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NOTE ON THE ASYMPTOTIC STABILITY OF SOLUTIONS OF DIFFERENTIAL SYSTEMS

Dao Thi Lien
Thai Nguyen Teacher Training College

Abstract. We shall discuss the asymptotic behavior of solutions of differential systems. Some new notions of stability and examples will be given and some stability conditions will be proved.

Consider the differential system

$$\frac{dx}{dt} = X(t, x) \tag{1}$$

$$X(t, 0) \equiv 0, t \in I = [a, +\infty), a \geq 0,$$

where $x \in \mathbb{R}^n, D = \{(t, x) \mid t \in I, \|x\| < H,\}$ $H > 0$ and suppose that function

$$X: \quad D \longrightarrow \mathbb{R}^n$$
$$(t, x) \longmapsto X(t, x)$$

is continuous and satisfies condition of uniqueness of solution in D. There is a vast literature on the theory and applications of Liapunov's second method (see, for example, [1], [2], [3], [4], [5], [6], [7], [8], [9]). Here we shall discuss on the "degree" of the asymptotic behavior of solution of differential system (1).

As well known, if there exist the numbers $N > 0, \gamma > 0$ such that

$$\|x(t, t_0, x_0)\| \leqslant N.\|x_0\|.e^{-\gamma(t-t_0)}, \quad \forall t \geqslant t_0, \tag{2}$$

the zero solution of the system (1) is exponential asymptotically stable.

However, there exist some motions which is not exponentially stable but it tends to zero more fast than $P(t) = \frac{c}{(t-t_0)^\lambda} \rightarrow 0$, as $t \rightarrow +\infty$, $(\lambda > 0)$

First, we give some definitions.

1. Definitions and examples

Definition 1.1. The trivial solution of (1) is said to be *quasi-asymptotically stable of order* $\lambda(\lambda \in \mathbb{R}_+)$ if given any $\epsilon > 0$ and any $t_0 \in I$ there exist a $\delta = \delta(t_0, \epsilon)$ and a $T = T(t_0, \epsilon)$, such that

$$\|x_1(t; t_0, x_0)\| \leqslant \epsilon(t - t_0)^{-\lambda} \tag{3}$$

for all $t \geq t_0 + T(t_0, \epsilon)$ if $\|x_0\| < \delta$, or there exist the numbers $N > 0$ and $T > 0$ such that

$$\|x(t, t_0, x_0)\| \leqslant N\|x_0\|(t - t_0)^{-\lambda}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX for all $t \geq t_0 + T$.

Definition 1.2. The trivial solution of (1) is said to be *quasi-uniform-asymptotically stable of order* λ if the numbers δ and T in Definition 1 are independent of t_0.

Definition 1.3. The trivial solution of (1) is said to be *equi-asymptotically stable of order* λ if it is stable in the sense of Liapunov and quasi-asymptotically stable of order λ.

Definition 1.4. The trivial solution of (1) is said to be *quasi-exponential asymptotically stable* if there exists a $\gamma > 0$ and given any $\epsilon > 0$ and any $t_0 \in I$ there exist a $\delta = \delta(t_0, \epsilon) > 0$ and a $T = T(t_0, \epsilon) > 0$, such that if $\|x_0\| < \delta$, then

$$\|x(t, t_0, x_0)\| < \epsilon . e^{-\gamma(t-t_0)}, \tag{4}$$

for all $t \geqslant t_0 + T$.

Example 1

Considering the equation

$$\frac{dx}{dt} = -\frac{7x}{2t}, \quad t \geqslant 1 \tag{5}$$

We see that $x \equiv 0$ is a solution of which. The general solution of (5) is

$$x(t) = \frac{x_0 t_0^{\frac{7}{2}}}{t^{\frac{7}{2}}}$$

for all $t \geq 1$, thus the trivial solution $x \equiv 0$ is quasi-asymptotically stable of order $\frac{7}{2}$.

