VNU. JOURNAL OF SCIENCE, Mathematics - Physics. T.XX, N$_0$4 - 2004

SPECIALIZATIONS OF REES RINGS AND INTEGRAL CLOSURES

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Abstract. The paper presents the specializations of Rees rings, associated graded rings and of integral closure of ideals. The preservation of some invariants of rings by specializations will also be concerned.

Introduction

Let k be an infinite field of arbitrary characteristic. Denote by K an extension field of k. Let $u = (u_1, \dots, u_m)$ be a family of indeterminates and $\alpha = (\alpha_1, \dots, \alpha_m)$ a family of elements of K. We denote the polynomial rings in n variables $x_1, \dots, x_n$ over $k(u)$ and $k(\alpha)$ by $R = k(u)[x]$ and by $R_\alpha = k(\alpha)[x]$, respectively.

The first step toward an algebraic theory of specialization was the introduction of the specialization of an ideal by W. Krull [2]. Let I be an ideal of R. The specialization of I with respect to the substitution $u \longrightarrow \alpha \in k^m$ is the ideal

$$I_\alpha = \{f(\alpha, x) |\ f(u, x) \in I \cap k[u, x]\} \subset k[x].$$

Following [2] the specialization of I with respect to the substitution $u \longrightarrow \alpha \in K^m$ is defined as the ideal I_α of R_α generated by elements of the set

$$\{f(\alpha, x) |\ f(u, x) \in I \cap k[u, x]\}.$$

A. Seidenberg [7] used specializations of ideals to prove that hyperplane sections of normal varieties are normal again under certain conditions. Using specializations of finitely generated free modules and of homomorphisms between them, we defined in [4] the specialization of a finitely generated module, and we showed that basic properties and operations on modules are preserved by specializations. In [3] we followed the same approach to introduce and to study specializations of finitely generated modules over a local ring [4] and of graded modules over graded ring [5]. We will give the definitions of specializations of Rees rings and associated graded rings, which are not finitely generated as R-modules and we want also to study specializations of integral closures of ideals.

In this paper, we shall say that a property holds for almost all α if it holds for all points of a Zariski-open non-empty subset of K^m. For convenience we shall often omit the phrase "for almost all α" in the proofs of the results[1].

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[1]The authors are partially supported by the National Basic Research Program

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1. Some results about specialization of graded modules

Let k be an infinite field of arbitrary characteristic. Denote by K an extension field of k. Let $u = (u_1, \dots, u_m)$ be a family of indeterminates and $\alpha = (\alpha_1, \dots, \alpha_m)$ a family of elements of K. Let $\mathfrak{m}$ and $\mathfrak{m}_\alpha$ be the maximal graded ideals of R and R_α, respectively.

The specialization of ideals can be generalized to modules. First, each element $a(u,x)$ of R can be written in the form

$$a(u,x) = \frac{p(u,x)}{q(u)}$$

with $p(u,x) \in k[u,x]$ and $q(u) \in k[u] \setminus \{0\}$. For any α such that $q(\alpha) \neq 0$ we define

$$a(\alpha,x) = \frac{p(\alpha,x)}{q(\alpha)}.$$

Let F be a free R-module of finite rank. The specialization F_α of F is a free R_α-module of the same rank. Let $\phi : F \longrightarrow G$ be a homomorphism of free R-modules. We can represent ϕ by a matrix $A = (a_{ij}(u,x))$ with respect to fixed bases of F and G. Set $A_\alpha = (a_{ij}(\alpha,x))$. Then A_α is well-defined for almost all α. The specialization $\phi_\alpha : F_\alpha \longrightarrow G_\alpha$ of ϕ is given by the matrix A_α provided that A_α is well-defined. We note that the definition of ϕ_α depends on the chosen bases of F_α and G_α.

