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SMALL MODULES AND QF-RINGS

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Abstract. It is shown that a semiperfect ring R is quasi-Frobenius if and only if R has finite right uniform dimension and every closed uniform submodule of $R(\omega)$ is a direct summand, where $R(\omega)$ denotes the direct sum of ω copies of the right R-module R and ω is the first infinite ordinal. This result extends the one of D. V. Huynh and N. S. Tung in [5. Theorem 1].

1. Introduction

Quasi-Frobenius rings (briefly, a QF-ring) were introduced by Nakayama in 1938. A ring R is a QF if it is a left artinian, left seflinjective ring. The class of QF-rings is one of the most interesting generlization of semisimple rings and have been studied by several authors (see, for example [4], [5], [7]). The number of characterization of QF-rings are so large that we are unable to give all the references here. In this paper, we will extend the result which was given by D. V. Huynh and N. S. Tung in [5]. Throughtout this note all rings R are associative rings with indentity and all modules are unitary right R-modules.

2. Preliminaries

A submodule N of a module M is called *small* in M, or a *small submodule* of M, denoted by $N \subseteq^0 M$, if for each submodule H of M, the relation $N + H = M$ implies $H = M$ (or equivalently for each proper submodule H of $M, M \neq N + H$). A module S is said to be a *small module*, if S is small in its injective hull. If S is not a small module, we say that S is non-small. By this definition we may consider the zero module as a non-small module although it is small in each non-zero module.

Small modules and non-small modules have been considered by many authors. In particular, Harada [3] and Oshiro [7] used these and related concepts of modules to characterize serveral interesting classes of rings including artinian serial rings and QF-rings.

Dually, a submodule E of a module M is *called essential* in M, or an *essential submodule* of M, if for any non-zero submodule T of M, $E \cap T \neq 0$. A non-zero module U is uniform if any non-zero submodule of U is essential in U.

Now let $A \subseteq B$ be a submodule of a module M such that A is essential in B. Then we say that B is an *essential extension* of A in M. A module C of M is called a *closed submodule* of M if C has no proper essension in M. By Zorn's Lemma, each submodule of M is contained essentially in a closed submodule of M.

If a module M has only one maximal submodule which contains all proper submodules of M, then M is called a *local module*.

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3. The results

Lemma 1. *i) If N is a non-zero small submodule of module M then N is a small module.*

ii) Let M be a local module such that any closed submodule of M is non-small. Then M is uniform.

iii) Let A, B be modules with $A \cong B$, then A is small if and only if B is small.

Proof. i) Since N is submodule of M, $E(M) = E(N) \oplus Y$ for some submodule Y of $E(M)$. Since $N \subseteq^0 M$, $N \subseteq^0 E(M)$. By [6, Lemma 4.2(2)] we have N is a small submodule of $E(N)$, therefore N is small module.

ii) Is obvious.

iii) Since $A \cong B$ there is an isomorphism

$$\varphi : E(A) \to E(B) \text{ with } \varphi(A) = B.$$

Hence the statement follows from [6, Lemma 4.2(3)].

Lemma 2. *[1, Chapter 27] Let R be a semiperfect ring, then R contains a complete set of primitive orthogonal idempotents $\{e_1, e_2, ..., e_n\}$ such that*

$$R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR \tag{1}$$

and each e_iR is a local module with local endomorphism ring. Moreover, the maximal submodule of each e_iR is a small submodule of e_iR.

We keep this decomposition of R thoughout the consideration below.

Lemma 3. *Let R be a semiperfect ring satifies one of two conditions:*

a) Every closed submodule of $R(\omega)$ is non-small.

b) R has finite right uniform dimension and every closed uniform submodule of $R(\omega)$ is non-small.

Then we have:

i) Every e_iR is uniform.

ii) Each e_iR is not embedded properly in e_jR, $j = 1, 2, ..., n$.

iii) Every closed uniform submodule of $R(\omega)$ is a direct summand.

Proof. Case 1: R satisfies a).

i) By (1), each closed submodule U of e_iR is closed in $R(\omega)$, hence U is non-small by a). If U is non-zero then $e_iR = U$, which shows that e_iR is uniform.

ii) By i) and (1) each e_iR is a closed uniform submodule of $R(\omega)$. Hence e_iR is non-small by a).

Then by Lemma 1, each e_iR can not embedded properly in e_jR, $(j = 1, ..., n)$.

iii) For convenience we write $R(\omega)$ in the form:

$$R(\omega) = \oplus_{\alpha \in I} P_\alpha\,, \tag{2}$$

where each P_α is isomorphism to some e_iR in $\{e_1R, \ldots, e_nR\}$ and I is an infinite countable set.