Example 2

Consider the equation

$$\frac{dx}{dt} = -x^2 t^2, \quad t \geqslant 1 \tag{6}$$

We have easily that for $t_0 \geq 1$

$$\int_{x_0}^{x} -\frac{dx}{x^2} = \int_{t_0}^{t} t^2 dt \quad \Leftrightarrow \quad x(t) = \frac{3x_0}{x_0(t^3 - t_0^3) + 3}$$

This implies

$$|x(t)| \leqslant \frac{3|x_0|}{|x_0|(t - t_0)^3}.$$

Hence there exist a $N > 0$ and a $T = T(t_0)$ such that

$$|x(t)| \leqslant \frac{N}{(t - t_0)^3}, \forall t \geq t_0 + T.$$

Therefore the zero solution of equation (6) is quasi-asymptotically of order 3.

2. Theorems

2.1. By $V(t, x)$ we denote a continuous scalar function, defined on an open set S and assume that $V(t, x)$ satisfies locally a Lipschitz condition with respect to x. Corresponding to $V(t, x)$, we define the function

$$V'_{(1)}(t, x) = \overline{\lim_{h \to 0^+}} \frac{1}{h}\{V(t + h, x + hX(t, x)) - V(t, x)\}.$$

Let $x(t)$ be a solution of (1) which stays in S, and denote by $V'(t,x(t))$ the right derivate of $V(t,x(t))$,i.e.,

$$V'(t,x(t)) = \overline{\lim_{h\to 0^+}} \frac{1}{h}\{V(t+h,x(t+h)) - V(t,x(t))\}.$$

We see easily ([7]) that

$$V'_{(1)}(t,x) = V'(t,x(t)).$$

By the same calculation , we obtain the relation

$$\lim_{h\to 0^+} \frac{1}{h}\{V(t+h,x(t+h)) - V(t,x)\} = \lim_{h\to 0^+} \frac{1}{h}\{V(t+h,x+hX(t,x)) - V(t,x)\}.$$

In the case $V(t,x)$ has continuous partial derivates of the first order, it is evident that

$$V'_{(1)}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}X(t,x)$$

Function $V(t,x)$ is called Liapunov one.

Theorem 2.1. *Suppose that there exists a Liapunov function $V(t,x)$ defined on D, satisfying the following conditions*

(i) $V(t,0) = 0$.

(ii) $\|x\|^{\frac{1}{\lambda}} \leqslant V(t,x), \quad \forall\lambda \in \mathbb{R}^*_+$

(iii) $V'_{(1)}(t,x) \leqslant -\frac{m.V(t,x)}{t}$, where $m \in \mathbb{N}, m \geq 2$. Then, the solution $x \equiv 0$ of (1) is equi-asymptotically stable of order $\lambda(m-1)$.

Proof. For any $0 < \epsilon < H$ we have $V(t,x) \geqslant \epsilon^{\frac{1}{\lambda}}$ for $t \in I = [a,+\infty)$ and x such that $\|x\| = \epsilon$ due to the condition (ii).

For the fixesd $t_0 \in I$, we can choose a $\delta = \delta(t_0,\epsilon) > 0$ such that $\|x_0\| < \delta$ implies $V(t_0,x_0) < \epsilon^{\frac{1}{\lambda}}$ because of the continuity of $V(t,x)$ and $V(t_0,0) = 0$.

Suppose that a solution $x = x(t,t_0,x_0)$ of (1) such that $\|x_0\| < \delta$ satisfies $\|x(t_1,t_0,x_0)\|$ ϵ at some t_1. From (iii), it follows that

$$V(t_1,x(t_1,t_0,x_0)) \leqslant V(t_0,x_0)$$

and hence

$$\epsilon^{\frac{1}{\lambda}} \leqslant V(t_1,x(t_1,t_0,x_0)) \leqslant V(t_0,x_0) < \epsilon^{\frac{1}{\lambda}}.$$

This is a contradiction and hence, if $\|x_0\| < \delta$ then $\|x(t,t_0,x_0)\| < \epsilon$, for all $t \geqslant t_0$ that is $x \equiv 0$ is stable in sense of Liapunov.