Definition. [3] Let L be an R-module. Let $F_1 \xrightarrow{\phi} F_0 \longrightarrow L \longrightarrow 0$ be a finite free presentation of L. Let $\phi_\alpha : (F_1)_\alpha \longrightarrow (F_0)_\alpha$ be a specialization of ϕ. We call $L_\alpha := \operatorname{Coker} \phi_\alpha$ a *specialization* of L (with respect to ϕ).

If we choose a different finite free presentation $F_1' \longrightarrow F_0' \longrightarrow L \longrightarrow 0$ we may get a different specialization L'_α of L, but L_α and L'_α are canonically isomorphic [4, Proposition 2.2]. Hence L_α is uniquely determined up to isomorphisms. The following lemmas show that the operations and the dimension of modules are preserved by specialization.

Lemma 1.1. [3, Proposition 3.2 and 3.6] *Let L be a finitely generated R-module and M, N submodules of L, and I an ideal of R. Then, for almost all α,*
(i) $(L/M)_\alpha \cong L_\alpha / M_\alpha$,
(ii) $(M \cap N)_\alpha \cong M_\alpha \cap N_\alpha$,
(iii) $(M + N)_\alpha \cong M_\alpha + N_\alpha$,
(iv) $(IL)_\alpha \cong I_\alpha L_\alpha$.

Let L be a finitely generated R-module. The dimension and depth of L are denoted by $\dim L$ and $\operatorname{depth} L$, respectively.

Lemma 1.2. [3] *Let L be a finitely generated R-module. Then, for almost all α, we have*
(i) $\operatorname{Ann} L_\alpha = (\operatorname{Ann} L)_\alpha$,
(ii) $\dim L_\alpha = \dim L$,
(iii) $\operatorname{depth} L_\alpha = \operatorname{depth} L$.

We recall now some facts from [5] which we shall need later. First we note that R is naturally graded. For a graded R-module L, we denote by L_t the homogeneous component of L of degree t. For an integer h we let $L(h)$ be the same module as L with grading shifted by h, that is, we set $L(h)_t = L_{h+t}$.

Let $F = \bigoplus_{j=1}^{s} R(-h_j)$ be a free graded R-module. We make the specialization F_α of F a free graded R_α-module by setting $F_\alpha = \bigoplus_{j=1}^{s} R_\alpha(-h_j)$. Let

$$\phi : \bigoplus_{j=1}^{s_1} R(-h_{1j}) \longrightarrow \bigoplus_{j=1}^{s_0} R(-h_{0j})$$

be a graded homomorphism of degree 0 given by a homogeneous matrix $A = (a_{ij}(u,x))$. Since

$$\deg(a_{i1}(u,x)) + h_{01} = \ldots = \deg(a_{is_0}(u,x)) + h_{0s_0} = h_{1i},$$

$A_\alpha = (a_{ij}(\alpha, x))$ is a homogeneous matrix with

$$\deg(a_{i1}(\alpha,x)) + h_{01} = \ldots = \deg(a_{is_0}(\alpha,x)) + h_{0s_0} = h_{1i}.$$

Therefore, the homomorphism

$$\phi_\alpha : \bigoplus_{j=1}^{s_1} R_\alpha(-h_{1j}) \longrightarrow \bigoplus_{j=1}^{s_0} R_\alpha(-h_{0j})$$

given by the matrix A_α is a graded homomorphism of degree 0.

Lemma 1.3. [5, Lemma 2.3] *Let L be a finitely generated graded R-module. Then L_α is a graded R_α-module for almost all α.*

Let $\mathbf{F}_\bullet$: $0 \longrightarrow F_\ell \xrightarrow{\phi_\ell} F_{\ell-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \longrightarrow L \longrightarrow 0$ be a minimal graded free resolution of L, where each free module F_i may be written in the form $\bigoplus_j R(-j)^{\beta_{ij}}$, and all graded homomorphisms have degree 0. The integers $\beta_{ij} \neq 0$ are called the *graded Betti numbers* of L. The following lemma shows that the graded Betti numbers are preserved by specializations.

