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AN ALGORITHM FOR SOLVING A CLASS OF BILINEAR INTEGER PROGRAMMING PROBLEM

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Abstract. In this paper we be concerned with a special class of bilinear integer programming problems (PI). Its objective function and constraints have variables which are multiplicative of two different variables. By restricting the integer condition of problem we shall study relaxation problem (PIR) and reduce (PI) to solve linear integer programming problems.

1. Introduction

Many real-world problems can be formulated as the following optimization problem:

$$\sum_{j=1}^{n}\Big(\sum_{i=1}^{m} c_i x_{ij}\Big) y_j \longrightarrow \max$$

subject to

$$\begin{cases} 0 < a_i \leqslant \sum_{j=1}^{n} x_{ij} y_j \leqslant A_i, \ i = 1,\dots,m, \\ 0 \leqslant x_{ij} \leqslant A_i, \ i = 1,\dots,m, \ j = 1,\dots,n, \\ \sum_{i=1}^{m} d_i x_{ij} \leqslant P_j, \ j = 1,\dots,n, \\ 0 \leqslant b_j \leqslant y_j \leqslant B_j, \ j = 1,\dots,n, \\ \sum_{j=1}^{n} M_j y_j \leqslant M, \\ x_{ij} \text{ and } y_j \text{ are integers}, i = 1,\dots,m; \ j = 1,\dots,n. \end{cases}$$

This is a bilinear integer problem. Its objective function and constraints have variable which are multiplicative of two different variables x_{ij}, y_j. A model of problem "Line up a luggage van" was given in [2].
Denoting

$$z_{ij} = x_{ij} y_j, \ z = (z_{ij}),$$

we have a program with linear objective function. However, the feasible solution set is

$$Z = X \bullet Y = \{ \ z \in D \subset \mathbb{R}^{n+m} : z = xy, x \in X, y \in Y\},$$

where

$$D = \{ \ z \in \mathbb{R}^{n+m} : 0 < a_i \leqslant \sum_{j=1}^{n} z_{ij} \leqslant A_i, i = 1,\dots,m\},$$

$$X=\{x \in \mathbb{R}^{n+m} : 0 \leqslant x_{ij} \leqslant A_i, i = 1,\dots,m; j = 1,\dots,n; \sum_{i=1}^{m} d_i x_{ij} \leqslant P_j, j = 1,\dots,n\}.$$

$$Y = \{\, y \in \mathbb{R}^n : 0 \leqslant b_j \leqslant y_j \leqslant B_j, j = 1,\dots,n, \sum_{j=1}^{n} M_j y_j \leqslant M\}.$$

In this paper we restrict the integer condition of problem and reduce to the following optimization problem (PI)

$$\text{(PI)} \quad \min c^T z \tag{1}$$

subject to

$$z \in D \tag{2}$$

$$z_j = x_j y_j, j = 1,2,\dots,p \tag{3}$$

$$x \in X, y \in Y \tag{4}$$

$$z \text{ integer}, \tag{5}$$

where

$$c = (c_j), c^T z = \sum_{j=1}^{p} c_j z_j,$$

$$D = \{\, z \in \mathbb{R}^p : \sum_{j=1}^{p} \beta_{ij} z_j \leqslant \beta_i,\ i = 1,\dots,m; 0 \leqslant z_j \leqslant \delta_j, j = 1,\dots,p\}, \tag{6}$$

$$X = \{\, x \in \mathbb{R}^p : 0 < a_j \leqslant x_j \leqslant A_j, j = 1,\dots,p\}, \tag{7}$$

$$Y = \{\, y \in \mathbb{R}^p : \sum_{j=1}^{p} \alpha_{ij} y_j = \alpha_i, i = 1,\dots,q; 0 \leqslant b_j \leqslant y_j \leqslant B_j, j = 1,\dots,p\}. \tag{8}$$

Without loss of generality assume that δ_j, β_{ij}, β_i are integers and $\beta_i \geq 0$. The problem, without integer constraints of z, was studied by T.V. Thieu [4]. In this paper, we are going to bring up an adjacent different method to transfer problem (PI) to the linear integer programming problems.