Now given $\alpha > 0$, we assume that $x(t,t_0,x_0)$ is a solution of (1) satisfying the condition $\|x_0\| \leqslant \alpha$. Applying Theorem 4.1 in [7], by (iii) we have

$$V(t,x(t,t_0,x_0)) \leqslant V(t_0,x_0)\left(\frac{t}{t_0}\right)^m \leqslant {t_0}^m V(t_0,x_0)(t-t_0)^{-m} \tag{7}$$

for all t sufficiently large. Let $M(t_0, \alpha) = \max_{\|x_0\| \leqslant \alpha} V(t_0, x_0)$ and let $T(t_0, \epsilon, \alpha)$ be such that

$$0 \leqslant \frac{M(t_0, \alpha)}{t - t_0} < \frac{\epsilon^{\frac{1}{\lambda}}}{t_0^m}$$

for all $t \geqslant t_0 + T(t_0, \epsilon, \alpha)$. Then from (7), it follows that for $t \geqslant t_0 + T(t_0, \epsilon, \alpha)$,

$$\|x(t, t_0, x_0)\|^{\frac{1}{\lambda}} \leqslant V(t, x(t, t_0, x_0)) \leqslant t_0{}^m V(t_0, x_0)(t - t_0)^{-m}$$

$$< t_0{}^m \frac{M(t_0, \alpha)}{t - t_0}(t - t_0)^{-m+1} < \epsilon^{\frac{1}{\lambda}}(t - t_0)^{-m+1}$$

$$\Longrightarrow \|x(t, t_0, x_0)\| < \epsilon(t - t_0)^{-\lambda(m-1)},$$

which proves equi-asymptotical stability of order $\lambda(m - 1)$ of the solution $x \equiv 0$, and the theorem is proved.

In the case $m = 2$ the zero solution is equi-asymptotically stable of order λ.

Theorem 2.2. *Suppose that there exists a Liapunov function $V(t, x)$ defined on D, satisfying condition (i) and (ii) of Theorem 2.1 and besides the following*

(iii)' $V'_{(1)}(t, x) \leqslant -CV(t, x)$, where $C > 0$ is a constant.

Then the solution$x \equiv 0$ of (1) is quasi-exponential asymptotically stable.

Proof. It is sufficient to prove the inequality

$$\|x(t, t_0, x_0)\| < \epsilon e^{-\alpha(t-t_0)},$$

for all t sufficiently large, with some positive number α. For this, we give $\beta > 0$ and assume that $x(t, t_0, x_0)$ is a solution of (1) satisfying the condition $\|x_0\| = \beta$.

Due to the theorem 4.1 in [7], by (iii)' the following inequality is valid

$$V(t, x(t, t_0, x_0)) \leqslant V(t_0, x_0)e^{-C(t-t_0)}, \tag{8}$$

for all t sufficiently large.

Let $M(t_0, \beta) = \max\{V(t_0, x_0), \|x_0\| = \beta\}, 0 < C_1 < C$ and let $T(t_0, \epsilon, \beta)$ such that

$$0 \leqslant \frac{M(t_0, \beta)e^{-C(t-t_0)}}{e^{-C(t-t_0)}} < \epsilon^{\frac{1}{\lambda}}$$

for all $t \geqslant t_0 + T(t_0, \epsilon, \beta)$. Then from (8) it follows that

$$\|x(t, t_0, x_0)\|^{\frac{1}{\lambda}} \leqslant V(t, x(t, t_0, x_0)) < \epsilon^{\frac{1}{\lambda}} e^{-C(t-t_0)}$$

$$\Longrightarrow \|x(t, t_0, x_0)\| \leqslant \epsilon e^{-\lambda C_1(t-t_0)},$$

for all $t \geqslant t_0 + T(t_0, \epsilon\beta)$, (here $\alpha = \lambda C_1$). The theorem is proved.

By the same arguments used in the proof of the above theorems we can prove the two following theorems for the quasi-uniform asymptotical stability of the zero solution.

Theorem 2.3. *Suppose that there exists a Liapunov function $V(t,x)$ defined on D which satisfies the following conditions*

(*i*) $\|x\|^{\frac{1}{\lambda}} \leqslant V(t,x) \leqslant b(\|x\|)$, *where $b(r)$ is a continuous increasing and positive definite function,* $\lambda \in \mathbb{R}_+$

(*ii*)$V'_{(1)}(t,x) \leqslant -\frac{mV(t,x)}{t}$, *where* $m \in \mathbb{N}, m \geq 2$.

Then, the solution $x \equiv 0$ of the equation (1) is quasi-uniform-asymptotically stable of order $\lambda(m-1)$.

