EXTENSION OF A RESULT OF HUNEKE AND MILLER

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Abstract. Let k be the ground field k and $x = (x_0, \dots, x_n)$ be indeterminates. Let I be a graded ideal of $k[x]$. In [2], [3] there are formulas to determine the Betti numbers and multiplicity of R/I. Now we want to give an extension and a new simple proof about a result of Huneke and Miller and we also consider the algebra with minimal multiplicity.[1]

Introduction

Let $R = k[x_1, \dots, x_n]$ be the polynomial ring over the field k. Let I be a graded ideal of R. R/I is said to have a pure resolution of type $(d_1, \dots, d_p)$ if its minimal resolution has the form

$$0 \longrightarrow \bigoplus_{j=1}^{\ell_p} R(-d_p) \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{\ell_1} R(-d_1) \longrightarrow R \longrightarrow R/I \longrightarrow 0, \; d_1 < \cdots < d_p.$$

In [2] Herzog and Kül have given a formul to determine the Betti numbers of R/I. The multiplicity of R/I is given by Huneke and Miller, see [3]. This paper presents an extension and a new simple proof about a result of Huneke and Miller and we also consider the algebra with minimal multiplicity.

1. Extension of a Huneke and Miller's result

We collect here a number of more or less standard definitions, results and notations of graded modules.

Let $R = \oplus_{i \geq 0} R_i$ be a graded ring, where R_0 is the ground-field k, and $M = \oplus_{t \in \mathbb{Z}} M_t$ a finitely generated graded R-module of dimension d. For evry $i \in \mathbb{Z}$, we denote by $M(i)$ the graded R-module with coincides with M as the underlying R-module and whose grading is given by $M(i)_j = M_{i+j}$ for all $j \in \mathbb{Z}$. Set $\ell(M_t) = \dim_k M_t$. Let $h(M,t)$ and $h_M(z)$ denote the Hilbert functions and the Hilbert series of M, which are defined to be

$$h(M,t) = \ell(M_t) \text{ for all } t \in \mathbb{Z},$$
$$h_M(z) = \sum_{t \in \mathbb{Z}} h(M,t) z^t.$$

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It is well known that $h(M,t) = \sum_{j=0}^{d-1}(-1)^{d-1-j}e_{d-1-j}\binom{t+j}{j}$ with $e_j \in \mathbb{Z}$ and $\forall t >> 0$. The *multiplicity* of M is defined as follows

$$e(M) = \begin{cases} e_0 & \text{if } d > 0, \\ \ell(M) & \text{if } d = 0. \end{cases}$$

Suppose that $0 \longrightarrow \bigoplus_{j=1}^{\ell_p} R(-d_{pj}) \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{\ell_0} R(-d_{0j}) \longrightarrow M \longrightarrow 0$ is a minimal graded free resolution of M. Since $h_R(z) = \sum_{t=0}^{\infty}\binom{t+n-1}{n-1}z^t = \frac{1}{(1-z)^n}$ and

$$\begin{aligned} h_M(z) &= \sum_{i=0}^{p}(-1)^i\sum_{j=1}^{\ell_i} h_{R(-d_{ij})}(z) = \sum_{i=0}^{p}(-1)^i\Big(\sum_{j=1}^{\ell_i} z^{d_{ij}}h_R(z)\Big) \\ &= \frac{1}{(1-z)^n}\sum_{i=0}^{p}(-1)^i\Big(\sum_{j=1}^{\ell_i} z^{d_{ij}}\Big) = \frac{g(z)}{(1-z)^d} \quad \text{with } g(z) \in k[z], \end{aligned}$$

there is $(1-z)^{n-d}g(z) = \sum_{i=0}^{p}(-1)^i(\sum_{j=1}^{\ell_i} z^{d_{ij}}) := S_M(z)$, see [1].

Theorem 1.1. [2, Corollary 4.1.14] *If M is finitely generated graded R-module of dimension d, then*

$$e_j = \frac{(-1)^{n-d}S_M^{(n-d+j)}(1)}{(n-d+j)!} \quad \text{and} \quad e(M) = \frac{(-1)^{n-d}S_M^{(n-d)}(1)}{(n-d)!}.$$

Let M be a finitely generated graded R-module. M is said to have a *pure resolution* of type $(d_0, d_1, \ldots, d_p)$ if its minimal resolution has the form

$$0 \longrightarrow \bigoplus_{j=1}^{\ell_p} R(-d_p) \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{\ell_1} R(-d_1) \longrightarrow \bigoplus_{j=1}^{\ell_0} R(-d_0) \longrightarrow M \longrightarrow 0, \ d_0 < \cdots < d_p.$$

The following theorem shows that the Betti numbers and multiplicity of the Cohen-Macaulay module M are completely determined by the twists d_i and Betti number ℓ_0.

