NON-LINEAR STABILITY OF STIFFENED LAMINATED COMPOSITE PLATES

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Abstract. This paper deals with the non-linear stability of the stiffened laminte composite plate subjected to biaxial loads. Numerical results are presented for illustrin theoritical analysis of stiffened and unstiffened laminated composite plates.

Key words. Stiffened laminated composite plate, Shape memory alloys (SMA), stbity

1. Introduction

Stiffened laminated composite plates are used extensively in Naval, AerospaeAu tomobile applications and in Civil engineering,v.v... Today, analysis of linear larinte composite plates has been studied by many authors. However, the analysis of no-lea laminated composite plates has received comparatively little attention [3, 4, 5], speall for analysis of non-linear stiffened laminated composite plates and shells subjete t compress bi-axial loads. This problem is studied in the present paper.

2. Governing equations of laminated plates

Let's consider a rectangular stiffened laminated composite plate, in which eah ye is a unidirectional composite material. This plate is subjected to a uniform compreio on each edge, with resultants P_x and P_y respectively (Figure 1), where P_x and P_y ar arbitrarily but as the plate is working in the elastic stage, so that every stress ta defined by every loading state respectively and doesn't depend on the process an Thus, we can put

$$P_y = \alpha P_x. \qquad (1)$$

The strain-displacement relations in the non-linear theory are of the form

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2, \\ \varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\Big(\frac{\partial w}{\partial y}\Big)^2, \\ \gamma_{xy} &= \Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big) + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}, \end{aligned} \qquad (2)$$

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$$k_x = -\frac{\partial^2 w}{\partial x^2},$$
$$k_y = -\frac{\partial^2 w}{\partial y^2},$$
$$k_{xy} = -2\frac{\partial^2 w}{\partial x \partial y},$$

where u, v, w are the midplane displacements along the x, y and z axes respectively.

