MESH-INDEPENDENCE PRINCIPLE AND CAUCHY PROBLEM FOR DIFFERENTIAL ALGEBRAIC EQUATIONS

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Abstract. In this paper, we apply the mesh-independence principle to differential algebraic equations.

1. Introduction

It was shown by the mesh-independence principle that if the Newton's method is used to analyse a nonlinear equation between some Banach spaces and some finite-dimensional discretization of that equation then the discretized process is asymptotically the same as that for the original iteration. As the result, the number of iterations steps needed for two processes to converge within a given tolerance is basically the same [1]. Consider the following equation:

$$F(z) = 0 \tag{1.1}$$

where, F is a nonlienar operator between Banach spaces $A, \hat{A}$. The Newton's method is defined as follow:

$$z_{n+1} = z_n - \left[F'(z_n)\right]^{-1} F(z_n), \quad n = 0, 1, 2, \ldots \tag{1.2}$$

Under certain conditions, equation (1.2) yields a sequence converging quadratically to a solution z^* of equation (1.1). Normally, the formal procedure defined by equation (1.2) is not suitable in infinite-dimensional spaces. Thus, in practice equation (1.1) is replaced by a family of discretized equations:

$$\Phi_h(\xi) = 0 \tag{1.3}$$

where h is some real number and Φ_h is a nonlinear operator between finite-dimensional spaces $A_h, \hat{A}_h$. If we define Δ_h to be the bounded linear operator $\Delta_h : A \to A_h$, then equation (1.3), under some appropriate assumprions, have solutions ξ^* which are the limit of the Newton sequence applied to equation (1.3). These solutions are obtained as follows:

$$\xi^* = \Delta_h z^* + 0(h^p)$$

and are started at $\Delta_h z_0$ that is:

$$\xi_0^h = \Delta_h z_0 \quad \xi_{n+1}^h = \xi_n^h - \left[\Phi_h'(\xi_n^h)\right]^{-1} \Phi_h(\xi_n^h), \quad n = 0, 1, 2, \ldots \tag{1.4}$$

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Observations in many computations indicates that for a sufficiently small h there is at most a difference of 1 between the number of steps needed for the two processes of equations (1.2) and (1.4) to converge within a given tolerance $\varepsilon > 0$. That is one aspect of the mesh-independence principle of Newton's method. Another aspect is that, if discretization satisfied certain conditions then:

$$\begin{aligned} \xi_n^h - \xi_n^* &= \Delta_h(z_n - z^*) + O(h^p) \\ \xi_{n+1}^h - \xi_n^h &= \Delta_h(z_{n+1} - z_n) + O(h^p) \\ \Phi_h(\xi_n^h) &= \hat{\Delta}_h F(z_n) + O(h^p) \end{aligned} \tag{1.5}$$

The aim of this paper is to apply the mesh-independence principle to differential algebraic equations. The paper consists of two sections dicussing the Newton's method for continuous problems and the Newton's method for discretized problems.

2. The mesh-independece principle

2.1. Newton method for continuous problems

$$\begin{cases} x'(t) = & y(x(t), y(t)) \\ y(t) = & f(x(t), y(t)) \\ y(0) = & y_0; x(0) = x_0 \\ y_0 = & f(x_0, y_0) \end{cases} \tag{2.1}$$
$$t \in [0, T] = J$$
$$x \in \mathbb{R}^m, y \in R^{n-m}, g : \mathbb{R}^n \to \mathbb{R}^m, \quad f : R^n \to R^{n-m}.$$

Without loss of generalty, we may assume that $y_0 = \theta; x_0 = \theta$.