Denote

$$I_i^+ = \{j : \alpha_{ij} \geq 0, i = 1,\dots,q; j = 1,\dots,p\};$$
$$I_i^- = \{j : \alpha_{ij} < 0, i = 1,\dots,q; j = 1,\dots,p\}.$$

From (3), (7), $x_j > 0$ we infer that $y_j = \dfrac{z_j}{x_j}$. We can write the constraints (8) as

$$\sum^{p} \frac{\alpha_{ij}}{x_j} z_j = \alpha_i, i = 1,\dots,q.$$

Define

$$t_{ij} = \frac{\alpha_{ij}}{x_j},\ t_j = (t_{ij}) \in R^q, \tag{9}$$

where t_{ij} satisfy the constraints

$$\begin{aligned}(\alpha_{ij}/A_j) \leqslant t_{ij} \leqslant (\alpha_{ij}/a_j),\ \text{for all } j \in I_i^+,\\ (\alpha_{ij}/a_j) \leqslant t_{ij} \leqslant (\alpha_{ij}/A_j),\ \text{for all } j \in I_i^-.\end{aligned} \tag{10}$$

Define

$$T_j = \{t_j \in \mathbb{R}^q : t_{ij} \text{ satisfy } (10)\}.$$

Choose $x_j^* \in [a_j, A_j], j = 1, \ldots, p$. From (9) we have

$$t_j^* = (t_{ij}^*) = (\alpha_{ij}/x_j^*).$$

Without any integer constraints (5), we have relaxation problems (PIR)

$$\text{(PIR)}\ \ \min c^T z \tag{1}$$

subject to

$$z \in D, \tag{2}$$

$$z_j = x_j y_j, j = 1, 2, \ldots, p, \tag{3}$$

$$x \in X, y \in Y, \tag{4}$$

As usual a triple (x, y, z) whose entries satisfy (2), (3) and (4) is called a *feasible solution* of (PIR), a feasible solution achieving the minimum of (1) is called an *optimal solution* of (PIR).

Choose $x^* = (x_j^*), t_j^* = (t_{ij}^*) = (\alpha_{ij}/x_j^*)$, we sólving linear programming

$$\text{(LP1)}\begin{cases}\min c^T z\\ \text{subject to}\\ z \in D\\ \sum_{j=1}^p t_{ij}^* z_j = \alpha_i, i = 1, \ldots, q.\end{cases} \tag{1}$$

Let $z^* = (z_j^*)$ be a basic optimal solution of the linear problem (LP1). By B we denote the basic associated with z^* and J the index set of B. From x^* and z^* we have $y^* = z^*/x^*$. So, (x^*, y^*, z^*) is a feasible solution of problem (PIR). Denote $c_B = (c_j), j \in J$

2. Main results

Proposition 1. *Let (x^*, y^*, z^*) be a feasible solution of (PIR). If it satisfies the constraints*

$$c_B B^{-1} t_j \leqslant c_j, t_j \in T_j, \text{ for all } j = 1, \ldots, p, \tag{12}$$

then (x^*, y^*, z^*) is an optimal solution of (PIR).

Proof. Assume to the contrary that there exists a feasible solution $(\tilde{x}, \tilde{y}, \tilde{z})$ of (PIR) which is better than (x^*, y^*, z^*), i.e. such that

$$c^T \tilde{z} \leqslant c^T z^*.$$

Since $\tilde{t}_j = (\tilde{t}_{ij}) = (\alpha_{ij}/\tilde{x}_j) \in T_j$ and constraints (12) we have

$$c_B B^{-1} \tilde{t}_j \leqslant c_j, \text{ for all } j = 1, \ldots, p.$$

We have $z = z^*, u = 0, (z^*, 0)$ which is an optimal solution of problem

$$\text{(LP2)} \begin{cases} \min c^T z \\ \text{subject to} \\ z \in D \\ \sum_{j=1}^p t^*_{ij} z_j + \sum_{j=1}^p \tilde{t}_{ij} u_j = \alpha_i, \ i = 1, \ldots, q. \end{cases} \tag{13}$$

Otherwise, with $z = 0, u = \tilde{z}$, $(0, \tilde{z})$ is a feasible solution of (LP2), with

$$c^T \tilde{z} \geq c^T z^*.$$

Hence $c^T \tilde{z} = c^T z^*$.

