SPACES WITH STAR-COUNTABLE QUASI-K-NETWORKS, LOCALLY COUNTABLE QUASI-K-NETWORKS

Tran Van An

Vinh University, Nghe An

Abstract. In this paper, we introduce some kinds of network, and investigate the relationships between them. Also, it is proved that the pseudo-open s-image of a Frechet space having a locally countable k-network is a Frechet space so doing.

1. Introduction

Spaces having star-countable k-networks, locally countable k-networks have considered by Y. Ikeda and Y. Tanaka in [2]. In that paper, the authors have studied the relationships among spaces with star-countable k-networks, spaces with locally countable k-networks. They have presented characterizations of spaces with star-countable k-networks, and spaces with star-countable closed k-networks. Also, the authors have shown that for some appropriate conditions, spaces with star-countable k-networks (or spaces with locally countable k-networks) are preserved by closed maps .

In this paper we deal with a spaces having star-countable quasi-k-networks, locally countable quasi-k-networks, star-countable k-networks, and locally countable k-networks consider relationships among these notions, and prove that a Frechet space with a locally countable k-network is preserved by pseudo-open s-map.

We assume that spaces are regular T_1, and maps are continuous and onto.

1.1. Definition. Let X be a topological space, and let $\mathcal{P}$ be a cover of X.

$\mathcal{P}$ is a *k-network*, if whenever $K \subset U$ with K compact and U open in X, then $K \subset \cup\mathcal{F} \subset U$ for a certain finite collection $\mathcal{F} \subset \mathcal{P}$.

$\mathcal{P}$ is a *strong-k-network*, if whenerver $K \subset U$ with K is compact and U is open in X, then there is a finite colletion $\mathcal{F} \subset \mathcal{P}$ such that for every $F \in \mathcal{F}$ there exists a closed set $C(F) \subset F$ satisfying $K \subset \bigcup_{F\in\mathcal{F}} C(F) \subset \cup\mathcal{F} \subset U$.

1.2. Definition. Let $\mathcal{P}$ be a cover of X.

$\mathcal{P}$ is called a *quasi-k-network*, if whenever $K \subset U$ with K is countably compact and U is open in X, then $K \subset \cup\mathcal{F} \subset U$ for a certain finite collection $\mathcal{F} \subset \mathcal{P}$.

$\mathcal{P}$ is called a *strong-quasi-k-network*, if whenever $K \subset U$ with K is countably compact $\mathcal{P}$, and U is open in X, then there is a finite colletion $\mathcal{F} \subset \mathcal{P}$ such that for every $F \in \mathcal{F}$ there exists a closed set $C(F) \subset F$ satisfying $K \subset \bigcup_{F\in\mathcal{F}} C(F) \subset \cup\mathcal{F} \subset U$.

1.3. Definition. A cover $\mathcal{P}$ of X is said to be *locally countable*, if for every $x \in X$ there is a neighbourhood V of x such that V meets only countable many members of $\mathcal{P}$.

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A cover $\mathcal{P}$ of X is said to be *point-countable*, if every $x \in X$ is in at most countably many elements of $\mathcal{P}$.

$\mathcal{P}$ is said to be *star-countable*, if every $P \in \mathcal{P}$ meets only countable many members of $\mathcal{P}$.

A cover $\mathcal{P}$ is said to be *closed (open)*, if every set $P \in \mathcal{P}$ is closed (respectively, open).

1.4. Definition. A topological space X is said to be *determined by a cover* $\mathcal{P}$ (or X has the *weak topology with respect to* $\mathcal{P}$), if a set $E \subset X$ is open (closed) in X if and only if $E \cap P$ is open (resp. closed) in P for every $P \in \mathcal{P}$.

A topological space X is called a *k-space*, if X is determined by the cover consisting of all compact subsets of X.

A topological space X is called a *quasi-k-space*, if X is determined by the cover consisting of all countably compact subsets of X.

A topological space X is said to be *Frechet*, if for every $A \subset X$ and $x \in \overline{A}$ there is a sequence $\{x_n\} \subset A$ such that $x_n \to x$.

1.5. Lemma. *Let X be a topological space, and $Y \subset X$. If X has a locally countable quasi-k-network (k-network), then so does Y.*
Proof. It directly follows from the definition.

