ON THE WEAK LAW OF LARGE NUMBERS FOR BLOCKWISE INDEPENDENT RANDOM VARIABLES

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Abstract. In this paper we give conditions of the weak law of large numbers for sequences of random variables which are blockwise independent. Some well-known results are extended.

1. Introduction and notations

In [2] and [5] it was shown that some properties of independent sequences of random variables can be applied to the sequences consisting of independent blocks. Particularly, it was proved in [5] that if $(X_i)_{i=1}^{\infty}$ is a independent sequence in blocks $[2^k, 2^{k+1})$, $EX_i = 0 (i \in N^*)$, then it satisfies the Kolmogorov's theorem: the condition $\sum_{i=1}^{\infty}(EX_i^2)i^{-2} < \infty$ implies the strong law of large numbers (s.l.l.n.), i.e.,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = 0 \text{ a.s.}$$

In [3], Gaposhkin found down sufficient conditions in which the s.l.l.n. is fulfilled for a blockwise independent sequence with arbitrary blocks. The strong law of large numbers for two-dimensional arrays of blockwise independent random variables was studied in [7].

In recent years, the weak law of large numbers has been theme of studying of many mathematicians. The aim of this paper is to establish the sufficient conditions in which the weak law of large numbers is fulfilled for blockwise independent sequences with arbitrary blocks. Furthermore, we extend some well-known results in this area.

Let $1 = \omega(1) < \omega(2) < \cdots < \omega(n) < \cdots$ be a sequence of positive integers and $\Delta_k = [\omega(k), \omega(k+1))$. We say that the sequence $\{X_i, i \in N^*\}$ of random variables is blockwise independent (blockwise orthogonal) with respect to blocks Δ_k, if, for every fix k, the sequence $\{X_i, i \in \Delta_k\}$ is independent (orthogonal). Denote

$$N_m = min\{N|\omega(N) \geq 2^m\}, \quad m = 0, 1, ...,$$
$$s_m = N_{m+1} - N_m + 1,$$
$$\varphi(i) = s_m \text{ if } i \in [2^m, 2^{m+1}).$$

Let $\Delta^{(m)} = [2^m, 2^{m+1}), \Delta_k^{(m)} = \Delta_k \cap \Delta^{(m)}$ for $m \geq 0$ and $k \geq 1$, $\delta_k^{(m)} = \sum_{k\in\Delta_k^{(m)}} X_k$. For each $m \geq 1$, we assume $\Delta_k^{(m)} \neq \emptyset$ for $p_m \leqslant k \leqslant q_m$. Since $\omega(N_m - 1) < 2^m, \omega(N_m) \geq 2^m, \omega(N_{m+1}) \geq 2^{m+1}$ for each $m \geq 1$, the number of nonempty blocks $\Delta_k^{(m)}$ is not larger than $s_m = N_{m+1} - N_m + 1$.

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2. Lemmas

To prove the main theorem

Lemma 2.1. *(Doob's Inequality) If* $\{X_i, \mathcal{F}_i\}_{i=1}^N$ *is a martingale difference,* $E|X|^p < \infty$ $(1 < p < \infty)$, *then*

$$E|\max_{k\leqslant N}\sum_{i=1}^{k} X_i|^p \leqslant (\frac{p}{p-1})^p E|\sum_{i=1}^{N} X_i|^p.$$

Lemma 2.2. *If* $q > 1$ *and* $\{x_n, n \geq 0\}$ *is a sequence of constants such that* $\lim_{n\to\infty} x_n = 0$, *then*

$$\lim_{n\to\infty} q^{-n}\sum_{k=0}^{n} q^{k+1}x_k = 0.$$

Proof. Let $s = q + \sum_{i=0}^{\infty} q^{-i}$. For any $\epsilon > 0$, there exists k_0 such that $|x_k| < \frac{\epsilon}{2s}$ for all $k \geq k_0$. Since $\lim_{n\to\infty} q^{-n} = 0$, so, there exists $n_0 \geq k_0$ such that for all $n \geq n_0$, we have

$$|q^{-n}\sum_{k=0}^{k_0} q^{k+1}x_k| < \frac{\epsilon}{2}.$$

It follows that

$$\begin{aligned}|q^{-n}\sum_{k=0}^{n} q^{k+1}x_k| &\leqslant |q^{-n}\sum_{k=0}^{k_0} q^{k+1}x_k| + |q^{-n}\sum_{k_0+1}^{n} q^{k+1}x_k| \\ &\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2s}(q + 1 + \frac{1}{q} + \cdots) \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for all } n \geq n_0.\end{aligned}$$

This completes the proof.