This shows that (x^*, y^*, z^*) is an optimal solution of (PIR).

Remark 1. To verify constraints (12), for every $j = 1, 2, \ldots, p$, we can to solve problem

$$\text{(LP3)} \begin{cases} \max(c_B B^{-1} t_j) \\ \text{subject to} \\ t_j \in T_j \end{cases}$$

and look for an optimal solution $t_j^{'} = (t_{ij}^{'})$. If $(c_B B^{-1} t_j^{'}) \leqslant c_j$, for every $j = 1, \ldots, p$, then constraints (12) are satisfied.

Suppose now that (x^*, y^*, z^*) does not satisfy the constraints (12), i. e. there exists $t_j^{'} \in T_j$ such that

$$c_B B^{-1} t_k^{'} > c_k. \tag{14}$$

Consider the linear program

$$(\text{LP4})\begin{cases} \min(c^T z + c_k v_k) \\ \text{subject to} \\ \sum_{j=1}^{p} t^*_{ij} z_j + t'_k v_k = \alpha_i, \ i = 1, \ldots, q \\ \sum_{j=1}^{p} \beta_{ij} z_j + \beta_{ik} v_k \leqslant \beta_i, i = 1 \ldots, m \\ 0 \leqslant z_j \leqslant \delta_j, \ j = 1, \ldots, p, \ v_k \geq 0, \end{cases} \tag{15}$$

where v_k is a nonnegative variable.

Assume that (LP4) has an optimal solution (z', v'_k).

Denote

$$\hat{z}_j = \begin{cases} z'_k + v'_k, \text{ if } j = k \\ z'_j, \text{ if } j = 1, \ldots, p, j \neq k \text{ and } z'_k + v'_k \neq 0 \\ z^*_j, \text{ if } j = 1, \ldots, p, j \neq k \text{ and } z'_k + v'_k = 0 \end{cases} \tag{16}$$

and

$$\hat{t}_j = \begin{cases} t^*_j, \text{ if } j \neq k, j = 1, \ldots, p \\ t'_j, \text{ if } j = k, z'_k + v'_k = 0 \\ \dfrac{z'_k t^*_k + v'_k t'_k}{z'_k + v'_k}, \ j = k, \ z'_k + v'_k \neq 0. \end{cases} \tag{17}$$

Proposition 2. *If there exists an index k satisfying (14), then (x^*, y^*, z^*) can be changed to a new feasible solution $(\hat{x}, \hat{y}, \hat{z})$ of the problem (PIR) which is either better than (x^*, y^*, z^*).*

Proof. From z^*, z', v'_k and applying (16), (17) we have $\hat{z}_j, \hat{t}_j$, $(j = 1, \ldots, p)$. Since T_j is the convex set, $\hat{t}_j \in T_j$. then we get

$$\sum_{j=1}^{p} \hat{t}_j \hat{z}_j = \sum_{j=1, j\neq k}^{p} t^*_j z^*_j + t'_k (z'_k + v'_k).$$

If $z'_k + v'_k = 0$, then

$$\sum_{j=1}^{p} \hat{t}_j \hat{z}_j = \sum_{j=1}^{p} t^*_j z^*_j = \alpha \text{ (because } z^*_k = 0 \text{ for } k \notin J\text{), where } \alpha = (\alpha_i).$$

If $z'_k + v'_k \neq 0$, then

$$\sum_{j=1}^{p} \hat{t}_j \hat{z}_j = \sum_{j=1, j\neq k}^{p} t^*_j z'_j + \frac{z'_k t^*_k + v'_k t'_k}{z'_k + v'_k} (z'_k + v'_k)$$
$$= \sum_{j=1}^{p} t^*_j z'_j + v'_k t'_k = \alpha.$$

