beta}, \text{ or} \quad (4.10)$$

$$2\frac{\lambda}{\beta}<-8-\sqrt{\frac{\omega^4}{\alpha^2\beta^2}+68}-\frac{\omega^2}{\alpha\beta}, \text{ or} \quad (4.11)$$

$$8-\sqrt{\frac{\omega^4}{\alpha^2\beta^2}+68}-\frac{\omega^2}{\alpha\beta}<2\frac{\lambda}{\beta}<-8+\sqrt{\frac{\omega^4}{\alpha^2\beta^2}+68}-\frac{\omega^2}{\alpha\beta}. \quad (4.12)$$

When any of the conditions (4.10), (4.11), (4.12) is satisfied, the preliminary requirement (4.7) can be assured. However, in order to fully satisfy (4.6), the graph of the function h(t) should be plotted, in which the set of parameter satisfied (4.7) is used. The criterion for (4.6) being fully satisfied is set such as it has one maximum and minimum only in a period when $\sin\omega t=0$.

To solve the above mentioned problem, h(t) is approximated by g(t) such as both functions are continuous and periodic.

$$g(t) = k + a_1 \cos\omega t. \tag{4.13}$$

When any of the conditions (4.10), (4.11), (4.12) is satisfied, h(t) and g(t) would have obtained the same maxima and minima when $\sin\omega t = 0$. Hence, it can be inferred that the function h(t) be approximated by g(t) when their maxima and minima are respectively equal.

When $\cos\omega t = -1$, we have

$$\frac{2\gamma\omega^2}{(\lambda-\beta)^2} - \frac{2\alpha\gamma + 3\lambda}{\lambda - \beta} + \frac{\omega^2 + 3\alpha\lambda + 2\alpha^2\gamma}{\omega^2 + \alpha\lambda - \alpha\beta} = k - a_1. \tag{4.14}$$

When $\cos\omega t = 1$, we have

$$\frac{2\gamma\omega^2}{(\lambda+\beta)^2} - \frac{2\alpha\gamma + 3\lambda}{\lambda + \beta} + \frac{\omega^2 + 3\alpha\lambda + 2\alpha^2\gamma}{\omega^2 + \alpha\lambda + \alpha\beta} = k + a_1. \tag{4.15}$$

From (4.14) and (4.15) it has

$$a_1 = \frac{\alpha\beta}{\omega^2}\left[\frac{2\alpha^2\gamma - \alpha\lambda}{\alpha^2\gamma} - \frac{\omega^2 + 3\alpha\lambda + 2\alpha^2\gamma}{\omega^2 + 2\alpha\lambda + \alpha^2\gamma}\right], \tag{4.16}$$

$$k = -\frac{1}{\omega^2}\left[2\omega^2 + \frac{(2\alpha^2\gamma - \alpha\lambda)\alpha\lambda}{\alpha^2\lambda} - \frac{(\omega^2 + 3\alpha\lambda + 2\alpha^2\gamma)(\omega^2 + \alpha\lambda)}{\omega^2 + 2\alpha\lambda + \alpha^2\gamma}\right]. \tag{4.17}$$

Based on (4.16), (4.17) it yields

$$\frac{\omega^2}{\alpha\beta} = -\frac{k + a_1\dfrac{\lambda}{\beta}}{\dfrac{\dfrac{\lambda}{\beta}}{\dfrac{\lambda^2}{\beta^2} - 1} + a_1}, \tag{4.18}$$

$$\left(\frac{\dfrac{\lambda}{\beta}}{\dfrac{\lambda^2}{\beta^2} - 1} + a_1\right)\left(\frac{\omega^2}{\alpha\beta} + \frac{\lambda}{\beta} - 1\right)\left(\frac{\omega^2}{\alpha\beta} + \frac{\lambda}{\beta} + 1\right) = \frac{\omega^2}{\alpha\beta} + \frac{\lambda}{\beta}. \tag{4.19}$$

With known values of a_1, k and ω, the value of $\frac{\lambda}{\beta}$ and $\frac{\omega^2}{\alpha\beta}$ can be found by solving the set of equation (4.18), (4.19).