Lemma 1.4. [5, Theorem 3.1] *Let $\mathbf{F}_\bullet$ be a minimal graded free resolution of L. Then the complex*

$$(\mathbf{F}_\bullet)_\alpha : 0 \longrightarrow (F_\ell)_\alpha \xrightarrow{(\phi_\ell)_\alpha} (F_{\ell-1})_\alpha \longrightarrow \cdots \longrightarrow (F_1)_\alpha \xrightarrow{(\phi_1)_\alpha} (F_0)_\alpha \longrightarrow L_\alpha \longrightarrow 0$$

is a minimal graded free resolution of L_α with the same graded Betti numbers for almost all α.

2. Specialization of Rees rings and associated graded rings

Let $y_1, \ldots, y_s$ be a sequence of distinct indeterminates. The polynomial ring of $y_1, \ldots, y_s$ with coefficients in R is denoted by $R[y]$. Let L be a finitely generated R-module. Then besides considering the polynomial ring $R[y]$ we may also consider polynomials in $y_1, \ldots, y_s$ with coefficients belong to L. The set $L[y]$ of all this polynomials has a natural structure as a module over $R[y]$. It is easily seen that $L[y] = L \otimes_R R[y]$. By a definition analogous to that used for the construction of L_α we may give a specialization $L[y]_\alpha$ of $L[y]$. Here we have

Lemma 2.1. *Let L be a finitely generated R-module. Then $L[y]_\alpha \cong L_\alpha[y]$ for almost all α.*

Proof. Let $R^p \xrightarrow{\varphi} R^q \longrightarrow L \longrightarrow 0$ be a finite free presentation of L. Since $R \longrightarrow R[y]$ and $R_\alpha \longrightarrow R_\alpha[y]$ are flat, we can deduce that the sequences

$$R^p[y] \xrightarrow{\varphi\otimes 1} R^q[y] \longrightarrow L[y] \longrightarrow 0 \text{ and } R^p_\alpha[y] \xrightarrow{\varphi_\alpha\otimes 1} R^q_\alpha[y] \longrightarrow L_\alpha[y] \longrightarrow 0$$

are finite free presentations of $L[y]$ and L_α, respectively. From the definition of specialization $L[y]_\alpha$, the following sequence is exact

$$R^p[y]_\alpha \xrightarrow{(\varphi\otimes 1)_\alpha} R^q[y]_\alpha \longrightarrow L[y]_\alpha \longrightarrow 0.$$

Because $R^h[y]_\alpha = R^h_\alpha[y]$ and $(\varphi \otimes 1)_\alpha = \varphi_\alpha \otimes 1$, therefore $L[y]_\alpha \cong L_\alpha[y]$.

Let I be an ideal of R. Denote the ring R/I by B. Let $\mathfrak{a}$ be an ideal of B. We set

$$B[\mathfrak{a}t] = \bigoplus_{j\geq 0} \mathfrak{a}^j t^j \subset B[t],$$

$$G(\mathfrak{a}, B) = \bigoplus_{j\geq 0} \mathfrak{a}^j t^j / \mathfrak{a}^{j+1} t^{j+1}.$$

Both, $B[\mathfrak{a}t]$ and $G(\mathfrak{a}, B)$ are graded rings. $B[\mathfrak{a}t]$ is called the *Rees ring* and $G(\mathfrak{a}, B)$ the *associated graded ring* of B with respect to $\mathfrak{a}$. If $\mathfrak{a}$ is generated by $a_1, \dots, a_s \in R/I$, then $B[\mathfrak{a}t] = B[a_1 t, \dots, a_s t]$. Note that $B_\alpha = R_\alpha / I_\alpha$.

Suppose that J is an ideal of R such that $I \subseteq J$ and $\mathfrak{a} = J/I$. Then $\mathfrak{a}_\alpha = J_\alpha / I_\alpha$ is a specialization of $\mathfrak{a}$ by Lemma 1.1.

