SPACE OF CONTINUOUS MAPS AND KN-NETWORKS

Dinh Huy Hoang, Nguyen Thieu Hoa

Department of Mathematics, Vinh University

Abstract. The aim of this paper is to establish conditions for which the space $C(X, Y)$ of continuous maps from space X into space Y has a point-countable kn-network. Also some properties related to point-countable covers of $C(X, Y)$ are proved.

1. Introduction

Since D. Burke, G.Gruenhage, E. Michael and Y. Tanaka [1,2,4] established the fundamental theory on point-countable covers in generalized metric spaces, many toplogists have investigated the point-countable covers with various characters, including k-networks, cs*-networks, p-k-networks,... were introduced and investigated. Recently, the above problem are considered in topological spaces. In this paper, we shall consider some conditions for spaces $C(X, Y)$ having a point-countable kn-networks and consider some properties of $C(X, Y)$ related to point-countable covers.

We assume that all spaces are regular and T_1. We begin with some basis definitions.

Let X be a space and $\mathcal{P}$ a cover of X. For every finite $\mathcal{F} \subset \mathcal{P}$, we denote by $\cup\mathcal{F}$ (respectively $\cap\mathcal{F}$) the set $\cup\{P : P \in \mathcal{F}\}$ (respectively $\cap\{P : P \in \mathcal{F}\}$).

1.1. Definition

(1) $\mathcal{P}$ is a *k-network* if, whenever $K \subset U$ with K compact and U open in$X$, then

$$K \subset \cup\mathcal{F} \subset U$$

for some finite $\mathcal{F} \subset \mathcal{P}$.

A compact (respectively open) k-network is a k-network consisting of compact subsets (respectively open subsets).

(2) $\mathcal{P}$ is a *network* if for every $x \in X$ and U open in X such that $x \in U$, then

$$x \in \cup\mathcal{F} \in U$$

for some finite $\mathcal{F} \subset \mathcal{P}$.

(3) $\mathcal{P}$ is *kn-network* if , whenever $K \subset U$ with K compact and U open in X, then

$$K \subset (\cup\mathcal{F})^o \subset \cup\mathcal{F} \subset U$$

for some finite $\mathcal{F} \subset \mathcal{P}$.

(4) $\mathcal{P}$ is called *point-countable* if for every $x \in X$, the set $\{P \in \mathcal{P} : x \in P\}$ is at most countable.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Definition 1.2. Let X be a space and $\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ be a family of subsets of X, satisfying following coditions for every $x \in X$,

(1) $x \in P$ for all $P \in \mathcal{P}_x$;

(2) If $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

$\mathcal{P}$ is called a *weak base* for X, if a subset G of X is open in X if and only if for each $x \in G$ there exists $P \in \mathcal{P}_x$ such that $P \subset G$.

A space X is a *gf-countable* space if X has a weak base $\mathcal{P}$ such that $\mathcal{P}_x$ is countable for every $x \in X$.

Definition 1.3. A space X is determined by a cover $\mathcal{P}$, or $\mathcal{P}$ determined X, if $U \subset X$ is open in X if and only if $U \cap P$ is open in P for every $P \in \mathcal{P}$.

If $\mathcal{P}$ is a collection of sets, then $\mathcal{P}^*$ (respectively $\mathcal{P}_*$) denotes $\{\cup F : \mathcal{F} \subset \mathcal{P}, \mathcal{F} \text{ finite}\}$ (respectively$\{\cap\mathcal{F} : F \subset \mathcal{P}, \mathcal{F} \text{ finite}\}$).

2. The main results

Let X and Y be spaces. Throught this paper by $\mathcal{U}$ we denote the topological base of Y and $C(X,Y)$ the space of continuous maps from X to Y equipped with the compact-open topology.

If $K \subset X$ and $U \subset Y$ then we denote

$$(K, U) = \{f \in C(X,Y) : f(K) \subset U)\}.$$

Theorem 2.1. *If X has a countable, compact k-network and Y has a point-countable base, then*

1) $C(X,Y)$ has a point-countable kn-network;

2) $C(X,Y)$ has a point-countable base;

3) $C(X,Y)$ is first countable.

Proof 1) Let $\mathcal{P}$ be a countable, compact k-network for X, $\mathcal{U}$ a point-countable base for Y and

$$\mathcal{V} = \{(G, U) : G \in \mathcal{P}^*, U \in \mathcal{U}\}.$$

We first prove that $\mathcal{V}$ is cover of $C(X,Y)$. Let $f \in C(X,Y)$ and $x \in X$. There exists $U \in \mathcal{U}$ such that $f(x) \in U$.