Theorem 1.2. *Let M be a finitely generated graded R-module of dimension d. If M is a Cohen-Macaulay module and has a pure resolution of type $(d_0, d_1, \ldots, d_p)$, then*

$$\ell_i = (-1)^i\frac{\ell_0\prod_{j=1}^{p}(d_0 - d_j)}{\prod_{j=0, j\neq i}^{p}(d_i - d_j)}, i = 1, \ldots, p, \quad e(M) = \frac{(-1)^p\ell_0}{p!}\prod_{j=1}^{p}(d_0 - d_j).$$

Proof. Since M is a Cohen-Macaulay module, there is $p = n-d$ and $S_M(z) = (1-z)^p g(z) = \sum_{i=0}^{p}(-1)^i\ell_i z^{d_i}$. Since $S_M^{(j)}(1) = 0$ for $j = 0, 1, \cdots, p-1$, and $S_M^p(1) = g(1)$, we obtain the following system of linear equations:

$$\begin{cases} \sum_{i=0}^{p}(-1)^i\ell_i = 0, \\ \sum_{i=0}^{p}(-1)^i\ell_i d_i.(d_i - 1)\ldots(d_i - j + 1) = 0, \\ j = 1, \cdots, p-1, \\ \sum_{i=0}^{p}(-1)^i\ell_i d_i.(d_i - 1)\ldots(d_i - p + 1) = g(1). \end{cases}$$

Set $y_i = (-1)^i \ell_i$, $i = 0, \dots, p$. Upon simple computation, we get

$$(A)\begin{cases} \sum_{i=0}^{p} y_i = 0, \\ \sum_{i=0}^{p} y_i d_i^j = 0, \\ j = 1, \cdots, p-1, \\ \sum_{i=0}^{p} y_i d_i^p = g(1). \end{cases}$$

Consider the following equation

$$\frac{g(1)}{(x-d_0)(x-d_1)\cdots(x-d_p)} = \frac{x_0}{x-d_0} + \frac{x_1}{x-d_1} + \frac{x_2}{x-d_2} + \cdots + \frac{x_p}{x-d_p}.$$

Then $(B):\ g(1) = x_0(x-d_1)\cdots(x-d_p) + x_1(x-d_0)(x-d_2)\cdots(x-d_p) + \cdots + x_p(x-d_0)(x-d_1)\cdots(x-d_{p-1})$. For $x = d_0, x = d_1, \cdots, x = d_p$, it can be verified that

$$(C)\begin{cases} x_0 = \frac{g(1)}{(d_0-d_1)\cdots(d_0-d_p)}, \\ x_1 = \frac{g(1)}{(d_1-d_0)\cdots(d_1-d_p)}, \\ x_2 = \frac{g(1)}{(d_2-d_0)(d_2-d_1)\cdots(d_2-d_p)}, \\ \dots\dots\dots\dots \\ x_p = \frac{g(1)}{(d_p-d_0)(d_p-d_1)\cdots(d_p-d_{p-1})}. \end{cases}$$

The tth power sum and the tth elementary symmetric polynomial of $d_0, \dots, d_p$ will be denoed by $M_t = d_0^t + \cdots + d_p^t$ and $E_t = E_t(d_0, \dots, d_p)$. Set $E_t^{(i)} = (E_t)_{d_i=0}$. Use the identity $E_t = E_t^{(i)} + d_i E_{t-1}^{(i)}$, and mathematical induction to caculate the coefficients of $x^p, \cdots, x$, from the above equation (B) it follows that

$$(D)\begin{cases} \sum_{i=0}^{p} x_i d_i^j = 0, \\ j = 0, \cdots, p-1, \\ x_0 d_0^p + x_1 d_1^p + \cdots + x_p d_p^p = g(1). \end{cases}$$

From $(A), (C)$ and (D) it follows that $y_i = x_i, i = 0, \dots, p$. Since $g(1) = x_0 \prod_{j=1}^{p}(d_0 - d_j)$, there are

$$\ell_i = (-1)^i \frac{\ell_0 \prod_{j=1}^{p}(d_0 - d_j)}{\prod_{j=0, j\neq i}^{p}(d_i - d_j)}, i = 1, \dots, p.$$

Consider

$$\frac{x(x-1)\cdots(x-p+1)}{(x-d_0)\cdots(x-d_p)} = \frac{z_0}{x-d_0} + \frac{z_1}{x-d_1} + \cdots + \frac{z_p}{x-d_p} \Longrightarrow x(x-1)\cdots(x-p+1) =$$
$$z_0(x-d_1)\cdots(x-d_p) + z_1(x-d_0)(x-d_2)\cdots(x-d_p) + \cdots + z_p(x-d_0)\cdots(x-d_{p-1}).$$