5. Algorithm for finding the approximated solution

Given that the following equation should be solved:

$$\ddot{u} + \omega^2(k + a_1 \cos\omega t)u = 0, \tag{5.1}$$

The following algorithm for finding its approximated solution should be followed:

- Solving the set of equation (4.18), (4.19) with the values of ω^2, k, a_1 given in (5.1), we obtain the values of $\frac{\lambda}{\beta}$, $\frac{\omega^2}{\alpha\beta}$.
- Checking the conditions (4.10), (4.11), (4.12). If none of them are satisfied, the approximated solution cannot be found by this proposed algorithm. If these conditions are satisfied we plot the graph of the function h(t) with the identified set of parameters.
- If the function h(t) does not posses a maximum and a minimum only when $\sin\omega t = 0$, the approximated solution cannot be found by this proposed algorithm.
- If the function h(t) satisfies the abovementioned condition, formula (3.1) with its respective parameters can be considered as the solution of (5.1).

Example 1.

Find the approximated solution of the following equation:

$$\ddot{u} - 4(0{,}00659 - 0{,}033415\cos 2t)u = 0\,. \tag{5.2}$$

Substitute

$$\omega = 2,\ k = -\,0{,}00659,\ a_1 = 0{,}033415, \tag{5.3}$$

into (4.18), (4.19), the results are

$$\frac{\lambda}{\beta} = 12,\ \frac{\omega^2}{\alpha\beta} = -\,3{,}361344538\,. \tag{5.4}$$

With the set of parameters (5.4), condition (4.10) is satisfied.

From (2.11) and (5.4) it can be inferred that:

$$\alpha\beta = -\,1{,}19;\ \alpha\lambda = -14{,}28;\ \alpha^2\gamma = 50{,}62558. \tag{5.5}$$

Based on (5.5), (5.3) the graphs of h(t), g(t) can be plotted as shown in Fig. 6. From there, it can be shown that the function h(t) has only a maxima and a minima when $\sin 2t = 0$. The functions h(t), g(t) have identical values of maxima and minima which are the approximation of each respective other. Therefore, it can be concluded that (3.2) with the conditions $u(0) = u_0, \dot{u}(0) = \dot{u}_0 = 0$ is the approximated solution of (5.1)

$$u = \frac{(\lambda + \beta)u_0}{\omega^2 + \alpha\lambda + \alpha\beta}\,\frac{\omega^2 + \alpha\lambda + \alpha\beta\cos\omega t}{\lambda + \beta\cos\omega t}\,. \tag{5.6}$$

The approximated solution (5.6) respective to the parameters identified in (5.5) has the form of

$$u = 1{,}348735u_0 \times \frac{10{,}28 + 1{,}19\cos 2t}{14{,}28 + 1{,}19\cos 2t}\,. \tag{5.7}$$

Substitute (5.7) into (5.2), it is observed that (5.7) is the approximated solution of (5.2).

Example 2.

Find the approximated solution of the following equation:

$$\ddot{u} + 4(0{,}001783728 - 0{,}007702649\cos 2t)u = 0 . \qquad (5.8)$$

Substitute

$$\omega = 2,\ k = -\,0{,}001783728,\ a_1 = 0{,}007702649, \qquad (5.9)$$

into (4.18), (4.19), the results are

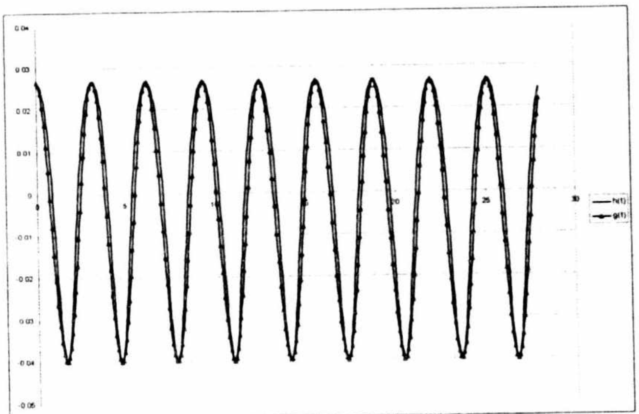
$$\frac{\lambda}{\beta} = 8{,}25,\ \frac{\omega^2}{\alpha\beta} = -\,0{,}535594672 . \qquad (5.10)$$

With the set of parameters (5.10), condition (4.10) is satisfied.