3. Main results

With the notations and lemmas listed above, main results may now be established.

Theorem 3.1. *Assumme that* $\{X_i\}_{i=1}^{\infty}$ *is a blockwise independent sequence such that* $EX_i = 0$, $EX_i^2 = c_i^2$, *and let* $\lim_{n\to\infty}\frac{1}{2^{2n+2}}\sum_{i=2^n}^{2^{n+1}-1} c_i^2\varphi(i) = 0$, *then*

$$\frac{\sum_{i=1}^{2^m-1} X_i}{2^m} \xrightarrow{\mathbb{P}} 0 \text{ as } m \longrightarrow \infty, \tag{3.1}$$

moreover

$$\frac{\sum_{l=1}^{m}\sum_{k=p_l}^{q_l}|\delta_k^{(m)}|}{2^m} \xrightarrow{\mathbb{P}} 0 \text{ as } m \longrightarrow \infty. \tag{3.2}$$

Proof.

Let us first estimate the norm in L^2. Put

$$\sigma_m = 2^{-m-1} \sum_{k=2^m}^{2^{m+1}-1} X_k, \quad \sigma'_m = 2^{-m-1} \sum_{k=p_m}^{q_m} |\delta_k^{(m)}|.$$

We have

$$|\sigma_m| \leqslant 2^{-m-1} \sum_{k=p_m}^{q_m} |\delta_k^{(m)}| = \sigma'_m.$$

Note that every blockwise independent sequence, is a blockwise orthogonal sequence provided $EX_i = 0$, therefore

$$\|\sigma_m\|^2 \leqslant \|\sigma'_m\|^2 \leqslant 2^{-2m-2} s_m \sum_{k=p_m}^{q_m} \|\delta_k^{(m)}\|^2$$

$$\leqslant 2^{-2m-2} s_m \sum_{k=2^m}^{2^{m+1}-1} c_k^2 = 2^{-2m-2} \sum_{k=2^m}^{2^{m+1}-1} c_k^2 \varphi(i) \longrightarrow 0 \quad (m \longrightarrow \infty)$$

which yields

$$\lim_{m\to\infty} \|\sigma_m\| = 0, \ \lim_{m\to\infty} \|\sigma'_m\| = 0.$$

On the other hand

$$\Big\|2^{-m} \sum_{k=1}^{2^m-1} X_k\Big\| = \Big\|2^{-m} \sum_{k=0}^{m-1} 2^{k+1} \sigma_k\Big\| \leqslant 2^{-m} \sum_{k=0}^{m-1} 2^{k+1} \|\sigma_k\|.$$

Using the the statement $\lim\limits_{m\to\infty} \|\sigma'_m\| = 0$, Lemma 2.2 and Markov's inequality, we reach the claim (3.1).

In the same way, using the statement $\lim\limits_{m\to\infty} \|\sigma'_m\| = 0$, Lemma 2.2 and Markov's inequality we prove (3.2)

The following theorem is a extension of the Markov's w.l.l.n.

Theorem 3.2. *If $\{X_i\}_{i=1}^{\infty}$ is a blockwise independent sequence, $EX_i = 0$, $EX_i^2 = c_i^2$, then the condition*

$$\lim_{n\to\infty} \frac{1}{2^{2n+2}} \sum_{i=2^n}^{2^{n+1}-1} c_i^2 \varphi(i) = 0 \tag{3.3}$$

implies w.l.l.n

$$\frac{\sum_{i=1}^n X_i}{n} \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \longrightarrow \infty \tag{3.4}$$

Proof. For $\Delta_k^{(m)} \neq \emptyset$, we set

$$r_k^{(m)} = \min\{r : r \in \Delta_k^{(m)}\};$$

$$r_k^{'(m)} = \max\{r : r \in \Delta_k^{(m)}\};$$

$$\gamma_k^{(m)} = \sup_{n\in\Delta_k^{(m)}} \Big| \sum_{i=r_k^{(m)}}^{n} X_i \Big|;$$

$$\gamma_m = 2^{-m-1} \sup_{p_m \leqslant k \leqslant q_m} \gamma_k^{(m)}.$$

Firstly, we will prove that

$$\gamma_m \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad m \longrightarrow \infty. \tag{3.5}$$