1.6. Lemma. [3, Lemma 1.1] *Let $\mathcal{P}$ be a star-countable cover of X. Then we have*

1. *X is a disjoint union of $\{X_\alpha : \alpha \in \Lambda\}$, where each X_α is a countable union of elements of $\mathcal{P}$.*
2. *If X is determined by $\mathcal{P}$, then X is the topological sum of the collection $\{X_\alpha : \alpha \in \Lambda\}$ in (1), and the cover $\mathcal{P}$ is locally countable.*

1.7. Proposition. [3, Proposition 1.7] *Let X be a Frechet space. Then the following are equivalent*

a. *X has a star-countable closed k-network;*
b. *X is a locally separable space with a point-countable k-network.*

1.8. Lemma. [6, Corollary 2.4] *Every k-space with a star-countable k-network is a paracompact σ-space.*

1.9. Theorem Balogh. [2, Theorem 4.1] *Every countably compact space with point-countable quasi-k-network (or k-network) is metrizable (and thus, compact).*

2. The main results

Firstly we present some relationships between a locally countable quasi-k-network, a locally countable strong-quasi-k-network, a locally countable closed quasi-k-network, a locally countable k-network, and a locally countable closed k-network.

2.1. Theorem. *Let X be a topological space. Then the following are equivalent*

a. *X has a locally countable strong-quasi-k-network;*
b. *X has a locally countable quasi-k-network;*
c. *X has a locally countable closed quasi-k-network;*

d. X has a locally countable k-network;

e. X has a locally countable closed k-network.

Proof. (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c) Let $\mathcal{P}'$ be a locally countable quasi-k-network. Denote $\mathcal{P} = \{\overline{P} : P \in \mathcal{P}'\}$. Then $\mathcal{P}$ is a locally countable closed quasi-k-network. Indeed, let K be a countably compact subset, and U an open subset such that $K \subset U$. Since X has a locally countable quasi-k-network, by Lemma 1.5 K also has a locally countable quasi-k-network. From Theorem Balogh, it follows that K is compact. For every $x \in K$, denote W_x an open neighbourhood of x such that $x \in W_x \subset \overline{W_x} \subset U$. Then the collection $\{W_x : x \in K\}$ covers K. As K is compact, there exists a finite subcollection $W_1, \dots, W_s$ so that $K \subset \bigcup_{i=1}^{s} W_i$. Because $\mathcal{P}'$ is a quasi-k-network, there is a finite collection $\mathcal{F} \subset \mathcal{P}'$ such that $K \subset \cup\mathcal{F} \subset \bigcup_{i=1}^{s} W_i$. It implies that $K \subset \bigcup\{\overline{P} : P \in \mathcal{F}\} \subset \bigcup_{i=1}^{s} \overline{W_i} \subset U$.

It is easily seen that if V_x is an open neighbourhood of x such that $V_x \cap \overline{P} \neq \phi$ for some $P \in \mathcal{P}'$, then $V_x \cap P \neq \phi$. Thus, from the local countablity of the collection $\mathcal{P}'$, it follows that the collection $\mathcal{P}$ is so locally countable.

(c) $\Rightarrow$ (d) is obvious.

(d) $\Rightarrow$ (e) Assume $\mathcal{P}'$ is a locally countable k-network. Let $\mathcal{P} = \{\overline{P} : P \in \mathcal{P}'\}$. Then $\mathcal{P}$ is a locally countable closed k-network. In fact, suppose that K is compact and U is an any open such that $K \subset U$. For every $x \in K$ by W_x we denote an open neighbourhood of x such that $x \in W_x \subset \overline{W_x} \subset U$. Then the collection $\{W_x : x \in K\}$ covers K. Because K is compact, there is a finite collection $W_1, \dots, W_s$ so that $K \subset \bigcup_{i=1}^{s} W_i$. Since $\mathcal{P}'$ is a quasi-k-network, there exists a finite collection $\mathcal{F} \subset \mathcal{P}'$ such that $K \subset \cup\mathcal{F} \subset \bigcup_{i=1}^{s} W_i$. It follows that $K \subset \bigcup\{\overline{P} : P \in \mathcal{F}\} \subset \bigcup_{i=1}^{s} \overline{W_i} \subset U$.

By using the method as in the proof of implication (b) $\Rightarrow$ (c), it follows that $\mathcal{P}$ is a locally countable.

