B-REGULARITY OF HARTOGS DOMAINS

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Abstract. Let Ω be a bounded set in $\mathbb{C}^n$ and $\varphi : \Omega \to [-\infty, \infty)$ an upper semicontinuous function on Ω. Consider the Hartogs domain $\Omega_\varphi = \{(z, w) \in \Omega \times \mathbb{C} : \log|w| + \varphi(z) < 0\}$. In this note, we give some necessary and sufficient conditions on B-regularity of Ω_φ.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$. An important question of (real) potential theory is to ask whether every continuous function on $\partial\Omega$ is the boundary values of some harmonic function on Ω. Using Perron's method, it can be shown that the answer to this question is affirmative if and only if at every point z of $\partial\Omega$, we can find a subharmonic function u in Ω such that $\overline{\lim}_{\xi\to z} u(\xi) = 0$ and $\overline{\lim}_{\xi\to y} u(\xi) < 0, y \neq z$ on $\overline{\Omega} \setminus \{z\}$. It is not hard to see that this characterization holds for every smoothly bounded domain in $\mathbb{R}^n$ (see [1]).

Now, it is natural to ask a similar problem in the complex setting. Namely, if Ω is a bounded domain in $\mathbb{C}^n$, under what conditions every continuous function f on $\partial\Omega$ can be extended to a plurisubharmonic function in Ω and continuous on $\overline{\Omega}$.

The Perron's method breaks down as the envelope

$$u_{f,\Omega} = \sup\{u(z) : u \in PSH(\Omega), \overline{\lim}\, u\,|_{\partial\Omega} \leqslant f\}.$$

may not be upper semicontinuous.

To solve this problem, Sibony in [3] introduced the following classes: B-regular compact set and B-regular domain. Roughly speaking, on a B-regular compact set every continuous function is the uniform limit of continuous plurisubharmonic functions on open neighbourhoods of the compact set and B-regular domains are domains such that every continuous function on $\partial\Omega$ can be extended continuously to $\overline{\Omega}$ to a plurisubharmonic function in Ω. Under very mild conditions, Sibony showed that Ω is B-regular if $\partial\Omega$ is B - regular.

The aim of this paper is to apply Sibony's results to study B - regularity of a concrete class of domain. Namely, the class of Hartogs domains. Let us defined what we mean by Hartogs domains. Let Ω be a bounded domain in $\mathbb{C}^n$ and φ a real valued, bounded from below, upper semicontinuous function on Ω. We set

$$\Omega_\varphi = \{(z, w) : \log|w| + \varphi(z) < 0\}.$$

It is clear that the unit ball and polydisks in $\mathbb{C}^{n+1}$ are Hartogs domains. However, we will see later that the unit polydisk is not B-regular (the boundary contains analytic structure)

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX while the unit ball is B-regular (being a strictly pseudoconvex domain). Thus, the problem of characterizing the B-regularity of Hartogs domains is of interest. In this paper, we give some necessary and sufficient conditions to ensure that. The present paper consists of two parts. In the first section, we give some definitions and facts about B-regular compact sets and B-regular domains. In the second section, we prove the following theorem which is the main result of the paper

Theorem 1. *Let Ω and φ as above. Assume that Ω_φ is B-regular. Then, we have:*

i. $\varphi \in PSH(\Omega) \cap C(\Omega)$.

ii. Ω *is B-regular.*

iii. $\lim_{z \to \zeta} \varphi(z) = +\infty, \quad \forall \zeta \in \partial\Omega$. *Conversely if Ω and φ satisfy (i), (ii) and (iii) and if the set*

$$X = \{z \in \Omega, \varphi \text{ is not strictly plurisubharmonic at } z\}$$

is locally B-regular then Ω_φ is B-regular.

We note that the condition on X can't remove. Indeed, if it is, we have a counterexample (see example 3.2).

2. Preliminaries

We first give some facts about B-regular compact sets.

Definition 2.1 Let K be a compact subset of $\mathbb{C}^n$. We say that u is a *plurisubharmonic function* on K, denote by $u \in PSH(K)$ if the following two conditions are satisfied

i. $u \in C(K)$.

ii. There exist a sequence of open neighbourhoods $\{U_j\}_{j=1}^{\infty}$ of K and a sequence $\{u_j\}_{j=1}^{\infty}$ such that $u_j \in PSH(U_j)$ and u is uniformly approximated by $\{u_j\}$ on K.