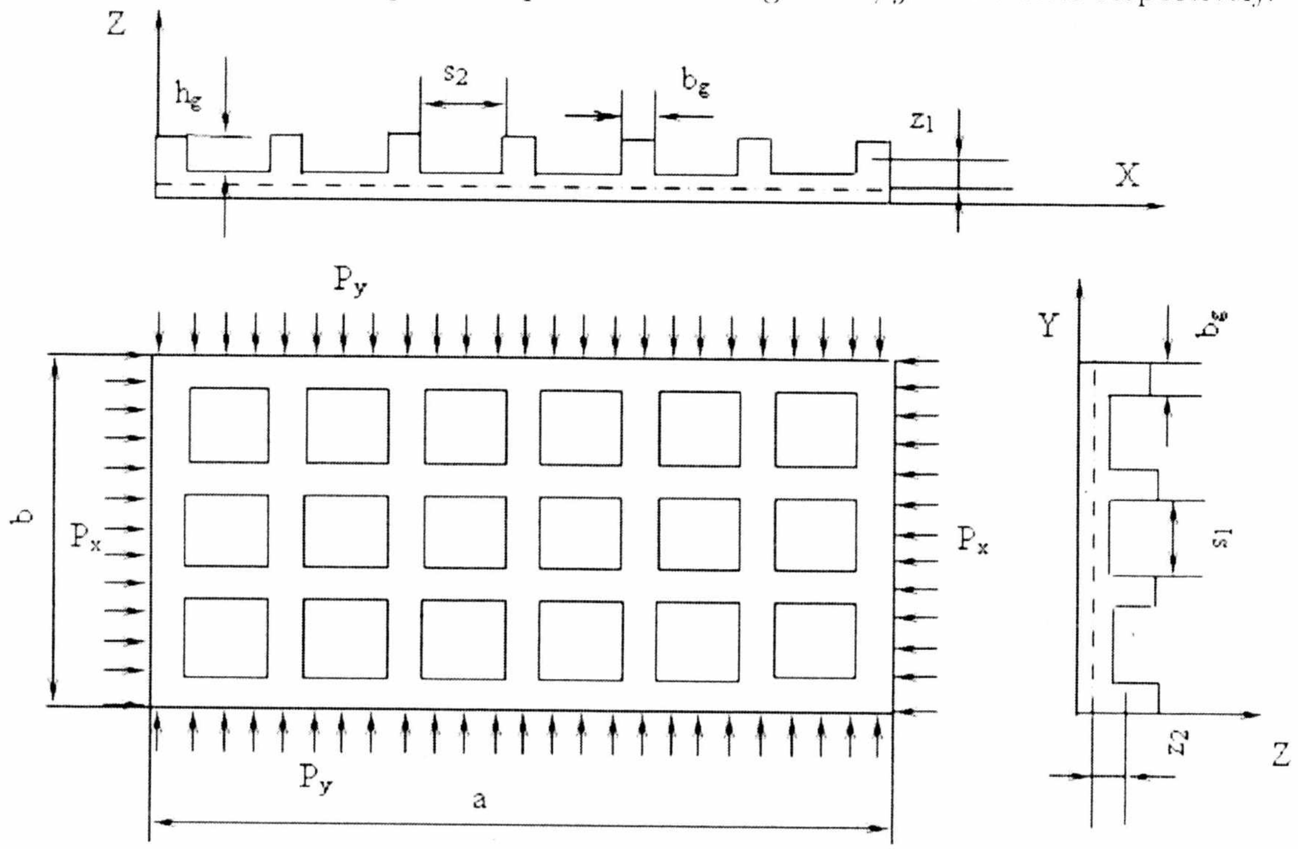

Fig. 1

Integrating the stress-strain equations through the thickness of plate we obtain the expressions for stress resultants and flexion moments:

$$\begin{aligned} N_x &= (A_{11} + E_1 A_1/s_1)\varepsilon_x + A_{12}\varepsilon_y + (E_1 A_1/s_1) z_1 k_x + P_x^c/s_1, \\ N_y &= (A_{22} + E_2 A_2/s_2)\varepsilon_y + A_{12}\varepsilon_x + (E_2 A_2/s_2) z_2 k_y + P_y^c/s_2, \\ N_{xy} &= A_{66}\gamma_{xy}, \\ M_x &= (D_{11} + E_1 I_1/s_1) k_x + D_{12} k_y + (E_1 A_1/s_1) z_1 \varepsilon_{x1}, \\ M_y &= (D_{22} + E_2 I_2/s_2) k_y + D_{12} k_x + (E_2 A_2/s_2) z_2 \varepsilon_y, \\ M_{xy} &= D_{66} k_{xy}, \end{aligned} \tag{3}$$

where

- A_{ij}, D_{ij} ($i, j = 1, 2$ and 6) are extending and bending stiffnesses of the plate without stiffeners,

- E_1, E_2 are the Young modulus of the longitudinal and transversal stiffeners, respectively,

- A_1, A_2 are the section areas of the longitudinal and transversal stiffeners, respectively,
- I_1, I_2 are the inertial moments of cross-section of the longitudinal and transversal stiffeners, respectively,
- s_1, s_2 are the distances between two longitudinal stiffeners and between two transversal stiffeners, respectively,
- z_1, z_2 are the distances from the mid-plane to the centroids of the longitudinal and transversal stiffeners, respectively,
- P_x^c, P_y^c are the recovery tensile force in the SMA wires.

The equilibrium equations of a plate according to [2] are

$$\begin{aligned}
&\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0,\\
&\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=0,
\end{aligned}\tag{4}$$

$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+N_x\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}+N_y\frac{\partial^2 w}{\partial y^2}+P_x\frac{\partial^2 w}{\partial x^2}+P_y\frac{\partial^2 w}{\partial y^2}=0,$$

Substituting (2) and (3) into (4) after some operations we obtain the equilibrium equations of the laminated plate

$$\begin{aligned}
&(A_{11}+E_1A_1/s_1)\frac{\partial^2 u}{\partial x^2}+A_{66}\frac{\partial^2 u}{\partial y^2}+(A_{12}+A_{66})\frac{\partial^2 v}{\partial x\partial y}-(E_1A_1/s_1)z_1\frac{\partial^3 w}{\partial x^3}+\\
&+(A_{11}+E_1A_1/s_1)\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+(A_{12}+A_{66})\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}+A_{66}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}=0,\\
&(A_{22}+E_2A_2/s_2)\frac{\partial^2 v}{\partial y^2}+A_{66}\frac{\partial^2 v}{\partial x^2}+(A_{12}+A_{66})\frac{\partial^2 u}{\partial x\partial y}-(E_2A_2/s_2)z_2\frac{\partial^3 w}{\partial y^3}+\\
&+(A_{22}+E_2A_2/s_2)\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}+(A_{12}+A_{66})\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}+A_{66}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2}=0.
\end{aligned}\tag{5}$$