There is $z_0 + \cdots + z_p = 1$ and for $x = d_0, \cdots, x = d_p$ we obtain

$$\begin{cases} z_0 = \frac{d_0(d_0-1)\cdots(d_0-p+1)}{(d_0-d_1)\cdots(d_0-d_p)}, \\ z_1 = \frac{d_1(d_1-1)\cdots(d_1-p+1)}{(d_1-d_0)\cdots(d_1-d_p)}, \\ \dots\dots\dots\dots \\ z_p = \frac{d_p(d_p-1)\cdots(d_p-p+1)}{(d_p-d_0)\cdots(d_p-d_{p-1})}. \end{cases}$$

Since $e(M) = \frac{(-1)^p S_M^p(1)}{p!}$, therefore

$$e(M) = \frac{(-1)^p S_M^{(p)}(1)}{p!} = \frac{(-1)^p}{p!}\sum_{i=0}^{p}(-)^i \ell_i d_i(d_i-1)\cdots(d_i-p+1)$$

$$= \frac{(-1)^p}{p!}\Big(\sum_{i=0}^{p}\frac{g(1)d_i(d_i-1)\cdots(d_i-p+1)}{\prod_{j\neq i}(d_i-d_j)}\Big) = \frac{(-1)^p g(1)}{p!}\sum_{i=0}^{p} z_i$$

$$= \frac{(-1)^p g(1)}{p!} = \frac{(-1)^p \ell_0 (d_0-d_1)d_0-d_2)\cdots(d_0-d_p)}{p!}.$$

Thus $e(M) = \frac{(-1)^p \ell_0}{p!}\prod_{j=1}^{p}(d_0-d_j)$.

Now we want to consider the case the finitely generated graded R-module M is not Cohen-Macaulay. In this case we have $p = \text{proj.}\dim M > n - \dim M = n - d$.

Theorem 1.3. *Let M be a finitely generated graded R-module of dimension d. If M has a pure resolution of type $(d_0, d_1, \ldots, d_p)$, then all ℓ_i are completely determined by $\ell_0, \ldots, \ell_{p-n+d}$.*

Proof. Set $y_i = (-1)^i \ell_i$ for $i = 0, \ldots, p$. By assumption, direct computation shows that

$$\begin{cases} \sum_{i=0}^{p} y_i = 0, \\ \sum_{i=0}^{p} y_i d_i^j = 0, \\ j = 1, \cdots, n-d-1, \\ \sum_{i=0}^{p} y_i d_i^{n-d+h} = g^{(h)}(1), \\ h = 0. \ldots, p-n+d. \end{cases}$$

Denote the summation of all products of n-h factors from $d_0, \ldots, d_{i-1}, d_{i+1}, \ldots, d_p$ by s_{ih} and set $a_h = g^{(h)}(1)$. By an argument analogous to that used for the proof of Theorem 1.2, we get the solution

$$\begin{cases} y_0 = \frac{1}{\prod_{j=1}^{p}(d_0-d_j)}\sum_{h=n-d}^{p} a_h s_{0h}, \\ y_i = \frac{1}{\prod_{j=0}^{p}(d_i-d_j)}\sum_{h=n-d}^{p} a_h s_{ih}, \\ i = 1, \ldots, p. \end{cases}$$

Thus

$$\begin{cases} \sum_{h=n-d}^{p} a_h s_{0h} = \ell_0 \prod_{j=1}^{p}(d_0-d_j), \\ \sum_{h=n-d}^{p} a_h s_{0h} = (-1)^i \ell_i \prod_{j=0}^{p}(d_i-d_j), \\ i = 1, \ldots, p-n+d. \end{cases}$$

From this system we can determine all ℓ_i when $\ell_0, \ldots, \ell_{p-n+d}$ are given.

Let I be a homogeneous ideal of R. R/I has a pure resolution of type $(d_1, \ldots, d_p)$ with its minimal resolution of the form

$$0 \longrightarrow \bigoplus_{j=1}^{\ell_p} R(-d_p) \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{\ell_1} R(-d_1) \longrightarrow R \longrightarrow R/I \longrightarrow 0,\ d_1 < \cdots < d_p.$$

The following result was proved by Huneke and Miller using residues theory of complex function, see [3]. As an immediate consequence of Theorem 1.2 we now want to give an another simple proof.

Corollary 1.4. [3, Theorem 1.2] *Let I be a homogeneous ideal of R. If R/I is Cohen-Macaulay and has a pure resolution of type $(d_1, \dots, d_p)$, then*

$$\ell_j = (-1)^{j+1} \prod_{i \neq j} (\frac{d_i}{d_i - d_j}) \ ; \ e(R/I) = \frac{1}{p!} \prod_{j=1}^{p} d_j.$$

Proof. Here $\ell_0 = 1, d_0 = 0$. By Theorem 1.2, we have $g(1) = d_1 \dots d_p$. Hence $\ell_j = (-1)^{j+1} \prod_{i \neq j} (\frac{d_i}{d_i - d_j})$ and $e(R/I) = \frac{\prod_{i=1}^{p} d_i}{p!}$.