(e) $\Rightarrow$ (a) Assume that X has a locally countable closed k-network $\mathcal{P}$. Then by Theorem Balogh, a subset A of X is countably compact if and only if A is compact. Therefore follows that $\mathcal{P}$ is a locally countable quasi-k-network of X.

2.2. Lemma. *Let X be a space having a locally countable quasi-k-network. Then*

1. *For every $x \in X$ there is a neighbourhood V with the following properties*
   (a) Every open set $W \subset V$ is a countable union of closed subsets;
   (b) V is Lindelof.
2. *Every $x \in X$ is a G_δ-set.*

Proof. 1) Assume that $\mathcal{P}'$ is a locally countable quasi-k-network. By the proof of Theorem 2.1, the collection $\mathcal{P} = \{\overline{P} : P \in \mathcal{P}'\}$ is a locally countable closed quasi-k-network. Hence for every $x \in X$ there is an open neighbourhood V of x such that V meets only countable many elements of $\mathcal{P}$. Denote $\mathcal{P}_x = \{Q \in \mathcal{P} : Q \subset V\}$. It follows that $\mathcal{P}_x$ is a countable collection and $V = \cup\{Q : Q \in \mathcal{P}_x\}$. Thus (a) is proved.

Let $\mathcal{U}$ be an any open cover of V. For every $y \in V$ there exists $U \in \mathcal{U}$ such that $y \in U$. Since $\mathcal{P}$ is a locally countable quasi-k-network in X, there is a $Q \in \mathcal{P}$ satisfying $y \in Q \subset U \cap V$. As shown above, the collection $\mathcal{P}_x = \{Q \in \mathcal{P} : Q \subset V\}$ is countable, and $V = \cup\{Q : Q \in \mathcal{P}_x\}$. For each $Q \in \mathcal{P}_x$, put an $U_Q \in \mathcal{U}$ such that $Q \subset U_Q$. Then the family $\mathcal{U}_x = \{U_Q \in \mathcal{U} : Q \in \mathcal{P}_x\}$ is a countable cover of V. Hence V is Lindelof.

2) Let x be an any point in X. By assertion (1) there exists a neighbourhood V of x such that every open subset of V is a countable union of closed subsets. Hence, we have $V \setminus \{x\} = \bigcup_{k=1}^{\infty} E_k$, where E_k is closed for each $k = 1, 2, \dots$. It follows that $\{x\} = \bigcap_{k=1}^{\infty}(V \setminus E_k)$. Thus, the set $\{x\}$ is a G_δ-set.

We now consider some relationships between a locally countable quasi-k-network, a σ-locally finite closed Lindelof quasi-k-network, a star-countable closed quasi-k-network and a star-countable quasi-k-network.

2.3. Theorem. *For an any topological space X, and the following conditions (a) – (d) we have (a) or (b) $\Rightarrow$ (c) $\Rightarrow$ (d).*

a. X has a locally countable quasi-k-network;

b. X has a σ-locally finite closed Lindelof quasi-k-network;

c. X has a star-countable closed quasi-k-network;

d. X has a star-countable quasi-k-network.

Proof. (b) $\Rightarrow$ (c). Assume that $\mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{P}_n$ is a σ-locally finite closed Lindelof quasi-k-network. It is only sufficient to prove that $\mathcal{P}$ is a star-countable. Indeed, put any $P \in \mathcal{P}$. Since $\mathcal{P}$ is σ-locally finite, for every $x \in P$, and for every $n \in I\!N$ there exists a neighbourhood V_x^n of x such that V_x^n meets only finite many elements $Q \in \mathcal{P}_n$. The collection $\{V_x^n : x \in P\}$ is a cover of P. Because P is Lindelof, there exists a countable subcollection $\{V_{x_k}^n\}_{k=1}^{\infty}$ covering P. As every set $V_{x_k}^n$ meets only finite many elements $Q \in \mathcal{P}_n$, P meets only countable many elements $Q \in \mathcal{P}_n$. Thus P meets only countable many elements of $\mathcal{P}$.

(c) $\Rightarrow$ (d) is trivial.

Now we prove that (a) $\Rightarrow$ (c). Assume that $\mathcal{P}'$ is a locally countable quasi-k-network. By the proof of Theorem 2.1, the collection $\mathcal{P} = \{\overline{P} : P \in \mathcal{P}'\}$ is a locally countable closed quasi-k-network. Hence for every $x \in X$ there is an open neighbourhood V_x of x such that V_x meets only countable many elements of $\mathcal{P}$. By Lemma 2.2 V_x is Lindelof. Put $\mathcal{P}^* = \{P \in \mathcal{P} : P$ is contained in V_x for some $x \in X\}$. Then $\mathcal{P}^*$ is a locally countable closed Lindelof quasi-k-network.