$$\begin{aligned}
&(D_{11}+E_1I_1/s_1)\frac{\partial^4 w}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+(D_{12}+E_2I_2/s_2)\frac{\partial^4 w}{\partial y^4}-\\
&-(P_x^c+P_x)\frac{\partial^2 w}{\partial x^2}-(P_y^c+P_y)\frac{\partial^2 w}{\partial y^2}-(E_1A_1/s_1)z_1\frac{\partial^3 u}{\partial x^3}-(E_2A_2/s_2)z_2\frac{\partial^3 v}{\partial y^3}-\\
&-(E_1A_1/s_1)z_1\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3}-(E_2A_2/s_2)z_2\frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial y^3}-\frac{1}{2}(A_{11}+E_1A_1/s_1)\frac{\partial^2 w}{\partial x^2}\Big(\frac{\partial w}{\partial x}\Big)^2-
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{2}A_{12}\frac{\partial^2 w}{\partial y^2}\Big(\frac{\partial w}{\partial x}\Big)^2-\frac{1}{2}A_{12}\frac{\partial^2 w}{\partial x^2}\Big(\frac{\partial w}{\partial y}\Big)^2-\frac{1}{2}(A_{22}+E_2A_2/s_2)\frac{\partial^2 w}{\partial y^2}\Big(\frac{\partial w}{\partial y}\Big)^2-\\
&-2A_{66}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}-(A_{11}+E_1A_1/s_1)\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}-2A_{66}\frac{\partial u}{\partial y}\frac{\partial^2 w}{\partial x\partial y}-A_{12}\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial y^2}-\\
&-A_{12}\frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial x^2}-2A_{66}\frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial x\partial y}-(A_{22}+E_2A_2/s_2)\frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial y^2}=0.
\end{aligned}$$

For a plate simply supported on all edges, the following boundary conditions are imposed

+ At edges $x = 0$ and $x = a$

$$w = 0, \quad v = 0, \quad M_x = 0; \tag{6}$$

+ At edges $y = 0$ and $y = b$

$$w = 0, \quad u = 0, \quad M_y = 0; \tag{7}$$

The boundary conditions discussed here can be satisfied if the buckling mode shape is represented by

$$\begin{aligned} u &= U_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \\ v &= V_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}, \\ w &= W_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \end{aligned} \tag{8}$$

where

- a, b : edges of plate in x and y axial directions respectively,
- m, n : the numbers of halfwave in the x and y axial directions respectively.

Substituting expressions (8) into the equilibrium equations (5) and applying the Galerkin proceduce yield the set of three algebraic equations with respect to the amplitudes U_{mn}, V_{mn}, W_{mn}, where the first two equations of this system are linear algebraic equations for U_{mn}, V_{mn}

$$\begin{aligned} a_1 U_{mn} + a_2 V_{mn} &= a_3 W_{mn} + a_4 W_{mn}^2, \\ a_5 U_{mn} + a_6 V_{mn} &= a_7 W_{mn} + a_8 W_{mn}^2, \end{aligned} \tag{9}$$

Getting from (9) expression U_{mn}, V_{mn} with respect to W_{mn} and substituting into third equation of (5) we obtain a non-linear equation with respect to W_{mn}:

$$a_9 W_{mn}^3 + a_{10} W_{mn}^2 + (a_{11} + \lambda P_x) W_{mn} = 0, \tag{10}$$

where a_i are coefficients which depend on the material, geometry and the buckling mode shape,

$$\begin{aligned} a_1 &= (A_{11} + E_1 A_1/s_1)\frac{m^2 b}{a} + A_{66}\frac{n^2 a}{b}, \\ a_2 &= a_5 = (A_{12} + A_{66})mn, \\ a_3 &= (E_1 A_1/s_1) z_1 \frac{m^3 \pi b}{a^2}, \\ a_4 &= -\frac{16}{9}\left[2(A_{11} + E_1 A_1/s_1)\left(\frac{m}{a}\right)^2 \frac{b}{n\pi} - (A_{12} - A_{66})\frac{n}{b\pi}\right], \\ a_6 &= (A_{22} + E_2 A_2/s_2)\frac{n^2 a}{b} + A_{66}\frac{m^2 b}{a}, \\ a_7 &= (E_2 A_2/s_2) z_2 \frac{n^3 a\pi}{b^2}, \\ a_8 &= -\frac{16}{9}\left[2(A_{22} + E_2 A_2/s_2)\left(\frac{n}{b}\right)^2 \frac{a}{m\pi} - (A_{12} - A_{66})\frac{m}{a\pi}\right], \end{aligned}$$