Definition Let $\mathfrak{a}$ be an ideal of B. We call $B_\alpha[\mathfrak{a}_\alpha t]$ and $G(\mathfrak{a}_\alpha, B_\alpha)$ as the *specializations* of $B[\mathfrak{a}t]$ and $G(\mathfrak{a}, B)$, respectively.

Proposition 2.2. *Let $\mathfrak{a}$ be a proper ideal of B. Then, for almost α, we have*

(i) $\dim_{B_\alpha[\mathfrak{a}_\alpha t]} G(\mathfrak{a}_\alpha, B_\alpha) = \dim_{B[\mathfrak{a}t]} G(\mathfrak{a}, B)$,

(ii) $\dim B_\alpha[\mathfrak{a}_\alpha t] = \dim B[\mathfrak{a}t]$.

Proof. (i) There is $\dim B_\alpha = \dim B$ by Lemma 1.2. Since $\dim_{B_\alpha[\mathfrak{a}_\alpha t]} G(\mathfrak{a}_\alpha, B_\alpha) = \dim B_\alpha$ and $\dim_{B[\mathfrak{a}t]} G(\mathfrak{a}, B) = \dim B$ from [9, Chapter IV Proposition 1.9], it follows that $\dim_{B_\alpha[\mathfrak{a}_\alpha t]} G(\mathfrak{a}_\alpha, B_\alpha) = \dim_{B[\mathfrak{a}t]} G(\mathfrak{a}, B)$.

(ii) Consider the B-algebra homomorphism $\phi : B[y_1, \dots, y_s] \longrightarrow B[\mathfrak{a}t]$, $y_i \longmapsto a_i t$. Denote by J the ideal of $B[y_1, \dots, y_s, t]$ generated by the polynomials $y_i - a_i t, i = 1, \dots, s$. By [10, Proposition 7.2.1] , there is

$$B[\mathfrak{a}t] = B[y_1, \dots, y_s]/J \cap B[y_1, \dots, y_s].$$

Using Lemma 2.1, we can specialize $B[\mathfrak{a}t]$. Semilarly, we have

$$B[\mathfrak{a}t]_\alpha = (B[y_1, \dots, y_s]/J \cap B[y_1, \dots, y_s])_\alpha \cong B_\alpha[y_1, \dots, y_s]/J_\alpha \cap B_\alpha[y_1, \dots, y_s] = B_\alpha[\mathfrak{a}_\alpha t].$$

Since $\dim B[\mathfrak{a}t]_\alpha = \dim B[\mathfrak{a}t]$ by Lemma 2.1, there is $\dim B_\alpha[\mathfrak{a}_\alpha t] = \dim B[\mathfrak{a}t]$.

Proposition 2.3. *Let $\mathfrak{a}$ be a proper ideal of B. Then, for almost α, we have*
(i) $\operatorname{depth}_{B_\alpha[\mathfrak{a}_\alpha t]} G(\mathfrak{a}_\alpha, B_\alpha) = \operatorname{depth}_{B[\mathfrak{a}t]} G(\mathfrak{a}, B)$,
(ii) $\operatorname{depth} B_\alpha[\mathfrak{a}_\alpha t] = \operatorname{depth} B[\mathfrak{a}t]$.

Proof. The proof is immediate from Lemma 1.2 and [4, Theorem 3.1].

Recall that a ring A is a *Cohen-Macaulay* ring, if $\dim A = \operatorname{depth} A$. The following corollary shows that the Cohen-Macaulay property of a Rees ring or an associated graded ring is preserved by specializations.

Corollary 2.4. *If $B[\mathfrak{a}t]$, (resp.$G(\mathfrak{a}, B)$), is Cohen-Macaulay, then $B_\alpha[\mathfrak{a}_\alpha t]$, (resp.$G(\mathfrak{a}_\alpha, B_\alpha)$), is again Cohen-Macaulay.*

Proof. By an easy computation, the proof follows from Propositions 2.2 and 2.3.