By the continuity of f, $f^{-1}(U)$ is open in X. Since $\{x\} \subset f^{-1}(U)$ and $\mathcal{P}$ is a k-network, it follows that there exists $P \in \mathcal{P}$ such that

$$\{x\} \subset P \subset f^{-1}(U).$$

This means that $f(P) \subset U$ and hence $f \in (P, U) \in \mathcal{V}$. Thus $\mathcal{V}$ is a cover of $C(X,Y)$ and so is $\mathcal{V}_*$.

We now show that $\mathcal{V}_*$ is a kn-network. Suppose $K \subset W$, where K is compact and W is open in $C(X,Y)$. If $f \in K$, then there exists the neighborhood V of f in $C(X,Y)$ such that

$$V = \bigcap_{i=1}^{k}(K_i, U_i) \subset W,$$

where K_i is compact in X and $U_i \in \mathcal{U}$ for $i = 1,..,k$.

Let $f \in V$, $K_i \subset f^{-1}(U_i)$ for $i = 1,...,k$. Since K_i is compact and $\mathcal{P}$ is a k-network, there exists $P_{1i}, P_{2i}, ..., P_{m_i i} \in \mathcal{P}$ such that

$$K_i \subset \bigcup_{j=1}^{m_i} P_{ji} \subset f^{-1}(U_i) \quad \text{for} \quad i = 1,...,k.$$

This yields

$$f(K_i) \subset f(\bigcup_{j=1}^{m_i} P_{ji}) \subset U_i \quad \text{for} \quad i = 1,...,k$$

and

$$f \in (\bigcup_{j=1}^{m_i} P_{ji}, U_i) \subset (K_i, U_i) \quad \text{for} \quad i = 1,...,k.$$

Let

$$P_i = (\bigcup_{j=1}^{m_i} P_{ji}, U_i) \quad \text{for} \quad i = 1,...,k$$

and

$$\tilde{P}_f = \bigcap_{i=1}^{k} P_i.$$

Then $P_i \in \mathcal{V}$, $\tilde{P}_f \in \mathcal{V}_*$ and

$$f \in \tilde{P}_f \subset \cap(K_i, U_i) = V \subset W.$$

Since K is compact and $\tilde{P}_f$ is open, there exist $f_1, f_2, ..., f_n \in K$ such that

$$K \subset (\bigcup_{i=1}^{n} \tilde{P}_{f_i})^o = \bigcup_{i=1}^{n} \tilde{P}_{f_i} \subset W.$$

Since$\tilde{P}_{f_i} \in \mathcal{V}_*$ for $i = 1,..n$, $\mathcal{V}_*$ is a kn-network for $C(X,Y)$.

It remains to show that $\mathcal{V}_*$ is point-countable. It is sufficient to prove that $\mathcal{V}$ is point-coutable. Let $f \in C(X,Y)$, $G \in \mathcal{P}^*$ and

$$\mathcal{F}_G = \{U \in \mathcal{U} : f \in (G,U)\}.$$

Then $\mathcal{F}_G \subset \mathcal{V}$. If $\mathcal{F}_G$ is uncountable, then there exists a uncountable subset $\mathcal{U}'$ of $\mathcal{U}$ such that

$$f \in (G,U) \quad \text{for every} \quad U \in \mathcal{U}'.$$

Hence, if $x \in G$, then $f(x) \in U$ for every $U \in \mathcal{U}'$. Since $\mathcal{U}$ is point-countable, we have a contradiction. It follows that $\mathcal{F}_G$ is countable. Since $\mathcal{P}$ is countable, $\mathcal{P}^*$ is countable. This yields the set $\{\mathcal{F}_G : G \in \mathcal{P}^*\}$ is countable and hence f is in at most countable many elements of $\mathcal{V}$. Thus, $\mathcal{V}$ is point- countable.

2) Since $\mathcal{V}_*$ is a open kn-network , $\mathcal{V}_*$ is a base for $C(X,Y)$. Thus $\mathcal{V}_*$ is a point-countable base for $C(X,Y)$.

3) Let $f \in C(X,Y)$ and

$$\mathcal{V}_f = \{W \in \mathcal{V}_* : f \in W\}.$$

Since $\mathcal{V}_*$ is point-countable, $\mathcal{V}_f$ is countable. Because $\mathcal{V}_*$ is a open kn-network, we conclude that $\mathcal{V}_f$ is a neighborhood base at f in $C(X,Y)$. Hence $C(X,Y)$ is a first countable space.

Remark 2.2. It is easy to show that the cover $\mathcal{P}$ of any space X is a point-countable base if and only if $\mathcal{P}$ is a point-countable, open kn-network. But a space with a point-coutable kn-network can not be a space with a point-countable base [8].