$$a_9 = \frac{3}{128}\Big[(A_{11} + E_1A_1/s_1)\frac{m^4b}{a^3} + 2\Big(A_{12} + \frac{2}{3}A_{66}\Big)\frac{(mn)^2}{ab} + (A_{22} + E_2A_2/s_2)\frac{n^4a}{b^3}\Big]$$
$$+ \frac{H_1(a_6a_4 - a_2a_8) + H_2(a_1a_8 - a_5a_4)}{a_1a_6 - a_2a_5},$$
$$a_{10} = \frac{8}{9}\Big[(E_1A_1/s_1)z_1\Big(\frac{m}{a}\Big)^3\frac{b}{n\pi^2} + (E_2A_2/s_2)z_2\Big(\frac{n}{b}\Big)^3\frac{a}{m\pi^2}\Big] +$$
$$+ \frac{H_1(a_3a_6 - a_2a_7) + H_2(a_1a_7 - a_3a_5) + H_3(a_6a_4 - a_2a_8) + H_4(a_1a_8 - a_5a_4)}{a_1a_6 - a_2a_5},$$
$$a_{11} = \frac{1}{4}\Big[(D_{11} + E_1I_1/s_1)\frac{m^4b}{a^3} + 2(D_{12} + 2D_{66})\frac{(mn)^2}{ab} + (D_{22} + E_2I_2/s_2)\frac{n^4a}{b^3}\Big] +$$
$$+ \frac{H_3(a_3a_6 - a_2a_7) + H_4(a_1a_7 - a_3a_5)}{a_1a_6 - a_2a_5},$$
$$\lambda = \frac{1}{4}\Big(\frac{m^2b}{a\pi^2} + \alpha\frac{n^2a}{b\pi^2}\Big),$$

$$H_1 = -\frac{16}{9}\Big[\Big(\frac{m}{a}\Big)^2\frac{b}{n\pi^3}(A_{11} + E_1A_1/s_1) + (A_{12} + 2A_{66})\frac{n}{b\pi^3}\Big],$$
$$H_2 = -\frac{16}{9}\Big[\Big(\frac{n}{b}\Big)^2\frac{a}{m\pi^3}(A_{22} + E_2A_2/s_2) + (A_{12} + 2A_{66})\frac{m}{a\pi^3}\Big],$$
$$H_3 = -\frac{1}{4}(E_1A_1/s_1)z_1\frac{m^3b}{a^2\pi},$$
$$H_4 = -\frac{1}{4}(E_2A_2/s_2)z_2\frac{n^3a}{b^2\pi}.$$

From (10) we can express compression load with respect to W_{mn} as follows

$$P_x = \varphi(W_{mn}) \tag{11}$$

The lower buckling load of the plate can be analysed by the minimum of $\varphi(W)_{mn}$, it means that:

$$\frac{\partial P_x}{\partial W_{mn}} = 0. \tag{12}$$

The value of W_{mn} corresponding to the lower buckling load is found from the equation (12) and then substituting into equation (11) we obtain lower buckling load P_x.

We can determine the minimum critical buckling force $P_{x\,\min}$ by the way of varying m, n and $P_{y\,\min} = \alpha P_{x\,\min}$.