Now we will show that the multiplicity of associated graded ring is preserved by specialization.

Proposition 2.5. *Let $\mathfrak{q} = (y_1, \dots, y_d)B$ be a parameter ideal of B, where $\dim B = d$. Then, for almost all α, we have*

$$e(\mathfrak{q}_\alpha; G(\mathfrak{a}_\alpha, B_\alpha)) = e(\mathfrak{q}; G(\mathfrak{a}, B)),$$

where $e(\mathfrak{q}_\alpha; G(\mathfrak{a}_\alpha, B_\alpha))$ and $e(\mathfrak{q}; G(\mathfrak{a}, B))$ are the multiplicities of $G(\mathfrak{a}_\alpha, B_\alpha)$ and $G(\mathfrak{a}, B)$ respectively.

Proof. By Lemma 1.2, $\dim B_\alpha = d$. By [7, Lemma 1.5] the ideal

$$\mathfrak{q}_\alpha = ((y_1)_\alpha, \dots, (y_d)_\alpha)B_\alpha$$

is again a parameter ideal on B_α and $e(\mathfrak{q}_\alpha; B_\alpha) = e(\mathfrak{q}; B)$ by [6, Theorem 1.6]. Because $e(\mathfrak{q}_\alpha; G(\mathfrak{a}_\alpha, B_\alpha)) = e(\mathfrak{q}_\alpha; B_\alpha)$ and $e(\mathfrak{q}; G(\mathfrak{a}, B)) = e(\mathfrak{q}; B)$, then the proof is complete.

3. Noether normalizations and integral closures by specializations

Consider the standard graded ring $R = k[x_1, \dots, x_n]$ with $\deg(x_j) = 1$ for all $j = 1, \dots, n$. Throughout this section, I will denote a homogeneous ideal of R and the residue class ring R/I will again denote by B. Put $\dim B = d$. Let us recall the notion of Noether normalization of a ring. Suppose that $f_1, \dots, f_d$ are polynomials of R. The subring $k(u)[f_1, \dots, f_d]$ is called a *Noether normalization* of B if $f_1, \dots, f_d$ are algebraically independent over k and B is a finitely generated $k(u)[f_1, \dots, f_d]$-module. The following proposition shalls show that a specialization of a Noether normalization of a ring is again a Noether normalization.

Proposition 3.1. *Assume that $\dim B = d$ and $f_1, \dots, f_d \in R$ are homogeneous polynomials of positive degrees. If the subring $k(u)[f_1, \dots, f_d]$ is a Noether normalization of B, then the subring $k(\alpha)[(f_1)_\alpha, \dots, (f_d)_\alpha]$ is also a Noether normalization of B_α.*

Proof. We have $\dim B_\alpha = \dim B = d$ by Lemma 1.2. By definition of specialization, $(f_1)_\alpha, \dots, (f_d)_\alpha$ are homogeneous polynomials with $\deg(f_j)_\alpha = \deg f_j$ for all $j = 1, \dots, d$.

By virtue of Lemma 1.1 one can deduce $(B/(f_1,\dots,f_d))_\alpha = B_\alpha/((f_1)_\alpha,\dots,(f_d)_\alpha)$. From [10, Proposition 2.3.1], it is well known that the subring $k(u)[f_1,\dots,f_d]$ is a Noether normalization of B if and only if $\dim_{k(u)} B/(f_1,\dots,f_d) < \infty$. Assume that the subring $k(u)[f_1,\dots,f_d]$ is a Noether normalization of S. Then $\dim B/(f_1,\dots,f_d) = 0$. By Lemma 1.2, $\dim(B/(f_1,\dots,f_d))_\alpha = 0$. Hence the subring $k(\alpha)[(f_1)_\alpha,\dots,(f_d)_\alpha]$ is also a Noether normalization of B_α.