By i) each P_α is uniform. Let U be a closed uniform submodule of $R(\omega)$. For each α we denote by π_α the projection of $R(\omega)$ onto P_α.

Then there exists a subset J of I which is maximal with respect to $U \cap R(J) = 0$ and $U \oplus R(J) \subseteq^0 R(\omega)$. We show that there exists only $k \in I$ such that $J = I \backslash \{k\}$.

Indeed, suppose there exists $k_1 \neq k_2 \notin J$ let

$$X_1 = P_{k_1} \cap (R(J) \oplus U),$$
$$X_2 = P_{k_2} \cap (R(J) \oplus U).$$

Since $R(J) \oplus U \subseteq^0 R(\omega)$, $X_1 \neq 0, X_2 \neq 0$. Let

$$X = (P_{k_1} \oplus P_{k_2}) \cap (R(J) \oplus U).$$

Then $X_1 + X_2 \subseteq X$ and $X \cap R(J) = 0$. Consider the projection $\pi : R(J) \to U$, from $X \cap R(J) = 0$ we infer that π/X is a monomorphism, hence $X \cong \pi(X) \subseteq U$. Thus X contains a submodule which is isomorphism with $X_1 \oplus X_2$, a contradiction to the fact that U is uniform.

Therefore, there is only index $k \in I$ such that $J = I \backslash \{k\}$. Then, we have $U \cap \ker \pi_k = 0$. Hence $U \cong \pi_k(U) \subseteq P_k$. By hypothesis, U is non-small, hence $\pi_k(U)$ is also non-small by Lemma 1. It follows that $P_k = \pi_k(U)$, since P_k is a local module.

From this, it is easy to see that $R(\omega) = U \oplus \ker \pi_k$ as desired.

Case 2: R satisfies b).

By the hypothesis b), U is non-small, since $e_i R$ is a local module then each proper submodule of $e_i R$ is small. Hence $U = e_i R$, otherword $e_i R$ is uniform, we have i).

i) and iii) can prove similarly.

The following theorem was given by D. V. Huynh and N. S. Tung in [5].

Theorem 4. *Let R be a semiperfect ring. Then the following statements are equavelent:*

i) R is a QF-ring.

ii) R has finite right uniform dimension, no non-zero projective right ideal of R is contained in the jacobson radical $J(R)$ of R and every closed uniform submodule of $R(\omega)$ is a direct summand.

Now we prove our main theorem.

Theorem 5. *A semiperfect ring R is a QF if and only if R has finite right uniform dimension and every closed uniform submodule of $R(\omega)$ is a direct summand.*

Proof. ($\Rightarrow$) Suppose that semiperfect ring R is a QF-ring. Then every closed submodule of $R(\omega)$ is non-small, by [6, Theorem 24.20]. By Lemma 3, each $e_i R$ is uniform, hence R has finite right uniform dimension and each closed uniform submodule of $R(\omega)$ is a direct summand.

($\Leftarrow$) Conversely, suppose semiperfect ring R has finite right uniform dimension and every closed uniform submodule of $R(\omega)$ is a direct summand. We prove R is a QF-ring.

Since R is a semiperfect ring, R has the form (1), where each $e_i R$ is local.

By the hypothesis, each closed uniform submodule of $R(\omega)$ is a direct summand, therefore it is non-small. By Lemma 3, we have each e_iR is uniform, i.e P_i is uniform for every $i \in I$ and each e_iR can not embedded properly in e_iR, $j = 1, 2, \dots, n$.

We first show that the decomposition $R(\omega) = \oplus_{\alpha \in I} P_\alpha$ complements direct summands, i.e for each direct summand A of $R(\omega)$, there is a subset I' of I such that $R(\omega) = A \oplus R(I')$ (see [6, Chapter 1]).

Thus, we assume now that A is a direct summand of $R(\omega)$, $A \neq R(\omega)$. By Zorn's Lemma, there is a subset H of I which is maximal with respect to $A \cap R(H) = 0$.

Since each P_α $(\alpha \in I)$ is uniform, it follows that

$$(A \oplus R(H)) \cap P_\alpha \neq 0 \text{ for every } i \in I.$$

Hence $B = A \oplus R(H)$ is essential in $R(\omega)$. To complete the proof, we will show that $B = R(\omega)$.