The norm in $\mathbb{R}^s$ spaces on $\mathbb{R}^{p\times q}$ spaces will be dentoted by the same symbol $\| \quad \|$, where $p, q, s \in N$ $\forall x \in C(J, S^s) : \|x\|_\infty = \max\limits_{0\leqslant t\leqslant T} |x(t)|$

$$\begin{aligned} Z &:= \left\{ z = (x, y) \in C(J, \mathbb{R}^n) : x \in C^1(J, \mathbb{R}^n), \;\; x(0) = \theta, y(0) = \theta \right\} \\ \|z\| &:= \|z\|_\infty + \|x'\|_\infty \\ W &:= C(J, \mathbb{R}^n) \end{aligned}$$

Hypotheses

H_1) (1.1) has a solution $z^* = (x^*, y^*) \in Z$ such that

$$G := (g, f)^T \in C^1(U(z^*, \rho))$$

where

$$U := U(z^*, \rho) = \{(x,y) \in \mathbb{R}^n : \exists t \in J : |x - x^*(t)| \leqslant \rho, \ \ |y - y^*(t)| \leqslant \rho\}$$

$H_2)$ $\left|\frac{\partial g}{\partial x}(z)\right| \leqslant \alpha,\ \left|\frac{\partial g}{\partial y}(z)\right| \leqslant \beta,\ \left|\frac{\partial f}{\partial x}(z)\right| \leqslant \gamma,\ \left|\frac{\partial f}{\partial y}(z)\right| \leqslant \delta\ \forall z \in U(z^*, \rho)$, where $\delta < 1$.

$$B := B(z^*, \rho) = \{z \in Z : \|z - z^*\| \leqslant \rho\} \quad \forall z \in B,\ \forall h = (h_1, h_2)^T \in Z,$$

$$F(z) := \begin{bmatrix} \dot{x} - g(x,y) \\ y - f(x,y) \end{bmatrix}$$

$F : Z \to W$; that is

$$F'(z).h = \begin{bmatrix} h_1' - \frac{\partial g}{\partial x}.h_1 - \frac{\partial g}{\partial y}h_2 \\ h_2 - \frac{\partial f}{\partial x}.h_1 - \frac{\partial f}{\partial y}h_2 \end{bmatrix}$$

The Newton's method for problem (1.1):

$$z_{k+1} = z_k - [F'(z_h)]^{-1}.F(z_k), \text{ with } h^{(k)} := \left(h_1^{(k)}, h_2^{(k)}\right)^T$$

$$\Leftrightarrow \begin{bmatrix} h_1^{(k)} - \frac{\partial g}{\partial x}(x_k, y_k)h_1^{(k)} - \frac{\partial g}{\partial y}(x_k, y_k)h_2^{(k)} \\ h_2^{(k)} - \frac{\partial f}{\partial x}(x_k, y_k).h_1^{(k)} .- \frac{\partial g}{\partial y}(x_k, y_k)h_2^{(k)} \end{bmatrix} = -\begin{bmatrix} x_k' - g(x_k, y_k) \\ y_k - f(x_k, y_k) \end{bmatrix} \qquad (2.2)$$

By the Gronwall's inequality and on the hypotheses: Let g, f has continuos Lipschits on the open domain U the $\frac{\partial g}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ by l we have the following attraction theorem for Newton's method described by (2.2)

Theorem 2.1. *Suppose that $(H_1), (H_2)$ are fulfilled. Then*

1) $\forall z \in B, \exists\ [F'(z)]^{-1}$ and $\|[F'(z)]^{-1}\| \leqslant C$
2) $\forall z, \overline{z} \in B : \|F'(z) - F'(\overline{z})\ \| \leqslant l\ \ \|z - \overline{z}\|$
3) *For* $\forall z_0 \in B^* := B[z^*, r^*], r^* = \frac{2}{3C.l}$

The Newton's method converges to $z^* : (x^*, y^*)$.

2.2 Newton method for discretized problems

With

$$h := \frac{T}{N} G_h := \{t_i = ih, i = \overline{0, N}\}, \quad \dot{G}_h = G_h \backslash \{0, T\}$$

$$Z_h = \{\zeta = (\zeta_0, \ldots, \zeta_N) : \zeta_0 = 0,\ \zeta_i = (\xi_i, \eta_i),\ \xi_i \in R^m, \eta_i \in R^{n-m}\ (i = \overline{0, N})\}$$

$$\|\zeta\|_h = \max_{0 \leqslant i \leqslant N} |\xi_i| + \max_{0 \leqslant i \leqslant N-1} \left|\frac{\xi_{i+1} - \xi_i}{h}\right| = \max_{0 \leqslant i \leqslant N} |\xi_i| + \max_{0 \leqslant i \leqslant N-1} \left|\frac{\Delta \xi_i}{h}\right|$$

$$W_h = \{\eta = (\eta_0, \ldots, \eta_{N-1}), \eta_i \in R^n\ (i = \overline{0, N})\};\ \|\eta\|_h = \max_{0 \leqslant i \leqslant N-1} |\eta_i|$$