In fact, since $\mathcal{P}^*$ is a subcollection of the locally countable collection $\mathcal{P}$, $\mathcal{P}^*$ also is locally countable. Moreover, every $Q \in \mathcal{P}^*$ is a closed subset of a certain Lindelof space V_x, hence Q is Lindelof. Now we prove that $\mathcal{P}^*$ is a quasi-k-network. Let K be countably compact, and U an any open set such that $K \subset U$. Since X has a locally countable closed quasi-k-network $\mathcal{P}$, by Theorem Balogh K is compact. For any $x \in K$, by V_x we denote an open neighbourhood of x such that V_x meets only countable elements of $\mathcal{P}$, and W_x an open neighbourhood of x such that $x \in W_x \subset \overline{W_x} \subset V_x \cap U$. The collection $\{W_x : x \in K\}$ is an open cover of K. Because K is compact, there exists a finite subcollection $W_{x_1}, \dots, W_{x_m}$ such that $K \subset \bigcup_{i=1}^{m} W_{x_i}$. For every $i = 1, \dots, m$, put $K_i = K \cap \overline{W_{x_i}}$. Then K_i is an countably compact set in V_{x_i}, $i = 1, \dots, m$. Since $\mathcal{P}$ is a locally countable closed quasi-k-network, for every $i = 1, \dots, m$ there is a finite collection $\{P_{ij} : j = 1, \dots, n_i\} \subset \mathcal{P}$ such that $K_i \subset \bigcup_{j=1}^{n_i} P_{ij} \subset V_{x_i}$. Thus the collection $\mathcal{F} = \{P_{ij} : i = 1, \dots, m; j = 1, \dots, n_i\} \subset \mathcal{P}^*$ is a finite subcollection of $\mathcal{P}^*$ satisfying $K \subset \cup\mathcal{F} \subset U$.

Since every $Q \in \mathcal{P}^*$ is Lindelof, $\mathcal{P}^*$ is a locally countable quasi-k-network, by using the argument presented in the proof of the implication $(b) \Rightarrow (c)$ it follows that $\mathcal{P}^*$ is star-countable.

2.4. Corollary. *If X is a k-space, then the following are equivalent*

a. *X has a locally countable quasi-k-network;*

b. *X has a locally countable k-network;*

c. *X has a star-countable closed quasi-k-network;*

d. *X has a star-countable closed k-network;*

e. *X has a σ-locally finite closed Lindelof quasi-k-network;*

f. *X has a σ-locally finite Lindelof quasi-k-network.*

Proof. (a) $\Leftrightarrow$ (b) It follows from Theorem 2.1.

$(a) \Rightarrow (c)$ It implies from Theorem 2.3.

$(c) \Rightarrow (d)$ is obvious.

$(d) \Rightarrow (e)$ Assume that X has a star-countable closed k-network $\mathcal{P}$. Denote $\mathcal{P}^*$ a collection of all finite unions of elements in $\mathcal{P}$. Then $\mathcal{P}^*$ is also a star-countable closed k-network. Since X has a star-countable closed k-network, by Theorem Balogh every countably compact subset of X is contained in a certain element of $\mathcal{P}^*$. Hence, by assumption X being a k-space it follows that X is determined by $\mathcal{P}^*$. By Lemma 1.6(a), X being a topological sum of $\{X_\alpha : \alpha \in \Lambda\}$, where $X_\alpha = \bigcup_{n=1}^{\infty} P_{\alpha_n}$, $P_{\alpha_n} \in \mathcal{P}^*$ for all $\alpha \in \Lambda, n \in I\!N$, and $\mathcal{P}^*$ is star-countable.

It is similar to the proof of the implication $(a) \Rightarrow (c)$ in Theorem 2.3, it follows that P_{α_n} is Lindelof for all $\alpha \in \Lambda, n \in I\!N$.

Put $\mathcal{P}_n = \{P_{\alpha_n} : \alpha \in \Lambda\}$. Then we get $\mathcal{P}^* = \bigcup_{n=1}^{\infty} \mathcal{P}_n$ with $\mathcal{P}_n$ is a locally finite collection for all $n \in I\!N$.