Corollary 2.3 If X has a countable, compact k-network and Y has the point-countable kn-network $\mathcal{G}$ such that if $y \in U$ with U open in X, then

$$y \in (\cup\mathcal{F})^o \subset \cup\mathcal{F} \subset U \quad \text{and} \quad y \in \cap\mathcal{F}$$

for some finite $\mathcal{F} \subset \mathcal{G}$. In particular $C(X,Y)$ has a point-countable kn-network.

Proof. By Theorem 2.1, it is sufficient to show that Y has a point-countable base.

For every $y \in Y$, put

$$\mathcal{G}_y = \{G \in \mathcal{G} : y \in G\},$$

$$\tilde{\mathcal{G}}_y = \{G^o : G \in (\mathcal{G}_y)^*\}$$

and

$$\tilde{\mathcal{G}} = \bigcup_{y \in Y} \tilde{\mathcal{G}}_y.$$

We will show that $\tilde{\mathcal{G}}$ is point-countable base for Y. Let $y \in Y$ and V be a neighborhood of y in Y. Then, there is a finite subset $\mathcal{F}$ of $\mathcal{G}$ such that

$$y \in (\cup\mathcal{F})^o \subset \cup\mathcal{F} \subset V \quad \text{and} \quad y \in \cap\mathcal{F}.$$

Put $G = \cup\mathcal{F}$. We have $G \in (\mathcal{G}_y)^*$ and

$$y \in G^o \subset G \subset V.$$

By $G^o \in \tilde{\mathcal{G}}$, $\tilde{\mathcal{G}}$ is a base of Y.

Since $\mathcal{G}$ is point-countable, $\mathcal{G}_y$ is countable. This yields $(\mathcal{G}_y)^*$ is countable and so is $\tilde{\mathcal{G}}_y$. Hence, $\tilde{\mathcal{G}}$ is point-countable.

Lemma 2.4. *If X is determined by a cover $\mathcal{P}$ and $\mathcal{P}$ is a refinement of $\mathcal{P}'$, then X is determined by $\mathcal{P}'$.*

Proof. Let $U \subset X$ such that $U \cap P'$ is open in P' for every $P' \in \mathcal{P}'$. We show that U is open in X. Let $P \in \mathcal{P}$. Then, since $\mathcal{P}$ is a refinement of $\mathcal{P}'$, there is a $P' \in \mathcal{P}'$ such that $P \subset P'$. Since $U \cap P'$ in P' is open, there exists G open in X such that $U \cap P' = G \cap P'$. Hence

$$U \cap P = U \cap (P' \cap P) = (U \cap P') \cap P = (G \cap P') \cap P = G \cap P.$$

It follows that $U \cap P$ is open in P for every $P \in \mathcal{P}$. Since X is determined by $\mathcal{P}$, U is open in X. Thus X is determined by $\mathcal{P}'$.

Theorem 2.5. *Let X be a locally compact space, $\mathcal{P} = \{P \subset X : P \text{is open and} \overline{P} \text{compact in } X\}$ and $\mathcal{V} = \{(P, U) : P \in \mathcal{P}, U \in \mathcal{U}\}$. Then*

1) $C(X,Y)$ is determined by $\mathcal{V}$;

2) $\mathcal{V}_*$ is a kn-network for $C(X,Y)$;

3) If X is a second countable and Y has a point-countable base, then $\mathcal{V}, \mathcal{V}_*$ are point-countable and $C(X,Y)$ is a countable gf-space.

Proof.

1) Put

$$\mathcal{V}' = \{(\overline{P}, U) : P \in \mathcal{P}, U \in \mathcal{U}\}.$$

It is obvious that $\mathcal{V}'$ is a refinement of $\mathcal{V}$ and $(\overline{P}, U)$ open in $C(X,Y)$ for every $(\overline{P}, U) \in \mathcal{V}'$. Hence, by Lemma 2.4, it is sufficient to show that $C(X,Y)$ is determined by $\mathcal{V}'$. Let W be an open subset in $C(X,Y)$. Then $W \cap V$ is open in V for every $V \in \mathcal{V}'$. Conversly, assume that $W \subset C(X,Y)$ such that $W \cap V$ is open in V for every $V \in \mathcal{V}'$. Then there exists an open subset G in $C(X,Y)$ such that

$$G \cap V = W \cap V.$$

But V is open in $C(X,Y)$, since $V \in \mathcal{V}'$. Hence $G \cap V$ open in $C(X,Y)$. Since $\mathcal{V}'$ is a cover of $C(X,Y)$, we get

$$W = \bigcup_{V \in \mathcal{V}'} (W \cap V) = \bigcup_{V \in \mathcal{V}'} (G \cap V).$$

Thus W is open in $C(X,Y)$ and hence, $C(X,Y)$ is determined by $\mathcal{V}'$.