The ring B is said to satisfy Serre's condition (S_r) if $\operatorname{depth} B_{\mathfrak{p}} \geq \min\{r, \dim B_{\mathfrak{p}}\}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Without loss of generality we can assume that $A = k(u)[x_1,\dots,x_d]$ is a Noether normalization of B. In this case B is a finitely generated graded A-module. Using the above proposition we are now in a position to prove the following result, see [6, Lemma 4.3].

Corollary 3.2. *If B satifies Serre's condition (S_r), so is B_α for almost all α.*

Proof. We consider B as a finitely generated graded A-module. Suppose that

$$\mathbf{F}_\bullet : 0 \longrightarrow A^{d_\ell} \xrightarrow{\phi_\ell} A^{d_{\ell-1}} \longrightarrow \cdots \longrightarrow A^{d_1} \xrightarrow{\phi_1} A^{d_0} \longrightarrow B \longrightarrow 0$$

is a minimal graded free resolution of B. Denote by $I_j(B)$ the ideal $I_{r_j}(\varphi_j), r_j = \operatorname{rank}\varphi_j$. By [10, Proposition 7.1.3], we know that B satifies (S_r) if and only if $\operatorname{ht} I_j(B) \geq j+r, j \geq 0$. By Proposition 3.1, $A_\alpha = k(\alpha)[x_1,\dots,x_d]$ is a Noether normalization of B_α and

$$\mathbf{F}_{\bullet\alpha} : 0 \longrightarrow A_\alpha^{d_\ell} \xrightarrow{(\phi_\ell)_\alpha} A_\alpha^{d_{\ell-1}} \longrightarrow \cdots \longrightarrow A_\alpha^{d_1} \xrightarrow{(\phi_1)_\alpha} A_\alpha^{d_0} \longrightarrow B_\alpha \longrightarrow 0$$

is a minimal graded free resolution of B_α by Lemma 1.4. Since $\operatorname{rank}(\varphi_j)_\alpha = \operatorname{rank}\varphi_j$ and $\operatorname{ht} I_j(B_\alpha) = \operatorname{ht} I_j(B)$ for all $j \geq 0$ by Lemma 1.2, therefore B_α satifies Serre's condition (S_r) by [10, Proposition 7.1.3].

The proplem of concern is now the preservation of the reduction number of B by specializations. First, let us recall the definition of reduction number of a graded algebra. Assume that $B = \oplus_{t\geq 0} B_t$ is a finitely generated, positively graded algebra over a field $B_0 = k_1$ and $z_1,\dots,z_d \in k_1[B_1]$ such that $A = k_1[z_1,\dots,z_d]$ is a Nother normalization of B. Let $v_1,\dots,v_s$ be a minimal set of homogeneous generators of B as an A-module

$$B = \sum_{j=1}^{s} Av_j,\ \deg v_j = m_j.$$

The *reduction number* $r_A(B)$ of B with respec to A is the supremum of all m_j.

Proposition 3.3. *Let A be a Noether normalization of B. Then $r_A(B) = r_{A_\alpha}(B_\alpha)$ for almost all α.*

Proof. As above, without loss of generality we can assume that $A = k(u)[x_1,\dots,x_d]$ is a Noether normalization of B. Let $v_1,\dots,v_s$ be a minimal set of homogeneous generators of B as an A-module

$$B = \sum_{j=1}^{s} Av_j,\ \deg v_j = m_j.$$

We have $\dim B_\alpha = d$ by Lemma 1.2. Then $A_\alpha = k(\alpha)[x_1, \ldots, x_d]$ is a Noether normalization of B_α by Proposition 3.1 and $B_\alpha = \sum_{j=1}^{s} A_\alpha(v_j)_\alpha$, $\deg(v_j)_\alpha = \deg v_j$ by definition of specialization. Hence $r_{A_\alpha}(B_\alpha) = \sup\{\deg(v_j)_\alpha\} = \sup\{\deg v_j\} = r_A(B)$.