3. Numerical examples

Let's consider a simply supported stifferned rectangular symmetrical composite plate: $a = 0.8\,\mathrm{m}$; $b = 0.5\,\mathrm{m}$;

The materials of the plate have Thornel 300 graphite fibers and Narmco 5205 Thermosetting Epoxy resin [5], the properties of these materials are

$E_1 = 127.4\,\text{GPa}$; $E_2 = 13\,\text{GPa}$; $G_{12} = 6.4\,\text{GPa}$; $\nu_{12} = 0.38$;

- The plate has six layers: $[45/-45/90/90/-45/45]$;

- Thickness of each layer: $t = 0.5\,\text{mm}$;

- The laminate plate is reinforced by longitudinal and trainsversal stiffeners, which were made of CPS, SMA and combined materials. In combined case: longitudinal and taisverse stiffeners are SMA and CPS material respectively;

Table1. Effect of thickness of the plate (With $\alpha = 1$)

h/b	Critical buckling loads P_x (N/m)			
	Unstiffeners	CPS Stiffeners	SMA+CPS Stiffeners	SMA Stiffeners
$4{,}2.10^{-3}$	2.8155e+003	5.6321e+003	6.1798e+008	2.2000e+009
$6{,}0.10^{-3}$	8.2084e+003	1.3027e+004	6.1799e+008	2.2000e+009
$7{,}8.10^{-3}$	1.8034e+004	2.5441e+004	6.1800e+008	2.2000e+009
$9{,}6.10^{-3}$	3.3622e+004	4.4277e+004	6.1802e+008	2.2000e+009
$11{,}4.10^{-3}$	5.6302e+004	7.0943e+004	6.1805e+008	2.2001e+009
$13{,}2.10^{-3}$	8.7404e+004	1.0684e+005	6.1808e+008	2.2001e+009
$15{,}0.10^{-3}$	1.2826e+005	1.5338e+005	6.1813e+008	2.2001e+009
$16{,}8.10^{-3}$	1.8019e+005	2.1196e+005	6.1819e+008	2.2002e+009

- The stiffeners have the same sizes, as follows: $b_g \times h_g = 4\,\text{mm} \times 6\,\text{mm}$;

- Diameter of SMA wire is: $d = 1.2 \times 10^{-3}\,\text{m}$;

- Spacing of longitudinal and transverse stiffeners are: $s_1 = s_2 = 0.1\,\text{m}$.

Table 2. Effect of orientations of the plate (With $\alpha = 1$, CPS Stiffeners)

The stacking Sequence	Critical buckling loads P_x (N/m)
30/-45/90/90/-45/30	0.93695e+004
0/90/0/0/90/0	0.93939e+004
45/-45/0/0-45/45	1.1554e+004
60/-45/30/30/-45/60	1.2680e+004
45/-45/90/90/-45/45	1.3027e+004
0/-45/90/90/-45/0	1.3599e+004

Table 3. Effect of transverse stiffeners on critical buckling loads (With $\alpha = 1$)

Spacing s_2	Critical buckling loads P_x (N/m)		
	CPS Stiffeners	CPS + SMA Stiffeners	SMA Stiffeners
0.05	1.6967e+004	6.1822e+008	3.7822e+009
0.1	1.3027e+004	6.1819e+008	2.2002e+009
0.15	1.1661e+004	6.1818e+008	1.6728e+009
0.2	1.0967e+004	6.1818e+008	1.4092e+009
0.25	1.0547e+004	6.1817e+008	1.2510e+009
0.3	1.0266e+004	6.1817e+008	1.1455e+009
0.35	1.0064e+004	6.1817e+008	1.0702e+009
0.4	9.9128e+003	6.1817e+008	1.0137e+009
0.45	9.7947e+003	6.1817e+008	9.6972e+008
0.50	9.7000e+003	6.1817e+008	9.3456e+008

4. Conclusions

When considering non-linear geometry of laminated composite plate reinforced by stiffeners, we obtain:

+ Critical force of CPS laminate plate not reinforced by stiffeners:

$$P_{x\,\min} = 8.2084e + 003 N/m;$$

+ Critical force of CPS laminate plate reinforced by CPS stiffeners:

$$P_{x\,\min} = 1.3027e + 004 N/m;$$

+ Critical force of CPS laminate plate reinforced by SMA and CPS stiffeners:

$$P_{x\,\min} = 6.1799e + 008 N/m;$$

+ Critical force of CPS laminate plate reinforced by SMA stiffeners:

$$P_{x\,\min} = 2,2002e + 009 N/m;$$

Critical force of the plate being reinforced by SMA stiffeners is higher than that of CPS stiffeners.

The plates, which are reinforced by SMA stiffeners work more stably than those of CPS stiffeners (Fig. 2) when geometrical parameters of plate are varied. In the case of plates under biaxial compression, the siffeners will influence strongly on the critical force(Fig. 3).