When the deseretization of (2.1)

$$\begin{cases} \xi_{k+1} = \xi_k + \dfrac{h}{2}(g_{k+1} + g_k) \\ \eta_{k+1} = f(\xi_{k+1}, \eta_{k+1}) \\ k = \overline{0, N-1}, \xi_0 = 0, \eta_0 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \xi_{k+1} - \xi_k - \dfrac{h}{2}(g_{k+1} + g_k) = 0 \\ \eta_{k+1} - _{k+1} = 0 \qquad k = \overline{0, N-1} \\ \xi = 0, \ \eta = 0 \end{cases}$$

$$\forall \zeta \in Z_h : \Phi_h(\zeta) := \begin{bmatrix} \xi_{k+1} - \xi_k - \dfrac{h}{2}(g_{k+1} + g_k) \\ \eta_{k+1} - f(\xi_{k+1}, \eta_{k+1}) \end{bmatrix}_{k=0}^{N-1} \qquad \xi_0 = 0, \eta_0 = 0$$

We have discretized equations

$$\Phi_h'(\zeta) = 0 \tag{2.3}$$

We obtain

$$\Phi_h'(\zeta) = \begin{bmatrix} \dfrac{1}{h} - \dfrac{1}{2}\dfrac{\partial g_1}{\partial \xi}, -\dfrac{1}{2}\dfrac{\partial g_1}{\partial \eta}, 0 \cdots & & \\ -\dfrac{\partial f_1}{\partial \xi}, 1 - \dfrac{\partial f_1}{\partial \eta}, 0 \cdots & & \\ 0, \cdots, -\dfrac{1}{h} - \dfrac{1}{2}\dfrac{\partial g_{N-1}}{\partial \xi}, \dfrac{1}{h} - \dfrac{1}{2}\dfrac{\partial g_N}{\partial \xi}, -\dfrac{1}{2}\dfrac{\partial g_N}{\partial \eta} & & \\ 0, \ldots, 0, \ldots, -\dfrac{\partial f_N}{\partial \xi}, 1 - \dfrac{\partial f_N}{\partial \eta} & & \end{bmatrix} \times \begin{vmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_N \end{vmatrix}$$

when $\zeta_i = (\xi_i, \eta_i)$, $\dfrac{\partial g_i}{\partial \xi} = \dfrac{\partial g}{\partial \xi}(\xi_i, \eta_i)$, $\dfrac{\partial f_i}{\partial \xi} = \dfrac{\partial f}{\partial \xi}(\xi_i, \eta_i)$. We have Newton's method

$$\begin{cases} \zeta_{n+1}^h(k) = \zeta_n^h(k) + \mu_n^h(k) \\ \Phi' \zeta_n^h(k).\mu_n^h(k) = -\Phi_n(\eta_n^h(k)) \end{cases} \quad k = \overline{1, N}, n = 0, 1, 2, \ldots$$

The Newton discretized sequence

$$\zeta_{n+1}^h = \zeta_n^h - \left[\Phi_h'(\zeta_n^h)\right]^{-1} \Phi_n(\zeta_n^h), \quad n = 0, 1, 2, \ldots$$

The discretization method to be considered here will be described by a family of triplets $\{\Phi_h, \Pi_h, \tau\}$. The first, we consider $\left[\Phi_h'(\zeta)\right]^{-1}$, with

$$C_0 = \alpha + \frac{\beta\gamma}{1-\delta} \quad \text{and} \quad \lambda : \frac{2\gamma}{1-\delta} < \lambda < \frac{\alpha}{\beta} + \frac{2\gamma}{1-\delta}$$

and

$$C : C > \max\left\{\frac{1}{\lambda(1-\delta) - 2\gamma}, \frac{1}{\lambda(1-\delta) - \gamma e^{2C_0 T}}\right\}$$

We consider

$$\|[\Phi_h'(\zeta)]^{-1}\| \leqslant C^*, \quad C^* = \max\{C.e^{2C\Gamma}, \lambda C\}$$