$(e) \Rightarrow (f)$ is trivial.

$(f) \Rightarrow (a)$ Assume that X is a k-space having a σ-locally finite Lindelof quasi-k-network $\mathcal{P}$. By using the proof presented in the implication $(b) \Rightarrow (c)$ in Theorem 2.3 it follows that $\mathcal{P}$ is star-countable. As in the proof of $(d) \Rightarrow (e)$ we get that X is determined by $\mathcal{P}$. Therefore, by applying Theorem 1.6(b) it follows that $\mathcal{P}$ is locally countable.

2.5. Definition. A space X is said to be ω-compact if every countable subset of X have an accumulation point.

2.6. Theorem. *Let X be a space. Then the following are equivalent*

a. X is compact metric;

b. X is an ω-compact space having a locally countable quasi-k-network;

c. X is an ω-compact first-countable space having a star-countable quasi-k-network;

d. X is a countably compact space having a point-countable quasi-k-network.

Proof. $(a) \Rightarrow (b)$ is obvious.

$(b) \Rightarrow (c)$. It follows from Theorem 2.3 that X has a star-countable quasi-k-network. Put any $x \in X$. Because X has a locally countable quasi-k-network, by Lemma 2.2 every point of X is a G_δ-set. Hence there exists a sequence of closed neighbourhoods $\{V_n\}$ of x such that $V_{n+1} \subset V_n$ for all $n \geq 1$, and $\{x\} = \bigcap_{n=1}^{\infty} V_n$. We shall prove that for every neighbourhood U of x there exists V_{n_o} such that $V_{n_o} \subset U$. Conversely, assume $V_n \not\subset U$ for all $n \geq 1$. Then for every $n \geq 1$ there exists $x_n \in V_n$ such that $x_n \notin U$. Since X is ω-compact, the set $\{x_n : n \geq 1\}$ have an accumulation point y. Because $x_m \in V_n$ for all $m \geq n$, and V_n is closed, it implies that $y \in V_n$ for all $n \geq 1$. It follows that $y \in \bigcap_{n=1}^{\infty} V_n$. Hence $y = x \in U$. On the other hand, as y is an accumulation point of $\{x_n : n \geq 1\}$, there exists $x_n \in U$. This is contrary to the choosing the sequence $\{x_n\}$ so that $x_n \notin U$ for all $n \geq 1$. Thus the collection $\{V_n\}$ is a countable neighbourhood base of x, and X is first-countable.

$(c) \Rightarrow (d)$. It follows from that a first-countable ω-compact space is countably compact, and a star-countable quasi-k-network is a point-countable quasi-k-network.

$(d) \Rightarrow (a)$. It follows from Theorem Balogh.

2.7. Definition. A map $f : X \to Y$ is *pseudo-open* if, for each $y \in Y$, $y \in \text{Int}f(U)$ whenever U is an open subset of X containing $f^{-1}(y)$.

2.8. Proposition. [5, Theorem 5.D.2] *If $f : X \to Y$ is pseudo-open, and X is a Frechet space, then so Y is .*

2.9. Definition. A map $f : X \to Y$ is *a s-map* if $f^{-1}(y)$ is separable for each $y \in Y$.

2.10. Lemma. [1, Corollary 5.1.26] *Every separable paracompact space is a Lindelof space.*

2.11. Lemma. [1, Corollary 3.1.5] *Let U be an open subset of a space X. If a family $\{F_s\}_{s\in S}$ of closed subsets of X contains at least one compact set - in particular, if X is compact - and if $\bigcap_{s\in S} F_s \subset U$, then there exists a finite set $\{s_1, \ldots, s_m\} \subset S$ such that $\bigcap_{i=1}^{m} F_{s_i} \subset U$.*

2.12. Proposition. *Let X be a space having a locally countable quasi-k-network. If X*

is ω-compact, or X is a locally compact space, then X is a first-countable space. That means that X is a Frechet space.

Proof. If X is ω-compact, then from the proof of $(b) \Rightarrow (c)$ in Theorem 2.6 it follows that X is first-countable.