Definition 2.2. Let K be a compact set in $\mathbf{C}^n$. A measure μ on K is called a *Jensen measure* with barycenter at $z \in K$ if μ is Borel positive, regular measure supported on K and satisfies

$$u(z) \leqslant \int_K u d\mu \ \forall u \in PSH(K).$$

We denote by J_z the set of Jensen measures with barycenter at z.

Definition 2.3.

a) Let K be a compact subset of $\mathbb{C}^n$. K is said to be *B- regular* if and only if for every continuous function u on K is uniform approximated on K by continuous plurisubharmonic functions on neighbourhoods of K.

b) A locally closed set X in $\mathbb{C}^n$ is said to be *locally B - regular* iff for every $z \in X$ there is a closed neighbourhood V_z such that $X \cap V_z$ is B - regular.

The following result, which is an easy consequence of a duality theorem due to Elwards, gives us a connection between Jensen measure and the B regularity.

Theorem 2.1. *Let K be a compact in $\mathbf{C}^n$. Then K is B regular if and only if $J_z = \{\delta_z\}$ for all $z \in K$, where δ_z is the Dirac measure at z.*

We give some examples of B-regular compact sets.

Example 2.1. If K is the unit circle then for every $u \in C(K)$, there is $\tilde{u}$ continuous on $\{|z| \leqslant 1\}$, harmonic in $\{|z| < 1\}$, such that $u = \tilde{u}$ on K.

For each $t > 1$, we let

$$\tilde{u}_t(z) = \tilde{u}(\frac{z}{t}).$$

Clearly, $\tilde{u}_t$ is harmonic in $\{|z| < t\}$ and $\tilde{u}_t \rightrightarrows u$ as t tends to 1. So, the unit circle is B-regular compact.

Example 2.2. If compact set K is B-regular in $\mathbb{C}^n$ then every compact subset K' of K is B-regular. Indeed, let $u \in C(K')$. By the Tietze extension theorem, we can find $\tilde{u} \in C(K)$ such that $\tilde{u} = u$ on K'. Because of B-regularity of K, there are open neighbourhoods $\{U_j\}$ of K and sequence $\{\tilde{u}_j \in PSH(U_j)\}$ whose limits is equal to $\tilde{u}$. Hence $\tilde{u}_j$ approximate u.

Now, we give some properties of B-regular compact sets based on [3].

Proposition 2.2.

a) K is $B-$ regular if and only if $-|z|^2 \in PSH(K)$.

b) $\cup_{n=1}^{\infty} K_n$ is B-regular if K_n is B-regular for each n.

We recall some definitions.

Definition 2.4. A bounded domain $\Omega \Subset \mathbb{C}^n$ is said to be *B-regular* if for every function $f \in C(\partial\Omega)$, there exists a function $u \in PSH(\Omega) \cap C(\overline{\Omega})$ such that $u |_{\partial\Omega}= f$.

Definition 2.5.

a) A domain Ω in $\mathbb{C}^n$ is said to be *pseudoconvex* if there exists a plurisubharmonic exhaustion function u on Ω, i.e $\{z : u(z) < c\}$ is relatively compact in Ω for all real c.

b) A domain Ω in $\mathbb{C}^n$ is said to be *hyperconvex* if there exist a negative plurisubharmonic exhaustion in Ω, that is $u \in PSH(\Omega), u < 0$ in Ω such that for every $c < 0$ we have $\{u(z) < c\} \Subset \Omega$.

Proposition 2.3. *For a bounded domain we have the following implications: $B-$ regularity $\Rightarrow$ hyperconvexity $\Rightarrow$ pseudoconvexity. Moreover, the inverse implications are false.*

Proof. Assume that Ω is a $B-$ regular domain. Then there exists $u \in PSH(\Omega) \cap C(\overline{\Omega})$ such that $u(z) = -|z|^2$ for $z \in \partial\Omega$. Clearly $v(z) = u(z) + |z|^2$ is a negative plurisubharmonic exhaustion for Ω. Next, if Ω is hyperconvex with a negative plurisubharmonic exhaustion function u then $-1/u$ is a plurisubharmonic exhaustion for Ω. Finally the bidisk is hyperconvex but not B regular and the Hartogs triangle $\{(z, w) : |z| < |w| < 1\}$ is pseudoconvex but not hyperconvex.

In [3], we have relations between B-regular domains and B-regular compact sets

Theorem 2.4. *If Ω is a bounded hyperconvex domain in $\mathbb{C}^n$ and $\partial\Omega$ is B-regular then Ω is B-regular.*

Conversely if Ω is a B-regular domain with C^1 boundary then $\partial\Omega$ is B-regular.