To study the specialization of integral closures of ideals we will recall the notion of reduction of an ideal, an object first isolated by Northcott and Rees, see [1]. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of B. $\mathfrak{a}$ is said to be a *reduction* of $\mathfrak{b}$ if $\mathfrak{a} \subset \mathfrak{b}$ and $\mathfrak{a}\mathfrak{b}^r = \mathfrak{b}^{r+1}$ for some nonnegative integer r and the least integer r with this property is called the reduction number of b with respect to $\mathfrak{a}$. This number is denoted by $r_{\mathfrak{a}}(b)$, and it is the largest non-vanishing degree of b. An element $z \in B$ is *integral* over $\mathfrak{a}$ if there is an equation

$$z^m + a_1 z^{m-1} + \cdots + a_m = 0,\ a_i \in \mathfrak{a}^i.$$

Denote the set of all elements of B, which are integral over $\mathfrak{a}$, by $\bar{\mathfrak{a}}$. $\bar{\mathfrak{a}}$ is called the *integral closure* of ideal $\mathfrak{a}$. Note that $z \in B$ is integral over $\mathfrak{a}$ if and only if $zt \in B[t]$ is integral over $B[\mathfrak{a}t]$. The set of all ideals of B which have $\mathfrak{a}$ as a reduction has a unique maximal member. That is $\bar{\mathfrak{a}}$ by [1, Corollary 18.1.6]. An ideal $\mathfrak{a}$ is said to be *integrally closed* if $\mathfrak{a} = \bar{\mathfrak{a}}$. To study specializations of integral closures we need the following

Lemma 3.4. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of B.*
(i) *If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\bar{\mathfrak{a}} \subseteq \bar{\mathfrak{b}}$.*
(ii) *If $\mathfrak{a}$ is a reduction of $\mathfrak{b}$, then $\mathfrak{b} \subseteq \bar{\mathfrak{a}}$.*
(iii) *If $\mathfrak{a}$ is a reduction of $\mathfrak{b}$, then $\bar{\mathfrak{a}} = \bar{\mathfrak{b}}$.*

Proof. (i) Assume that $\mathfrak{a} \subseteq \mathfrak{b}$. Suppose that $z \in \bar{\mathfrak{a}}$. There is an equation

$$z^m + a_1 z^{m-1} + \cdots + a_m = 0,\ a_i \in \mathfrak{a}^i.$$

Since $\mathfrak{a}^i \subseteq \mathfrak{b}^i$, therefore $z \in \bar{\mathfrak{b}}$. Hence $\bar{\mathfrak{a}} \subseteq \bar{\mathfrak{b}}$.
(ii) Assume that $\mathfrak{a}$ is a reduction of $\mathfrak{b}$, then each element of $\mathfrak{b}$ is integral over $\mathfrak{a}$ by [1, Proposition 18.1.5]. Thus $\mathfrak{b} \subseteq \bar{\mathfrak{a}}$.
(iii) Assume that $\mathfrak{a}$ is a reduction of $\mathfrak{b}$. Then $\mathfrak{a} \subseteq \mathfrak{b}$. Thus $\bar{\mathfrak{a}} \subseteq \bar{\mathfrak{b}}$ by (i). Because $\mathfrak{a}$ is a reduction of $\mathfrak{b}$, therefore $\mathfrak{b} \subseteq \bar{\mathfrak{a}}$ by (ii). Thus $\bar{\mathfrak{b}} \subseteq \overline{(\bar{\mathfrak{a}})}$. We need prove $\overline{(\bar{\mathfrak{a}})} = \bar{\mathfrak{a}}$. Since $\mathfrak{a}$ is a reduction of $\bar{\mathfrak{a}}$ and $\bar{\mathfrak{a}}$ is a reduction of $\overline{(\bar{\mathfrak{a}})}$ by [1, Corollary 18.1.6], $\mathfrak{a}$ is a reduction of $\overline{(\bar{\mathfrak{a}})}$. It implies $\bar{\mathfrak{a}} = \overline{(\bar{\mathfrak{a}})}$ from the maximality of integral closure of $\mathfrak{a}$.