Assume now that X is a locally compact space, and x is an arbitrary point in X. Because X has a locally countable quasi-k-network, by Lemma 2.2 every point of X is a G_δ-set. Hence, there exists a sequence of compact closed neighbourhoods $\{V_n\}$ of x such that $V_{n+1} \subset V_n$ for all $n \geq 1$, and $\{x\} = \bigcap_{n=1}^{\infty} V_n$. Assume that U is an any open neighbourhood of x, i.e $\{x\} = \bigcap_{n=1}^{\infty} V_n \subset U$. From Lemma 2.11, it follows that there exists a neighbourhood V_{n_o} such that $V_{n_o} \subset U$. Thus the family $\{V_n\}$ is a countable neighbourhood base of x, and X is a first-countable space.

2.13. Proposition. [2, Theorem 7.1.(g)] *Let X be a Frechet space with a point-countable k-network. If $f : X \to Y$ is a quotient s-map, then Y has a point-countable k-network.*

2.14. Theorem. *Let $f : X \to Y$ be a pseudo-open s-map. If X is a Frechet space having a locally countable k-network, then so does Y.*

Proof. As it is well-known, every Frechet space is a k-space, by Proposition 1.7, and Corollary 2.4 in order to prove Theorem 2.14, it is sufficient to show that if X is a Frechet space with a locally countable k-network, $f : X \to Y$ a pseudo-open s-map, then Y is a Frechet space having a star-countable closed k-network.

Indeed, since X is Frechet, and f is pseudo-open, it follows from Proposition 2.8 that Y is a Frechet space. Since every locally countable k-network is point-countable, and every pseudo-open map is quotient, by Proposition 2.13 we get that Y has a point-countable k-network.

As every Frechet space is a k-space, and X has a locally countable k-network $\mathcal{P}$, by Lemma 2.4 and Lemma 1.8 X is paracompact. For each $y \in Y$, since f is a s-map, $f^{-1}(y)$ is a separable closed subset of paracompact space X. By Lemma 2.10, it follows that $f^{-1}(y)$ is Lindelof. Put any $z \in f^{-1}(y)$, since $\mathcal{P}$ is a locally countable k-network in X, by Lemma 2.2 there exists an open Lindelof neighbourhood V_z of z such that V_z meets only countable many elements of $\mathcal{P}$. The family $\{V_z : z \in f^{-1}(y)\}$ is an open cover of $f^{-1}(y)$. Because $f^{-1}(y)$ is Lindelof, there exists a countable family $\{V_{z_n} : n \geq 1\}$ covering $f^{-1}(y)$. Denote $U = \bigcup_{n=1}^{\infty} V_{z_n}$, we have $f^{-1}(y) \subset U$, and by the proof of Lemma 2.2 it follows that the collection $\mathcal{Q} = \{P \in \mathcal{P} : P \subset U\}$ is countable, and $U = \cup\{\overline{P} : P \in \mathcal{Q}\}$. For each $P \in \mathcal{Q}$ take $x_P \in \overline{P}$. Then the set $A = \{x_P : P \in \mathcal{Q}\}$ is countable, and $\overline{A} = U$. Denote $B = f(A)$, then B is countable. Because f is continuous it implies that $\overline{B} = f(U)$. And since f is a pseudo-open map, we get $y \in \mathrm{Int} f(U)$. Thus $f(U)$ is a separable neighbourhood of y.

Hence, Y is a locally separable Frechet space with a point-countable k-network. By Proposition 1.7 Y is a Frechet space having a star-countable closed k-network. It follows from Corolary 2.4 that Y is a Frechet space with a locally countable k-network.

Since every Frechet space is a k-space, by Corollary 2.4 and Theorem 2.14 we obtain

2.15. Corollary. *Let $f : X \to Y$ be a pseudo-open s-map. If X is a Frechet space satisfying the one of the following*

a. X has a locally countable quasi-k-network;

b. X has a locally countable k-network;

c. X has a star-countable closed quasi-k-network;

d. X has a star-countable closed k-network;

e. X has a σ-locally finite closed Lindelof quasi-k-network;

f. X has a σ-locally finite Lindelof quasi-k-network

then so Y has respectively.

From the latter, Proposition 2.12 and Theorem 2.14, we have

2.16. Corollary. *Let X be a space having a locally countable quasi-k-network, $f : X \to Y$ a pseudo-open s-map. Then each one of the following (a)-(d) implies that Y has a locally countable quasi-k-network*

a. X is an ω-compact space;

b. X is a locally compact space;

c. X is a first-countable space;

d. X is a Frechet space.

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