Finally, let Ω be a bounded domain in $\mathbb{C}^n$, $u \in PSH(\Omega)$, u is called strictly plurisubharmonic at $z_0 \in \Omega$ if we can find a neighbourhood V of z_0 such that $u(z) = \lambda|z|^2 + \varphi(z)$ where λ is some positive real and $\varphi(z) \in PSH(V)$.

3. Proof of the main theorem

3.1. *Necessary condition*

i). First assume that Ω_φ is B-regular. By Proposition 2.3 we have Ω_φ is hyperconvex. In particular, it is pseudoconvex. So $\varphi \in PSH(\Omega)$. We claim that $\varphi \in C(\Omega)$. Otherwise, we can find $z_0 \in \Omega$ such that $\underline{\lim}_{z\to z_0} \varphi(z) < \varphi(z_0)$. Hence, there exist a sequence $\{z_n\} \subset \Omega, z_n \to z_0$ and $\lim\limits_{n\to\infty} \varphi(z_n) = c < \varphi(z_0)$, where $c = \underline{\lim}_{z\to z_0} \varphi(z)$. We can suppose that $c < \varphi(z_0) - \epsilon$ where $\epsilon > 0$ is sufficiently small. Then $\{(z_0, w) : |w| \leqslant e^{-\varphi(z_0)-\epsilon}\} \subset \partial\Omega_\varphi$. By $\lim\limits_{n\to\infty} \varphi(z_n) = c < \varphi(z_0)$, we have $\{(z_n, w) \in \Omega \times \mathbb{C} : |w| = e^{-\varphi(z_n)-\epsilon}\} \subset \Omega_\varphi$, for n large enough. Set

$$u(z,w) := \begin{cases} 0 & z = z_0, |w| = e^{-\varphi(z_0)-\epsilon} \\ 1 & z = z_0, w = 0 \end{cases}$$

Again using Tietze theorem we extend u to a continuous function on $\partial\Omega_\varphi$. By B-regularity of Ω_φ there exists $v \in PSH(\Omega_\varphi) \cap C(\overline{\Omega_\varphi})$ such that $v = u$ on $\partial\Omega_\varphi$. In particular, $v(z_n, 0) \leqslant \max\limits_{|\zeta|=e^{-\varphi(z_n)-\epsilon}} v(z_n, \zeta)$. Let n tends to infinity, note that v is continuous, we get a contradiction.

ii). We have $\partial\Omega \times \{0\} \subset \partial\Omega_\varphi$. Let u be an arbitrary continuous function on $\partial\Omega$. Put $v(z,0) = u(z), z \in \partial\Omega$. Then v is a continuous function on $\partial\Omega_\varphi$. By the B-regularity of Ω_φ we can find $\tilde{v} \in PSH(\Omega_\varphi) \cap C(\overline{\Omega_\varphi})$ such that $\tilde{v} = u$ on $\partial\Omega \times \{0\}$. Thus, $\tilde{u}(z) = \tilde{v}(z,0)$ is continuous on $\overline{\Omega}$, plurisubharmonic in $\Omega, \tilde{u}$ coincides with $u(z)$ on $\partial\Omega$. This implies Ω is B-regular.

iii). Suppose that there is $\zeta \in \partial\Omega$ satisfies $\lim\limits_{z\to\zeta} \varphi(z) < \infty$. Let

$$D = \{(\zeta, w) : |w| = e^{-\underline{\lim}_{z\to\zeta} \varphi(z)}\}.$$

It is clear that $D \subset \partial\Omega_\varphi$. We choose a function $u = 0$ on $D, u(\zeta, 0) = 1$, and extend u continuity on $\partial\Omega_\varphi$. By B-regularity of Ω_φ, there is $v \in PSH(\Omega_\varphi) \cap C(\overline{\Omega_\varphi})$ such that $v = u$ on $D \cup \{(\zeta, 0)\}$. Using similar argument as in proof of i) and maximal principle, we get a contradiction.

3.2. *Sufficient condition*

Let Ω and φ be as above and we have to show that Ω_φ is $B-$ regular. Since Ω is B-regular, using Proposition 2.3, it implies that Ω is hyperconvex. Hence we find a negative plurisubhamonic function u in Ω such that $u = 0$ on $\partial\Omega$. Set $v(z, w) = u(z)$. Then v is a plurisubharmonic function in $\Omega_\varphi, v = 0$ on $\{(z, w) \in \partial\Omega_\varphi : z \in \partial\Omega\}$ and

$\tilde{u}(z,w) = \max(v(z,w), \log|w| + \varphi(z))$ is a negative plurisubharmonic exhaustion function in Ω_φ and $\tilde{u} = 0$ on $\partial\Omega_\varphi$. This shows that Ω_φ is hyperconvex. We claim that $\partial\Omega_\varphi$ is B-regular, so Ω_φ is also B-regular, by Theorem 2.4.