Note that Theorem 1.2 is as considered an extension of Corollary 1.4.

Remark 1.5. *If we present $(x-1)(x-2)\cdots(x-p+1) = x^p - s_1 x^{p-1} + s_2 x^{p-2} - \dots + (-1)^{p-1} s_{p-1} x$, then*

$$(-1)^i \ell_i d_i (d_i - 1) \cdots (d_i - p + 1) = (-1)^i \ell_i [d_i^p - s_1 d_i^{p-1} + s_2 d_i^{p-2} - \dots + (-1)^{p-1} s_{p-1} d_i].$$

Since $\sum_{i=1}^{p} y_i d_i^j = 0, j = 1, \dots, p-1$, and since $d_0 = 0$, we obtain

$$\begin{aligned} e_j &= \frac{(-1)^{n-d} S_{R/I}^{(n-d+j)}(1)}{(n-d+j)!} = \frac{(-1)^p S_{R/I}^{(p+j)}(1)}{(p+j)!} \text{ by Theorem 1.1} \\ &= \frac{(-1)^p}{(p+j)!} \sum_{i=1}^{p} (-1)^i \ell_i d_i (d_i - 1) \cdots (d_i - p - j + 1) \\ &= \frac{(-1)^p}{(p+j)!} \sum_{i=1}^{p} (-1)^i \ell_i d_i (d_i - 1) \cdots (d_i - p - j + 1) \\ &= \frac{1}{(p+j)!} \sum_{i=1}^{p} (-1)^{p+i} \ell_i d_i^p. \end{aligned}$$

In particular, for $j = 0$ there is $e(R/I) = \frac{1}{p!} \sum_{i=1}^{p} (-1)^{p+i} \ell_i d_i^p$, (formula of Peskine-Szpiro), see [3].

Assume that $I \neq 0$ is a homogeneous ideal of R. Denote by $v(R/I) = h(R/I; 1)$ the embedding dimension of R/I. Abhyankar proved that if R/I is Cohen-Macaulay then

$$v(R/I) - \dim R/I + 1 \ \square \ e(R/I).$$

Recall that a Cohen-Macaulay local ring R/I is called a *ring with minimal multiplicity* if $v(R/I) - \dim R/I + 1 = e(R/I)$. We will say that R/I has *h-linear resolution* if R/I has the pure resolution of type $(h, h+1, \dots, h+p-1)$.

Proposition 1.6. *Assume that the ring R/I is Cohen-Macaulay and has a h-linear resolution of type $(h, h+1, \cdots, h+p-1), p = n - \dim R/I$. R/I is the ring with minimal multiplicity if and only if $h = 1$ or $h = 2$.*

Proof. Since R is the Cohen-Macaulay ring, there is $\mathrm{ht}(I) = n - \dim R/I = p$ by [1, Corollary 2.1.4]. Because R/I has a h-linear resolution of type $(h, h+1, \cdots, h+p-1)$, therefore $I_j = 0$ for all $j < h$ and $\dim_k I_h = \binom{h+p-1}{h}$. If $h = 1$, then $e(R/I) = \frac{p!}{p!} = 1$ and $v(R/I) = \binom{n-1+1}{1} - \dim_k I_1 = n - p$. Hence R/I is the ring with minimal multiplicity, because

$$v(R/I) - \dim R/I + 1 = (n-p) - (n-p) + 1 = e(R/I).$$

By Theorem 1.2, we have

$$\begin{aligned} e(R/I) &= \frac{h(h+1)\cdots(h+p-1)}{p!} \geq \frac{2.3.\cdots(p+1)}{p!} = p+1 \\ &= \binom{n-1+1}{n-1} - (n-p) + 1 = v(R/I) - \dim R/I + 1. \end{aligned}$$

Also, in the case $h \geq 2$, the Cohen-Macaulay local ring R/I is a ring with minimal multiplicity if and only if $h = 2$. Hence, the Cohen-Macaulay ring R/I, which has a h-linear resolution of type $(h, h+1, \cdots, h+p-1), p = n - \dim R/I$, is the ring with minimal multiplicity if and only if $h = 1$ or $h = 2$.

References

1. D. Einsenbud, S. Goto, Linear free resolution and minimal multiplicity, *J. Algebra* **88**(1984), 89-133.
2. J. Herzog, E. Kunz, Der kanonische Module eines Cohen-Macaulay-Rings, *Lecture Notes in Math.*, Vol. 238, Springer-Verlag 1971.
3. C. Huneke, M. Miller, A note on the multiplicity of Cohen-Macaulay algebras with pure resolution, *Canad. J. Math.*, **37**(1985), 1149-1162.