From (2.11) and (5.10) it can be inferred that

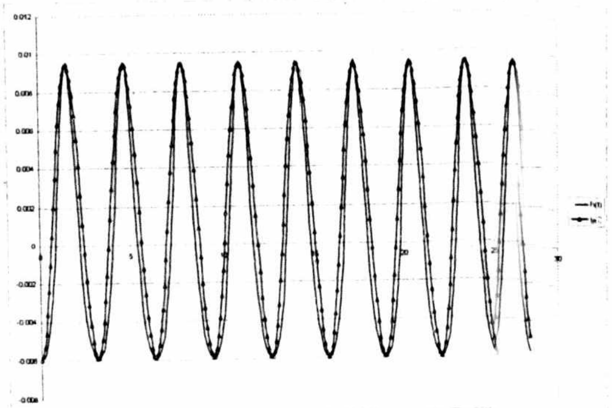
$$\alpha\beta = 7{,}48;\ \alpha\lambda = 61{,}71;\ \alpha^2\gamma = 938{,}04. \qquad (5.11)$$

Based on (5.9), (5.10) the graphs of h(t), g(t) can be plotted as shown in Fig. 7. From there, it can be shown that the function h(t) has only a maximum and a minimum when $\sin 2t = 0$. The functions h(t), g(t) have identical values of maxima and minima, which are the approximation of each respective other. Therefore, it can be concluded that (3.2) with the conditions $u(0) = u_0, \dot{u}(0) = \dot{u}_0 = 0$ is the approximated solution of (5.1)



ω	$\alpha\beta$	$\alpha\lambda$
2	-1.19	-14.28
$\alpha^2\gamma$	k	a_1
50.62558	-0.00659	0.033415

Fig.6 Graph of function h(t), g(t) with $\frac{\lambda}{\beta} = 12$



ω	$\alpha\beta$	α
2	7.48	[illegible]1.71
$\alpha^2\gamma$	k	a
938.04	0.001783728	-0.007702649

Fig.7. Graph of function h(t), g(t) with $\frac{\lambda}{\beta} = 8{,}25$

$$u_1 = \frac{(\lambda + \beta)u_0}{\omega^2 + \alpha\lambda + \alpha\beta} \cdot \frac{\omega^2 + \alpha\lambda + \alpha\beta\cos\omega t}{\lambda + \beta\cos\omega t} . \qquad 5.12)$$

The approximated solution (5.6) respective to the parameters identified in (5.5) has the form of

$$u_1 = 0{,}94534u_0 \times \frac{65{,}71 + 7{,}48\cos 2t}{61{,}71 + 7{,}48\cos 2t}. \tag{5.13}$$

Substitute (5.13) into (5.8), it is observed that (5.13) is the approximated solution of (5.8).

6. Discussion

In order to satisfy (4.6), the condition (4.7) plays only a role of preliminary requirement, but it is possible to establish a more precise condition however more complex in calculation.

The accuracy of above mentioned approximate method depends on the ratio $\frac{\lambda}{\beta}$.

From obtained results for u(t), the displacement w(x, t) of beams can be found

Acknowledgement. This research is completed with the financial support of the National Council for Natural Sciences.

References

1. Nguyen Van Dao, Tran Kim Chi, Nguyen Dung, "Chaotic phenomenom in a nonlinear Mathieu osillator", *Proceeding of the Seventh National Congress on Mechanics*, Hanoi, 18-20 December 2002, T.1, pp 40 – 49.
2. Weidenhammer, F. "*Biegeschwingugen des Stable unter axial pulsierender Zufallslast*". VDI-Brinchte Nr: 101 – 107, 1996.
3. Dao Huy Bich, Nguyen Đang Bich, "On the method solving a class of non-linear differential equations in mechanics", *Proceedings of the sixth National Congress on Mechanics*, Hanoi Dec, 1997, pp 11 – 17.
4. Granino A. Korn, Theresa M. Korn, *Mathematical handbook for scientist and engineers*, McGraw-Hill Book Company, 1968.