Let $p = (z_0, w_0) \in \partial\Omega_\varphi$ be a arbitrary boundary point.

If $z_0 \in \partial\Omega$, by $\lim\limits_{z\to z_0} \varphi(z) = \infty$, we have $w_0 = 0$. Because of B-regularity of Ω, we can find $u \in PSH(\Omega) \cap C(\overline{\Omega})$ such that $u(z_0) = 0$ and $u < 0$ on $\overline{\Omega} \setminus \{z_0\}$. Put $v(z,w) = u(z)$. It is clear that $v(z,w)$ is a strong plurisubharmonic barrier at (z_0, w_0).

If $z_0 \notin \partial\Omega$ then $log|w_0| + \varphi(z_0) = 0$. Note that $\varphi \in C(\Omega)$, there is $r > 0$ such that

$$\overline{B}(p,r) \cap \partial\Omega_\varphi = \{(z,w) \in \overline{B}(p,r) : \log|w| + \varphi(z) = 0\}.$$

We claim that $\overline{B}(p,r) \cap \partial\Omega_\varphi$ is B-regular. So $B(p,r) \cap \Omega_\varphi$ is B-regular, by Proposition 2.3. So Ω_φ is locally B-regular.

Assume that $(z,w) \in \tilde{A}$, where $\tilde{A}$ is a compact subset of $\overline{B}(p,r) \cap \partial\Omega_\varphi, \pi(\tilde{A}) \cap X = \emptyset, \pi$ is projection from $\mathbb{C}^{n+1}$ to $\mathbb{C}^n, \pi(z,w) = z$. Then $\varphi(z)$ is strictly plurisubharmonic at z, we can find a small neighbourhood V of z and $\psi(z) \in PSH(V)$ such that $\varphi(z) = \lambda|z|^2 + \psi(z), \lambda$ is some positive real. On $\tilde{A} \cap (\overline{V} \times \mathbb{C})$, we have

$$0 = \varphi(z) + log|w| = \psi(z) + \lambda|z|^2 + log|w|$$

This implies that $\psi(z) + log|w| = -\lambda|z|^2$. Consequenly, $-\lambda|z|^2 \in PSH(\tilde{A} \cap (\overline{V} \times \mathbb{C}))$. Using Proposition 1.3 in [3], we obtain a strong plurisubharmonic barrier at (z,w). This concludes that $\tilde{A}$ is B-regular.

Next, if $(z,w) \in \tilde{\tilde{A}}$, where $\tilde{\tilde{A}}$ is an arbitrary compact subset of $\overline{B}(p,r) \cap \partial\Omega_\varphi, (.) \subset X$. We use following lemma of Poletsky [2]

Lemma 3.1. *Suppose that K is a compact set in an open set V such that there is a continuous plurisubharmonic function u on V equal to zero on K and greater than zero on $V \setminus K$. If v is a plurisubharmonic function defined on a neighbourhood $W \subset V$ of K and bounded below on K then there is a plurisubharmonic function $\tilde{v}$ on V which coincides with v on K.*

Using Poletsky's lemma, we prove the following

Lemma 3.2. *Let X be a compact set in $\mathbf{C}^{n+1}$ and π_1, π_2 be the projection*

$$\pi_1 : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^n, \pi_1(z,w) = z,$$
$$\pi_2 : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}, \pi_2(z,w) = w.$$

such that $0 \notin \pi_2(X)$. Assume that $Y = \pi_1(X)$ is B- regular and for every $y \in Y, \pi_1^{-1}(y)$ is circle. Then X is B-regular.

Proof. Suppose that $a \in X$ and $\mu \in J_a(X)$ is an arbitrary Jensen measure with barcentre a. For every $v \in PSH(\pi_1(X))$, we have $v \circ \pi_1 \in PSH(X)$. Hence,