Indeed, using Lemma 2.1 for the martingale

$$S_n = \sum_{i=r_k^{(m)}}^{n} X_i \ (r_k^{(m)} \leqslant n \leqslant r_k^{'(m)})$$

we have

$$\|\gamma_k^{(m)}\|^2 \leqslant 4 \Big\| \sum_{i\in\Delta_k^m} X_i \Big\|^2 = 4 \sum_{i\in\Delta_k^m} \|X_i\|^2 = 4 \sum_{i\in\Delta_k^m} c_i^2.$$

Consequently,

$$\|\gamma_m\|^2 \leqslant 2^{-2m-2} \sum_{k=p_m}^{q_m} \|\gamma_k^{(m)}\| \leqslant 4 \sum_{i=2^m}^{2^{m+1}-1} \frac{c_i^2}{i^2} \leqslant 2^{-2m-2} \sum_{i=2^m}^{2^{m+1}-1} c_i^2 \longrightarrow 0 \quad (n \to \infty)$$

Using the Markov's inequality, we get

$$\gamma_m \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad m \longrightarrow \infty.$$

Now assume that $n \in \Delta_k^{(m)}$. Using (3.5) and Theorem 3.1 we obtain

$$\begin{aligned}
0 &\leqslant \left| \frac{X_1 + X_2 + \cdots + X_n}{n} \right| \\
&\leqslant \left| \frac{X_1 + X_2 + \cdots + X_{2^m-1}}{2^m} \right| + \left| \frac{X_{2^m} + X_{2^m+1} + \cdots + X_n}{2^m} \right| \\
&\leqslant \left| \frac{X_1 + X_2 + \cdots + X_{2^m-1}}{2^m} \right| + \frac{\sum_{k=p_m}^{q_m} |\delta_k^{(m)}| + 2^{m+1}\gamma_m}{2^m} \\
&= \left| \frac{X_1 + X_2 + \cdots + X_{2^m-1}}{2^m} \right| + 2\sigma'_m + 2\gamma_m \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \longrightarrow \infty
\end{aligned}$$

(Because $m \to \infty$ as $n \to \infty$.)

This yields (3.4) and the proof is completed

Corollary 3.3. *If $\{X_i\}_{i=1}^{\infty}$ is a blockwise independent sequence, $EX_i = 0$, $EX_i^2 = c_i^2$ and $\lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} c_i^2 \varphi(i) = 0$, then*

$$\frac{\sum_{i=1}^{n} X_i}{n} \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Proof. Clearly, from the condition

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} c_i^2 \varphi(i) = 0$$

follows

$$\lim_{n\to\infty} \frac{1}{2^{2n+2}} \sum_{i=2^n}^{2^{n+1}-1} c_i^2 \varphi(i) = 0,$$

which completes the proof.

Corollary 3.4. *If $\omega(k) = 2^k$, then for a Δ_k-independent sequence (X_i) with $EX_i = 0$ and $\lim\limits_{n\to\infty} \frac{1}{2^{2n+2}} \sum_{i=2^n}^{2^{n+1}-1} c_i^2 = 0$, we have*

$$\frac{\sum_{i=1}^{n} X_i}{n} \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Proof. It is easy to see that the case $\varphi(i) = 1$ and the statement of Corollary 2.3. leads to $\lim\limits_{n\to\infty} \frac{1}{2^{2n+2}} \sum_{i=2^n}^{2^{n+1}-1} c_i^2 \varphi(i) = 0.$

Similarly to corollaries 2.3 and 2.4, we get

Corollary 3.5. *(Markov's weak law of large numbers) If $\{X_i\}_{i=1}^{\infty}$ is a independent sequence of random variables with $EX_i = 0$, $EX_i^2 = c_i^2$ and $\lim\limits_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} c_i^2 = 0$, then*

$$\frac{\sum_{i=1}^{n} X_i}{n} \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \longrightarrow \infty.$$

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