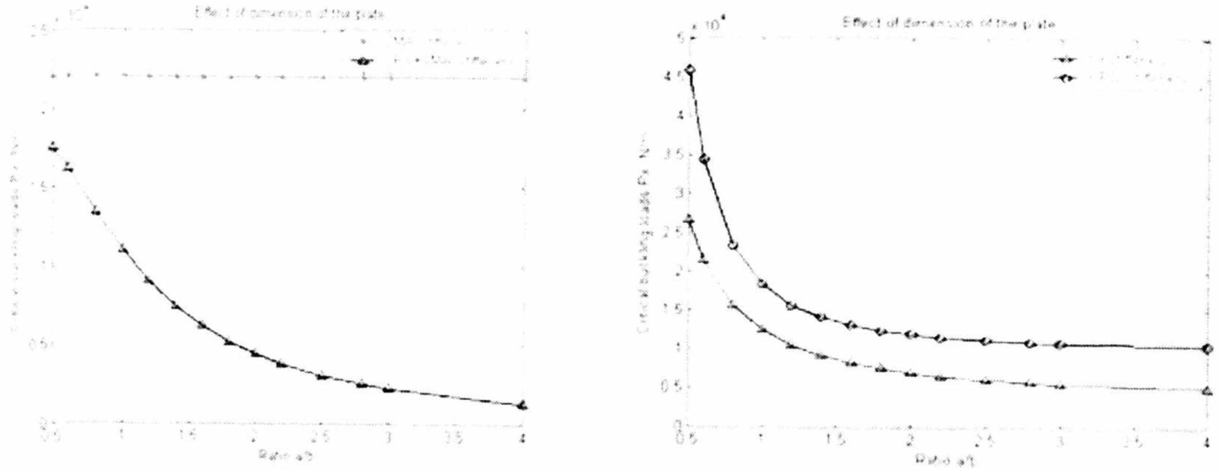

Fig. 2. Effect of dimension on critical loads (with $\alpha = 1$)

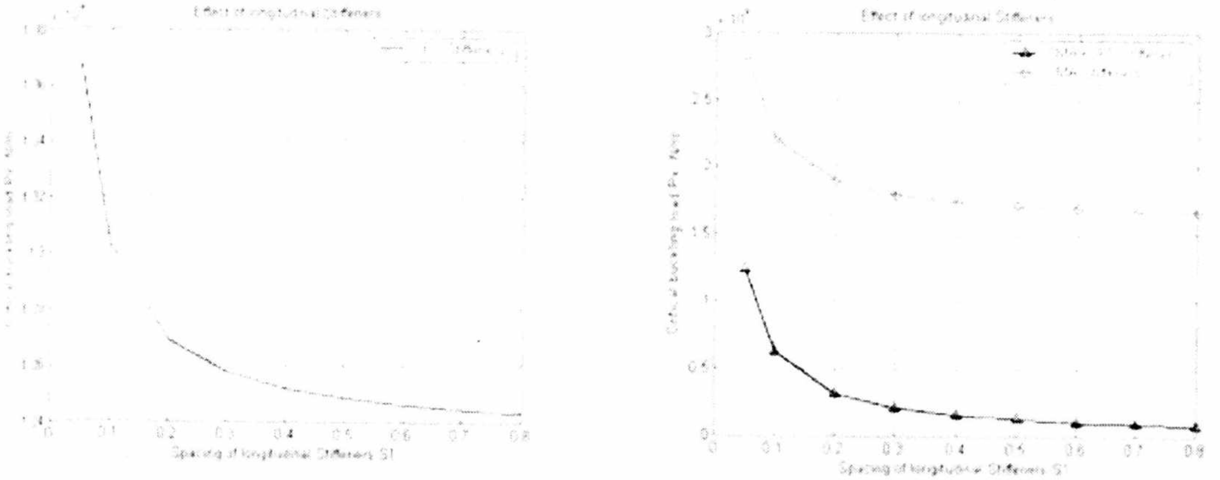

Fig. 3. Effect of longitudinal stiffeners on critical buckling loads (with $\alpha = 1$)

Depending on arranged layers of the plate, we can receive different critical buckling force. In this example, we received minimum critical buckling force coressponding to the case [30/-45/90/90/-45/30].

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