Lemma 3.5. *Let $\mathfrak{a}$ be an ideal of B. Then $(\bar{\mathfrak{a}})_\alpha \subseteq \overline{\mathfrak{a}_\alpha}$ and $\overline{(\bar{\mathfrak{a}})_\alpha} = \overline{\mathfrak{a}_\alpha}$ for almost all α.*

Proof. Note that if $\mathfrak{b}$ is an ideal of B and $\mathfrak{a}$ is a reduction of $\mathfrak{b}$, then there is an positive integer r such that $\mathfrak{a}\mathfrak{b}^r = \mathfrak{b}^{r+1}$. Hence $\mathfrak{a}_\alpha \subseteq \mathfrak{b}_\alpha$ and $\mathfrak{a}_\alpha \mathfrak{b}_\alpha^r = \mathfrak{b}_\alpha^{r+1}$ by Lemma 1.1 (iv). Also, $\mathfrak{a}_\alpha$ is a reduction of $\mathfrak{b}_\alpha$. Since $\mathfrak{a}$ is a reduction of $\bar{\mathfrak{a}}$, therefore $\mathfrak{a}_\alpha$ is a reduction of $(\bar{\mathfrak{a}})_\alpha$ by above note. Hence $(\bar{\mathfrak{a}})_\alpha \subseteq \overline{\mathfrak{a}_\alpha}$ and $\overline{\mathfrak{a}_\alpha} = \overline{(\bar{\mathfrak{a}})_\alpha}$ follows from Lemma 3.4 (iii).

Theorem 3.6. *Let $\mathfrak{a}$ be an ideal of B. The integral closure of the Rees ring $B[\mathfrak{a}_\alpha t]$ is the integral closure of a specialization of the integral closure of the Rees ring $B[\mathfrak{a}t]$.*

Proof. We know that the inegral closure of $B[\mathfrak{a}t]$ is the graded subring $T = \oplus_{j \geq 0} \overline{\mathfrak{a}^j} t^j$. By above definition, the specialization of T is the graded subring $T_\alpha = \oplus_{j \geq 0} (\overline{\mathfrak{a}^j})_\alpha t^j$ and the integral closure of T_α is $\oplus_{j\geq 0}\overline{(\mathfrak{a}^j)_\alpha} t^j$. Because $\overline{(\mathfrak{a}^j)_\alpha} = \overline{\mathfrak{a}^j}_\alpha$ by Lemma 3.5 and α and j may be commute, i.e. $(\mathfrak{a}^j)_\alpha = (\mathfrak{a}_\alpha)^j = \mathfrak{a}_\alpha^j$, therefore $\oplus_{j\geq 0}\overline{\mathfrak{a}_\alpha^j} t^j$ is the integral closure of a specialization $\oplus_{j\geq 0}\overline{(\mathfrak{a}^j)}_\alpha t^j$ for almost all α.

Proposition 3.7. *Let $\mathfrak{q}$ be a paramer ideal of B. Then $e(\overline{\mathfrak{q}_\alpha}; B_\alpha) = e(\overline{\mathfrak{q}}; B)$ for almost all α*

Proof. It is well-known that $e(\overline{\mathfrak{q}_\alpha}; B_\alpha) = e(\mathfrak{q}_\alpha; B_\alpha)$ and $e(\overline{\mathfrak{q}}; B) = e(\mathfrak{q}; B)$ by [7]. The proof is immediate from the equation $e(\mathfrak{q}_\alpha; B_\alpha) = e(\mathfrak{q}; B)$ by [6, Theorem 1.6].

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