Theorem 2.4. *Suppose that there existss a Liapunov function $V(t,x)$ defined on D which satisfies the following conditions*

(*i*) $\|x\|^{\frac{1}{\lambda}} \leqslant V(t,x) \leqslant b(\|x\|)$, *where $b(r)$ is a continuous increasing and positive definite function,* $\lambda \in \mathbb{R}_+$

(*ii*) $V'_{(1)}(t,x) \leqslant -cV(t,x)$, *where $c > 0$ is a constant. Then the solution $x \equiv 0$ of (1) is quasi-exponential asymptotically stable.*

2.2. Let consider now the linear system

$$\frac{dx}{dt} = A(t)x, \tag{9}$$

where $A(t)$ is a continuous $n \times n$ matrix on I and $t_0 \in I$. Note

$$S_\gamma^h = \{x \in \mathbb{R}^n : \gamma \leqslant \|x\| \leqslant h\},$$

where $0 < h < H, \gamma > 0$. We have the following converse theorem for the system (9).

Theorem 2.5. *Suppose that there exist a $M > 0$ and $\lambda \in \mathbb{R}_+^*$ such that*

$$\|x(t,t_0,x_0)\| \leqslant M\|x_0\|(t-t_0+1)^{-\lambda}, \tag{10}$$

for all $t \geqslant t_0$, where $x(t,t_0,x_0)$ is a solution of (9). Then there exists a Liapunov function $V(t,x)$ which satisfies the following conditions

(*i*) $\|x\|^{\frac{1}{\lambda}} \leqslant V(t,x) \leqslant M^{\frac{1}{\lambda}}\|x\|^{\frac{1}{\lambda}}$.

(*ii*) $\|V(t,x) - V(t,x')\| \leqslant L\|x - x'\|, \forall x, x' \in S_\gamma$

(*iii*) $V'_{(9)}(t,x) \leqslant 0$.

Proof. Let $V(t,x)$ be defined by

$$V(t,x) = \sup_{\tau \geqslant 0} \|x(t+\tau,t,x)\|^{\frac{1}{\lambda}}(\tau+1)^{-\lambda}$$

It is clear that

$$V(t,x) \geqslant \|x(t,t,x)\|^{\frac{1}{\lambda}} = \|x\|^{\frac{1}{\lambda}}.$$

On the other hand, because of (10) we have

$$\|x(t+\tau,t,x)\| \leqslant M\|x\|(\tau+1)^{-\lambda}$$

for all $\tau \geqslant 0$.

Hence

$$V(t,x) = \sup_{\tau\geqslant 0} \|x(t+\tau,t,x)\|^{\frac{1}{\lambda}}(\tau+1)^{-\lambda} \leqslant \sup_{\tau\geqslant 0} M^{\frac{1}{\lambda}}\|x\|^{\frac{1}{\lambda}}(\tau+1)^{-\lambda-1} = M^{\frac{1}{\lambda}}\|x\|^{\frac{1}{\lambda}}$$

The condition (i) is proved. Since the system is linear, we have the relation

$$x(\tau,t,x) - x(\tau,t,x') = x(\tau,t,x-x') \tag{11}$$

Hence, for all $x, x' \in S_\gamma{}^h$ the inequalities following will be hold

$$\begin{aligned}
&\big|V(t,x) - V(t,x')\big| = \\
&= \big|\sup_{\tau\geqslant 0} \|x(t+\tau,t,x)\|^{\frac{1}{\lambda}}(\tau+1)^{-\lambda} - \sup_{\tau\geqslant 0} \|x(t+\tau,t,x')\|^{\frac{1}{\lambda}}(\tau+1)^{-\lambda}\big| \\
&\leqslant \sup_{\tau\geqslant 0} \big\{\big|\|x(t+\tau,t,x)\|^{\frac{1}{\lambda}} - \|x(t+\tau,t,x')\|^{\frac{1}{\lambda}}\big|\big\}(\tau+1)^{-\lambda} \\
&\leqslant \sup_{\tau\geqslant 0} L_1\{\big|\|x(t+\tau,t,x)\| - \|x(t+\tau,t,x')\|\big|\}(\tau+1)^{-\lambda} \\
&\leqslant \sup_{\tau\geqslant 0} L_1\{\|x(t+\tau,t,x) - x(t+\tau,t,x')\|\}(\tau+1)^{-\lambda}
\end{aligned}$$