Suppose on the contrary that $B \neq R(\omega)$. Then there exists an element $k \in I$ such that $P_k \subseteq B$.

Since P_k is uniform and B is essential in $R(\omega)$, $T = P_k \cap B$ is an uniform submodule of B with $T \neq P_k$. Let T^* be a maximal essential extension of T in B.

Therefore, B is isomorphirm to a direct summand of $R(\omega) \oplus R(\omega) \cong R(\omega)$. From this it is to easy to see that B also has property as $R(\omega)$, i.e, each closed unifom submodule of B is a direct summand in B. On the otherhand, $R(\omega)$ is a projective right R-module. By [1, Theorem 27.11], T^* is isomorphirm to some e_iR in $\{e_iR, \dots, e_nR\}$. Since $R(\omega) = P_k \oplus R(I \backslash \{k\})$ we have by modularity

$$P_k + T^* = P_k \oplus T_1,$$

where $T_1 = (P_k + T^*) \cap R(I \backslash \{k\})$.

If $T_1 = 0$ we have $P_k + T^* = P_k$, so T^* is contained in P_k and then by the previous remark on e_iR we must have $P_k = P^* \subseteq B$, a contradiction. From this we have $T_1 \neq 0$. Moreover, from the definition of T_1 we have $T_1 \subseteq R(I \backslash \{k\})$ and since $T \subset P_k$, $T \cap R(I \backslash \{k\}) = 0$. Since T^* is a maximal essential extension of T in B, $T^* \cap R(I \backslash \{k\}) = 0$, it follows $T_1 \cap T^* = 0$.

Let M be the maximal submodule of P_k. Because T^* is not embedded in M, $T^* \oplus T_1 \subseteq M \oplus T_1$. In particular, the factor module $(P_k \oplus T_1)/T_1$ is a local module with the maximal submodule $(M \oplus T_1)/T_1$. Therefore

$$(T^* \oplus T_1)/T_1 = (P_k \oplus T_1)/T_1,$$

implying $T^* \oplus T_1 = P_k \oplus T_1$. Hence $P_k + T^* = T^* + T_1$. Now by modularity we have

$$\begin{aligned} B \cap (P_k + T^*) &= (B \cap P_k) + T^* = T + T^* \\ &= T^* = B \cap (T^* \oplus T_1) \\ &= T^* \oplus (B \cap T_1). \end{aligned}$$

Consequently $B \cap T_1 = 0$, a contradiction to the fact that $T_1 \neq 0$ and B is esstial in $R(\omega)$. Thus $B = R(\omega)$, as desired.

By [6, Theorem 2.25], every local direct summand of $R(\omega)$ is a direct summand. We use this to show below that every closed submodule of $R(\omega)$ is a direct summand.

Let A be a non-zero submodule of $R(\omega)$, with $0 \neq a \in A$. Then aR is a xyclic submodule. Hence there exits a finite subset F of I such that $aR \subseteq \oplus_{j\in F} R_j$.

From this it follows that aR has finite uniform dimension, so A contains a uniform submodule.

Let Q be a non-zero closed submodule of $R(\omega)$. Then Q contains a closed uniform submodule U which is also closed in $R(\omega)$. Hence U is a direct summand of $R(\omega)$, by the hypothesis.

Let

$$\mathcal{K} = \{A = \oplus_{k\in K} U_k \,|\, U_k \text{ is a uniform submodule of } Q, A = \oplus_{k\in K} U_k \text{ is a local direct summand}\}.$$

By the above argument $\mathcal{K} \neq \emptyset$.

From this we may use Zorn's Lemma to that $\mathcal{K}$ contains a maximal element $L = \oplus_{k\in K} U_k$. Since $L = \oplus_{k\in K} U_k \subseteq R(\omega)$ and every local direct summand is a direct summand, L is a direct summand of $R(\omega)$. Say $R(\omega) = L \oplus P$ for some submodule P of $R(\omega)$.

By modularity, we have $Q = L \oplus (P \cap Q)$, lets $P' = P \cap Q$, clearly, P' is closed in Q. If $P' \neq 0, P'$ contains a uniform direct summand V of $R(\omega)$. Then it is clear that $L \oplus V = (\oplus_{k\in K} U_k) \oplus V$ is a direct summand of $R(\omega)$ with $L \oplus V \subseteq Q$, a contradiction to the maximality of L in $\mathcal{K}$. Thus $P' = 0$ and so $Q = L$ is a direct summand of $R(\omega)$. This shows that $R(\omega)$ is CS. Consequently R is QF by [4, Corollary 2].

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