$$v(\pi_1(a)) \leqslant \int\limits_X v(\pi_1(x))d\mu(x).$$

but by definition of image measure,

$$\int\limits_X v(\pi_1(x))d\mu(x) = \int\limits_{\pi_1(X)} vd(\pi_{1*}\mu).$$

Then the image measure $\pi_{1*}\mu$ is also Jensen measure with barycenter $\pi_1(a)$ on $\pi_1(X)$. By $\pi_1(X)$ is B-regular, we have $\pi_{1*}\mu = \delta_{\pi_1(a)}$, where δ denotes the Dirac measure at $\pi_1(a)$. Hence μ is support on $\pi_1^{-1}(\pi_1(a)) \cap X$. Note that the circle $\pi_1^{-1}(\pi_1(a)) \cap X \setminus \{0\}$ satifies the hypothesis of lemma 3.1 with $u = \left(\frac{1}{|z|^2} - 1\right)^2$. Using Poletsky' lemma, and choose a stricty negative subharmonic function v on neighbourhood of circle such that $v(a) = 0$, we extend v to a subharmonic function $\tilde{v}$ on $\mathbb{C} \setminus \{0\}$. Since $0 \notin \pi_2(X)$ then we may assume that $\tilde{v} \in PSH(X)$. Thus

$$0 = v(a) = \tilde{v}(a) \leqslant \int\limits_X \tilde{v}d\mu \leqslant 0.$$

Hence $a \in J(X)$, where $J(X)$ is Jensen boundary of X due to [3]. Using Proposition 1.3 in [ ] we obtain that X is B-regular. The proof is finished.

Now, we continue the proof of the main theorem.

It is obvious that $\tilde{A} \subset \{(z, w) \in \Omega \times \mathbb{C} : log|w| + \varphi(z) = 0\}$. Consider the projection $\pi : \tilde{A} \to \Omega$. Then $\pi(\tilde{A})$ is B-regular (because X is locally B-regular) and every fibre $\{w : (z, w) \in \tilde{A}\}$ is circle. An application of Lemma 3.2 we conclude that $\tilde{A}$ is B-regular. Since $\overline{B}(p, r) \cap \partial\Omega_\varphi$ is union of all compact sets of the froms $\tilde{A}$ and $\tilde{A}$, by a Sibony result (see Proposition 2.2). $\overline{B}(p, r) \cap \partial\Omega_\varphi$ is B-regular. So, by applying a result of Sibony we have $B(p, r) \cap \Omega_\varphi$ is B-regular.

We are going to apply this theorem to obtain an another class of B regular domains.

Example 3.1. Let $\triangle$ be a unit disc in $\mathbb{C}$ and $\triangle'$ be compact subset of a circle in $\triangle$. From Example 1, we known that $\triangle'$ is B-regular in $\mathbb{C}$. We want to find $\varphi \in SH(\triangle) \cap C(\triangle)$ such that $\varphi = \infty$ on $\partial\Delta$ and $X = \{z, \varphi$ is not strict plurisubharmonic at $z\} \subset \triangle'$. So we have $\triangle_\varphi$ is a Hartogs domain and is B-regular. Let g be a continuous non negative function on $\triangle$ which vanishes precisely on $\triangle'$. Put $\tilde{\varphi} = G_\triangle * g$ where $*$ denote convolution operator and G is Green function of Δ or is fundamental solution of homogeneonaly Dirichlet problem on $\Delta \setminus \Delta'$. Then we have $\tilde{\varphi} \in SH(\triangle)$ satisfying $\Delta\tilde{\varphi} = g$. It is easy to look for ψ which is strictly subharmonic on $\triangle$, $\psi(z) = \infty$ on $\partial\triangle$ (e.g. $\psi(z) = \frac{1}{1-|z|^2} + |z|^2$). Let $\varphi = \max(\tilde{\varphi}, \psi)$. Then we have our desired function.

Example 3.2. Let $\Omega = \{|z| < 1\}$, $\varphi \in SH(\Omega) \cap C(\Omega)$ such that

$$\varphi = 0 \text{ on } |z| < \frac{1}{2},$$

$$\lim_{z \to \xi} \varphi(z) = +\infty \text{ for every } \xi \in \partial\Omega.$$

Then the set $X \supset \left\{|z| < \frac{1}{2}\right\}$ and hence X is not locally B-regular. It is clear that the conditions i), ii), iii) are satified but Ω_φ is not B-regular. Indeed,

$$\partial\Omega_\varphi \supset \{(z, w) : |z| < \frac{1}{2}, |w| = 1\}$$

so $\partial\Omega_\varphi$ have analytic structure, consequently, Ω_φ is not B-regular. (We choose a continuous function u=1 on $\{z = 0, |w| = 1\}$, u=0 on $\{|z| = \frac{1}{4}, |w| = 1\}$ and extend u to a continuous function on $\partial\Omega_\varphi$. Using similar argument as in proof of i) we get a contradiction, so Ω_φ is B-regular).

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