where L_1 is a positive number. This implies by (11)

$$\big|V(t,x) - V(t,x')\big| \leqslant \sup_{\tau\geqslant 0} L_1\{\|x(t+\tau,t,x-x')\|\}(\tau+1)^{-\lambda}$$

$$\leqslant \sup_{\tau\geqslant 0} L_1 M\|(x-x')\|(\tau+1)^{-2\lambda} = L_1 M\|(x-x')\|,$$

for all $x, x' \in S_\gamma^h$. By putting $L = L_1 M$ we have (ii).

Now we shall prove the continuity of $V(t,x)$. The conditions (i), (ii) imply that $V(t,x)$ is continue at 0. We shall prove this in $x \neq 0$. Take a number $\delta \geqslant 0$, we have

$$\begin{aligned}
&\big|V(t+\delta,x') - V(t,x)\big| \leqslant \\
&\leqslant \big|V(t+\delta,x') - V(t+\delta,x)\big| + \big|V(t+\delta,x) - V(t+\delta,x(t+\delta,t,x))\big| + \\
&+ \big|V(t+\delta,x(t+\delta,t,x)) - V(t,x)\big|
\end{aligned} \tag{12}$$

Since $V(t,x)$ is Lipschitz in x and $x(t+\delta,t,x)$ is continuous in δ, the first two term of (12) are small when $\|x-x'\|$ and δ are small.

Let us consider the third term. Since

$$x(t+\delta+\tau,t+\delta,x(t+\delta+\tau,t,x) = x(t+\tau,t,x)$$

We have

$$\begin{aligned}
&|V(t+\delta, x(t+\delta,t,x)) - V(t,x) = \\
&= \big|\sup_{\tau\geqslant 0}\|x(t+\delta+\tau, t+\delta, x(t+\delta,t,x))\|^{\frac{1}{\lambda}}(\tau+1)^{-\lambda} - \sup_{\tau\geqslant 0}\|x(t+\tau,t,x)\|^{\frac{1}{\lambda}}(\tau+1)^{-\lambda}\big| \\
&= \big|\sup_{\tau\geqslant 0}\{\|x(t+\delta+\tau,t,x))\|^{\frac{1}{\lambda}}(\tau+1)^{-\lambda} - \sup_{\tau\geqslant 0}\|x(t+\tau,t,x))\|^{\frac{1}{\lambda}}\|(\tau+1)^{-\lambda}\big| \\
&\leqslant \sup_{\tau\geqslant 0}\{\big|\|x(t+\delta+\tau,t,x))\|^{\frac{1}{\lambda}} - \|x(t+\tau,t,x))\|^{\frac{1}{\lambda}}\big|\}(\tau+1)^{-\lambda} \\
&\leqslant \sup_{\tau\geqslant 0} L_1\{\|x(t+\delta+\tau,t,x))\| - \|x(t+\tau,t,x))\|\}(\tau+1)^{-\lambda} \\
&\leqslant \sup_{\tau\geqslant 0} L_1\{\|x(t+\delta+\tau,t,x))\| - \|x(t+\tau,t,x))\|\}(\tau+1)^{-\lambda}
\end{aligned} \tag{13}$$

Since $x(t+\delta+\tau,t,x))$ is continuous, then for $\delta > 0$ sufficiently small the right hand part of (13) will be arbitrary small. Hence we have the continuity of $V(t,x)$.

Finally, we shall establish condition (iii). It is clear that for $h > 0$

$$\begin{aligned}
V(t+h, x(t+h,t,x)) &= \sup_{\tau\geqslant 0}\|x(t+h+\tau, t+h, x(t+h,t,x))\|^{\frac{1}{\lambda}}(\tau+1)^{-\lambda} \\
&\leqslant \sup_{\tau\geqslant 0}\|x(t+\tau,t,x)\|^{\frac{1}{\lambda}}(\tau+1)^{-\lambda} = V(t,x)
\end{aligned}$$

That is

$$\frac{V(t+h,x(t+h,t,x)) - V(t,x)}{h} \leqslant 0$$

Thus $V'_{(9)}(t,x) \leqslant 0$. This completes the proof.

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