2) Let K be a compact subset of $C(X,Y)$ and let W be an open subset of $C(X,Y)$ such that $K \subset W$. Then, for every $f \in K$, there exists a neighborhood V of f in $C(X,Y)$ such that

$$V = \bigcap_{i=1}^{m} (K_i, U_i) \subset W,$$

where K_i is compact in X and $U_i \in \mathcal{U}$ for $i = 1, ..., m$. Since X is regular, locally compact and K_i is compact, there exists $V_i \in \mathcal{P}$ such that

$$K_i \subset V_i \subset \overline{V_i} \subset f^{-1}(U_i) \quad \text{for} \quad i = 1, 2, .., m.$$

This yields

$$f \in \bigcap_{i=1}^{m} (\overline{V_i}, U_i) \subset \bigcap_{i=1}^{m} (V_i, U_i) \subset \bigcap_{i=1}^{m} (K_i, U_i).$$

Put

$$P_f = \bigcap_{i=1}^{m} (U_i, V_i) \quad \text{and} \quad P'_f = \bigcap_{i=1}^{m} (\overline{V_i}, U_i).$$

Then $P_f \in \mathcal{V}_*$, P'_f is open in $C(X, Y)$ and $P'_f \subseteq P_f$. By the compactness of K, there exists $f_1, f_2, ..., f_n \in K$ such that

$$K \subset \bigcup_{i=1}^{n} P'_{f_i} \subset \bigcup_{i=1}^{n} P_{f_i} \subset W. \tag{1}$$

As P'_{f_i} is open for every $i = 1, .., n$, we have

$$K \subset \Big(\bigcup_{i=1}^{n} P_{f_i}\Big)^{o} \subset \bigcup_{i=1}^{n} P_{f_i} \subset W.$$

Hence, $\mathcal{V}_*$ is a kn-network for $C(X, Y)$.

3) Let $\mathcal{B}$ be a countable base of X and let $x \in B$ with $B \in \mathcal{B}$. Since X is locally compact and regular, there is a $P \in \mathcal{P}$ such that

$$x \in P \subset \overline{P} \subset B.$$

Hence, we can assume that $\mathcal{P}$ is countable. By a similar argument as the proof of Theorem 2.1, we conclude that $\mathcal{V}$ is point-countable and hence is so $\mathcal{V}_*$.

We now show that $(\mathcal{V}')_*$ is a weak base for $C(X, Y)$. For every $f \in C(X, Y)$ by $\mathcal{V}'_f$ we denote the set $\{Q \in (\mathcal{V}')_* : f \in Q\}$. Then, we have

$$(\mathcal{V}')_* = \cup\{(\mathcal{V}')_f : f \in C(X, Y)\}.$$

It follows from (1) that $(\mathcal{V}')_*$ is an open k-network for $C(X, Y)$. Since $(\mathcal{V}')_*$ is a k-network and it is closed under finite intersections, $\mathcal{V}'_f$ is a network and it is closed under finite intersections. Let W be a subset of $C(X, Y)$ such that for every $f \in W, Q \subset W$ for some $Q \in \mathcal{V}'_f$. From $Q \in \mathcal{V}'_f$, we can suppose

$$Q = \bigcap_{i=1}^{n} (\overline{P}_i, U_i),$$

where $P_i \in \mathcal{P}, U_i \in \mathcal{U}$ for every $i = 1, ..., n$. By the compactness of $\overline{P_i}$ and the openning of U_i for $i = 1, ..., n$, Q is open in $C(X,Y)$ and hence, so is W. This yields $(\mathcal{V}')_*$ is a weak base for $C(X,Y)$. Since $(\mathcal{V})_*$ is point-countable, $(V')_*$ is point-countable. Hence, $\mathcal{V}'_f$ is countable for every $f \in C(X,Y)$. Thus $C(X,Y)$ is a countable gf-space.

References

1. D.K. Burke and E. Michael, On a theorem of V.V Flippov, *Isarel J. Math.* **11**(1972),394-397.
2. D.K. Burke and E. Michael, On certain poin-countable covers, *Pacific Journal of Math.* **64**(1)(1976),79-92.
3. H. Chen, *Compact-covering maps and k-networks,* preprint (2003).
4. G. Gruenhage, E. Michael and Y. Tanaka, Spaces determined by point-countable covers, *Pacific Journal of Math.* **113**(2)(1984) 303-332.
5. P.O', Meara, On paracompactness in function spaces with the compact-open topology, *Proc. Amer Math. Soc,* **29**(1971), 183-189.
6. Y. Tanaka, Point-countable covers and k-networks, *Topology-proc.* **12**(1987),327-349.
7. Y. Tanaka, *Theory of k-networks II,* Q and A in General Topology,**19**(2001), 27-46.
8. P.Yan and S. Lin, Point-countable k-networks, cs*-network and α_4-spaces, *Topology Proc,* **24**(1999), 345-354.