Put·

$$Z_0 := \{z \in Z : x \in C^2(J, R^m), \quad y \in C^1(J, R^{n-m})\}$$
$$|\dot{x}|_\infty \leqslant \{B_0, \quad \|\dot{x}\|_\infty \leqslant B_1, \quad \|\dot{y}\| \leqslant B_2\}.$$

When: $\forall z \in Z_0$ we have

$$\tau F(z) = \begin{bmatrix} \dot{x}(t_1) - g(x(t_1), y(t_1)) \\ y(t_1) - f(x(t_1), y(t_1)) \\ \cdot \\ \dot{x}(t_N) - g(x(t_N), y(t_N)) \\ y(t_N) - f(x_N, y_N) \end{bmatrix}$$

$$\Pi_h z = (z_1, z_2, \dots, z_N), \quad z_i := z(t_i)$$

$$\Phi_h[\Pi_h z] = \begin{bmatrix} \dfrac{x_1 - x_0}{h} - \dfrac{1}{2}[g(x_1, y_1) + g(x_0, y_0)] \\ y_1 - f(x_1, y_1) \\ \dots \dots \\ \dfrac{z_N - x_{N-1}}{h} - \dfrac{1}{2}[g(x_N, y_N) + g(x_{N-1}, y_{N-1})] \\ y_N - f(x_N, y_N). \end{bmatrix}$$

Using finite incrment formular we find that

$$\|\tau(F(z)) - \Phi_h(\Pi_h z)\| \leqslant C_0^* h, \quad C_0^* := \left[B_0 + \frac{\alpha B_1 + \beta B_2}{2}\right] h,$$

with $u = (u_1, u_2) \in Z_0$

$$\tau(F'(z)u) = \begin{bmatrix} u_1'(t_1) - \frac{\partial g_1}{\partial x} u_1(t_1) - \frac{\partial g_1}{\partial y} u_2(t_1) \\ u_2(t_1) - \frac{\partial f_1}{\partial x} u_1(t_1) - \frac{\partial f_2}{\partial y} u_2(t_1) \\ \dots \dots \\ u_1'(t_N) - \frac{\partial g_N}{\partial x} u_1(t_N) - \frac{\partial g_N}{\partial y} u_2(t_N) \\ u_2(t_N) - \frac{\partial f_N}{\partial x} u_1(t_N) - \frac{\partial f_N}{\partial y} u_2(t_N) \end{bmatrix}$$

Here

$$\frac{\partial g}{\partial x}(x(t_i), y(t_i)) := \frac{\partial g_i}{\partial x}, \cdots$$

$$\Pi_h(u) = \big(u_1(t_1), u_2(t_1), u_1(t_2), u_2(t_2), \dots, u_1(t_N), u_2(t_N)\big)^T$$

We have

$$\tau(F'(z)u) - \Phi_h'(\Pi_h z)\Pi_h u$$
$$= \begin{bmatrix} u_1'(t_1) - \frac{1}{h}u_1(t_1) - \frac{1}{2}\frac{\partial g_1}{\partial x} u_1(t_1) - \frac{1}{2}\frac{\partial g_1}{\partial y} u_2(t_2) \\ 0 \dots \dots \quad \dots 0 \dots \dots \dots 0 \\ u_1'(T_N) + \frac{1}{2}\frac{\partial g_{N-1}}{\partial x} u_1(t_{N-1}) + \frac{1}{h}u_1(t_{N-1}) + \frac{1}{2}\frac{\partial g_{N-1}}{\partial y} u_2(t_{N-1}) \\ -\frac{1}{2}\frac{\partial g_N}{\partial x} u_1(t_N) - \frac{1}{h}u_1(t_N) - \frac{1}{2}\frac{\partial g_N}{\partial y} u_2(t_N) \\ 0 \dots \dots \quad \dots 0 \dots \dots \dots 0 \end{bmatrix}$$

We consider

$$\|\tau(F'_{(t)}u) - \Phi'_h(\Pi_h z)\Pi_h u\| \leqslant C_1^* h, \quad , C_1^* = B_0 + \alpha B_{1+} + \beta B_2 + 2l\|\dot{z}\|_\infty \|u\|_\infty .$$

By the Lipschitz continuity of

$$\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$$

with constant l, we find that

$$\|\phi'_h(\zeta) - \phi'_h(\overline{\zeta})| \leqslant 2l\|\zeta - \overline{\zeta}|, \quad h > \quad \forall \zeta, \overline{\zeta} \in \overline{B}(\Pi_h z^*, \rho)$$

Consider a Lipschitz uniform discretization $\{\Phi_h, \Pi_h, \tau_h\}$ which is bounded, stable and cosistent of order 1. With the notation introduced in the previous section we may formulate the main result as the following lemma.