We have

$$\sum_{j=1}^{p} \beta_{ij}\hat{z}_j = \sum_{j=1,j\neq k}^{p} \beta_{ij}\hat{z}_j + \beta_{ik}\hat{z}_k$$
$$= \sum_{j=1,j\neq k}^{p} \beta_{ij}z_j^* + \beta_{ik}(z_k' + v_k'), \text{ if } z_k' + v_k' = 0$$
$$= \sum_{j=1}^{p} \beta_{ij}z_j^* \leqslant \beta_i \text{ (because } j \notin J), \text{ if } z_k^* = 0.$$

From (16) we have

$$\sum_{j=1}^{p} \beta_{ij}\hat{z}_j = \sum_{j=1,j\neq k}^{p} \beta_{ij}z_j' + \beta_{ik}(z_k' + v_k'), \text{ if } z_k' + v_k' \neq 0.$$

It follows that $\sum_{j=1}^{p} \beta_{ij}\hat{z}_j \leqslant \beta_i$. (see (15)). It is easy to see that $\hat{z}_j \geq 0$. From $\hat{t}_j$ and (9) we find $\hat{x}$ and $\hat{z}$, from (3) find $\hat{y}$. This shows that $(\hat{x}, \hat{y}, \hat{z})$ is a feasible solution of (PIR). It is easily verified that $(z^*, 0)$ is a feasible solution of (LP4), but from (14) then $(z^*, 0)$ is not an optimal solution of (LP4). It follows that $c^T z^* > c^T z' + c_k v_k' = c^T \hat{z}_j$, i. e. $(\hat{x}, \hat{y}, \hat{z})$ is better than (x^*, y^*, z^*).

Before presenting the algorithm, we have some remarks.

Remark 2. Relaxation problems (PIR) haven't integer constraints. Since D is a polyhedron, using the Gomory cut method (or the coordinate cut [1]) for solving linear integer programming, it follows that after a finite number of steps we receive an optimal integer solution.

Remark 3. Since D is a polyhedron, using methods of linear programming, after a finite number of steps we receive an optimal solution of program (LP1).

Remark 4. The solving (LP3) is an easy task because T_j, for every $j = 1, 2, \ldots, p$, is a rectangle (from (10)).

Remark 5. (LP4) and (LP1) differ from one to another only by a new column. Hence, to solve (LP4) we can use the solution of (LP1). Applying the reoptimization technique of linear programming, we have the solution of (LP4).

. The algorithm for solving problem (PI)

From the above results we are now in a position to derive an algorithm for solving problem (PI). The algorithm consists of the following steps.

Step 1. Take $x^* \in X$, determine $t_j^* \in T_j$ from (9), (10). Solve the linear program (LP1). Let z^* be a basic optimal solution, with basic B and the index set J of B.

Step 2. For every $j = 1, 2, \ldots, p$, we solve the linear program (LP3) and obtaining an optimal solution $t_j' = (t_{ij}')$.

If $(c_B B^{-1} t_j^{'}) \leqslant c_j$, with $j = 1, \ldots, p$, then (z^*, x^*) is an optimal solution of relaxation problems (PIR). Go to Step 4.

Otherwise, there exists a first index k satisfy $(c_B B^{-1} t_k) > c_k$. Go to Step 3.

Step 3. Solve the linear program (LP4), let $(z^{'}, v_k^{'})$ be an optimal solution. From (16) and (17) we change to a new feasible solution $(\hat{x}, \hat{y}, \hat{z})$ which is better than (x^*, y^*, z^*) Go to Step 1.

Step 4. If z^* is integer then (x^*, y^*, z^*) is an optimal solution of (PI). Otherwise to add a cut constraint and go to Steps 1.

Proposition 3. *The above algorithm terminates after a finite number of steps.*

Proof From remarks 1, 2, 3, 4, 5 we have the proposition.

References

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