Lemma. *Suppopose that for Cauchy problem (2.1), exists solution* $z^* := (x^*, y^*) \in Z; G := (g, f)^T$ *continuously differentiability on the open domain* U *of* z^*, *with*

$$U := U(z^*, \rho) = \{(x, y) \in R^n : \exists\, t \in J : |x - x^*(t)| < \rho, |y - y^*(t)| < \rho\}.$$

The differentiations of f *and* g *are satified:*

$$\left|\frac{\partial g}{\partial x}(z)\right| \leqslant \alpha, \quad \left|\frac{\partial g}{\partial y}(z)\right| \leqslant \beta, \quad \left|\frac{\partial f}{\partial x}(z)\right| \leqslant \gamma, \quad \left|\frac{\partial f}{\partial y}(z)\right| \leqslant \delta,$$

with: $\delta < 1, \ \forall z \in U(z^*, \rho)$.

with:$\delta < 1, \forall Z \in U(z^*, \rho)$. When the discrete family $\{\Phi_h, \Pi_h, \tau\}, h > 0$ satisfying conditions which is Lipschitz bounded, stable, and consistent of order 1:

$$\|\Phi'_h(\zeta) - \Phi'_h(\overline{\zeta})\| \leqslant 2l\|\zeta - \overline{\zeta}\| = L\|\zeta - \overline{\zeta}\| h > 0, \quad \forall \zeta, \overline{\zeta} \in \overline{\beta}(\Pi_h z^*, \rho)$$

and

$$L = 2l; \|\Pi_h z\| \leqslant \|z\|, \quad h > 0, \quad z \in Z_0 \ \forall z \in Z_0 \cap B^*$$

if: $B^* = B(z^*, r^*)$ with radius $r^* = \dfrac{2}{3Cl}$ we have

$$\begin{gathered}
\|[\Phi'_h(\Pi_h z)]^{-1}\| \leqslant C^* \\
\|\tau(F(z) - \Phi_h(\Pi_h z)\| \leqslant C_0^* h, \quad \forall z \in Z_0 \cap B^*, \quad h > 0 \\
\|\tau(F'(z)u - \Phi'_h(\Pi_h z)\Pi_h u\| \leqslant C_1^* h, \quad \forall z \in Z_0 \cap B^*, \quad u \in Z_0, \quad h > 0.
\end{gathered}$$

From this result we may formulate the mesh-independence principle as follows.

Theorem 2.2. *With the hypotheses of the lema, then problem (2.3) has a locally unique solution*

$$\zeta^* = \Pi_h(z^*) + 0(h)$$

for all $h > 0$ satisfying:

$$0 < h \leqslant h_0 = \left[\frac{1}{2C^*C_0} \min\left(\rho, (C^*\varepsilon L)^{-1}\right)\right].$$

Moreover, there exist constant $h_1 \in (0, h_0), r_1 \in (0, r^*)$ such that discrete process (2.4) converges to ξ_h^* and that:

$$\zeta_n^h = \Pi_h z_n + 0(h), \quad n = 0, 1, 2, \dots$$
$$\Phi_h(\zeta_n^h) = \tau F(z_n) + 0(h), \quad n = 0, 1, 2, \dots$$
$$\zeta_n^h - \zeta_h^* = \Pi_h(z_n - z^*) + 0(h), \quad n = 0, 1, 2, \dots,$$
$$|\min\{n > 0, \; \|z_n - z^*\| < \varepsilon\} - \min\{n > 0 : \|\zeta_n^h - \zeta_h^*\| < \varepsilon\}| \leqslant 1.$$

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