ON THE HAHN - DECOMPOSITION AND THE RADON - NIKODYM THEOREM FOR SUBMEASURES IN $\mathbb{R}^D$

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Abstract. In this note we characterize the pairs of submeasures in $\mathbb{R}^d$ possessing a certain Hahn - decomposition property and prove the sufficient condition of the Radon - Nikodym Theorem for submeasures in $\mathbb{R}^d$ having the *stable property* (SP).

1. Introduction

As we have seen, the Hahn - decomposition of a signed measure is one of the main tools in the measure theory. In addition, it is a base for proving the Radon - Nikodym Theorem, a fundamental theorem in the measure theory, probability theory and mathematical statistics. The standards of the Hahn - decomposition and the Radon - Nikodym Theorem have been extended by Graf to a new research area, that is capacity[4].

In this note we are going to extend the Hahn - decomposition and the Radon - Nikodym Theorem in measure spaces to one space of submeasures in $\mathbb{R}^d$ which have the stable property. The paper is organized as follows. In section 2 we give the notion of submeasure in $\mathbb{R}^d$ and prove some properties of them. In section 3 we characterize those pairs of submeasures in $\mathbb{R}^d$ which possess a certain Hahn - decomposition property. Section 4 is devoted to the Radon - Nikodym derivaties for submeasures in $\mathbb{R}^d$ which have the stable property.

2. Submeasures in $\mathbb{R}^d$

We first recall the various notion from [7] which will appear in the paper.

Let $\mathcal{K}(\mathbb{R}^d)$, $\mathcal{F}(\mathbb{R}^d)$, $\mathcal{G}(\mathbb{R}^d)$ and $\mathcal{B}(\mathbb{R}^d)$ denote the families of compact sets, closed sets, open sets and Borel sets in R^d, respectively.

2.1. Definition. A set - function $T : \mathcal{B}(\mathbb{R}^d) \longrightarrow [0;\infty)$ is called a *submeasure* in $\mathbb{R}^d$ if the following conditions hold:

1. $T(\emptyset) = 0$;
2. $T(A \cup B) \leqslant T(A) + T(B)$ for any Borel sets A, B;
3. $T(A) = \sup\{T(K) : K \in \mathcal{K}(\mathbb{R}^d), K \subset A\}$ for any Borel set $A \in \mathcal{B}(\mathbb{R}^d)$;
4. $T(K) = \inf\{T(G) : G \in \mathcal{G}(\mathbb{R}^d), G \supset K\}$ for any compact set $K \in \mathcal{K}(\mathbb{R}^d)$.

From the definition it follows that any submeasure in $\mathbb{R}^d$ is a non - decreasing and finite subadditive set - function on Borel sets of $\mathbb{R}^d$. Morever, we have

2.2. Proposition. ([7]). *Let T be a submeasure in $\mathbb{R}^d$. If $A \in \mathcal{B}(\mathbb{R}^d)$ with $T(A) = 0$, then*

$$T(B) = T(A \cup B) \quad \textit{for every } B \in \mathcal{B}(\mathbb{R}^d).$$

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2.3. Proposition. ([7]). *Any capacity is upper semi - continuous on compact sets, .i.e, if $K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$ is a decreasing sequence of compact sets in $\mathbb{R}^d$ and $\bigcap_{n=1}^{\infty} K_n = K$, then $\lim\limits_{n\to\infty} T(K_n) = T(K)$ for any capacity T.*

2.4. Proposition. *Any submeasure is lower semi - continuous on open sets, .i.e, if $G_1 \subset G_2 \subset \cdots \subset G_n \subset \cdots$ is a increasing sequence of open sets in $\mathbb{R}^d$ and $\bigcup_{n=1}^{\infty} G_n = G$, then $\lim\limits_{n\to\infty} T(G_n) = T(G)$ for any submeasure T.*

Proof. For given $\epsilon > 0$, by (3.) in the definition of submeasures, there exists $K \in \mathcal{K}(\mathbb{R}^d), K \subset G$ such that

$$T(K) > T(G) - \epsilon.$$

We claim that, there exists $n_0 \in \mathbb{N}$ such that

$$K \subset G_n \qquad \text{for every} \qquad n \geq n_0.$$

Indeed, assume, on the contrary, that $K \setminus G_n \neq \emptyset$ for all n. Since K is a compact set and G_n are open sets, $\{K \setminus G_n\}$ is a decreasing sequence of non - void compact sets. Hence

$$\bigcap_{n=1}^{\infty} (K \setminus G_n) = K \setminus (\bigcup_{n=1}^{\infty} G_n) = K \setminus G \neq \emptyset$$

giving a contradition to $K \subset G$. The claim is proved. It follows

$$T(G_n) \geq T(K) > T(G) - \epsilon, \text{ for every } n \geq n_0.$$

Therefore

$$\lim_{n\to\infty} T(G_n) > T(G) - \epsilon.$$

Since ϵ is arbitrary, then we have

$$\lim_{n\to\infty} T(G_n) \geq T(G).$$

Combinating the last inequality with $T(G) \geq \lim\limits_{n\to\infty} T(G_n)$ we get

$$\lim_{n\to\infty} T(G_n) = T(G).$$

The proposition is proved.

From Proposition 2.4 we have the following corollary

2.5. Corollary. *Any submeasure T in $\mathbb{R}^d$ possesses the countable subadditivity on $\mathcal{G}(\mathbb{R}^d)$ and $\mathcal{K}(\mathbb{R}^d)$.*

Proof. Firstly, let $\{G_n\}_{n=1}^{\infty}$ be a sequence of open sets in $\mathbb{R}^d$ and let T be a capacity in $\mathbb{R}^d$. For $n \in \mathbb{N}$, set

$$B_n = \bigcup_{k=1}^{n} G_k.$$

Then $\{B_n\}_{n=1}^{\infty}$ is a increasing sequence of open sets and $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} G_n$, by Proposition 2.4 we have

$$\begin{aligned} T\left(\cup_{n=1}^{\infty} G_n\right) = T\left(\cup_{n=1}^{\infty} B_n\right) &= \lim_{n\to\infty} T(B_n) = \lim_{n\to\infty} T\left(\cup_{k=1}^{n} G_k\right) \\ &\leqslant \lim_{n\to\infty} \left(\sum_{k=1}^{n} T(G_k)\right) = \sum_{n=1}^{\infty} T(G_n). \end{aligned}$$

We will show that T has σ - subadditive property on $\mathcal{K}(\mathbb{R}^d)$. Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of compact sets in $\mathbb{R}^d$. Given $\epsilon > 0$, for every n, by (4.) in definition of submeasures, there exists $G_n \in \mathcal{G}(\mathbb{R}^d)$ such that $G_n \supset K_n$ and

$$T(G_n) < T(K_n) + \frac{\epsilon}{2^n}.$$

Hence

$$\begin{aligned} T\left(\bigcup_{n=1}^{\infty} K_n\right) \leqslant T\left(\bigcup_{n=1}^{\infty} G_n\right) \leqslant \sum_{n=1}^{\infty} T(G_n) &< \sum_{n=1}^{\infty} \left(T(K_n) + \frac{\epsilon}{2^n}\right) \\ &= \sum_{n=1}^{\infty} T(K_n) + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, we get

$$T\left(\bigcup_{n=1}^{\infty} K_n\right) \leqslant \sum_{n=1}^{\infty} T(K_n).$$

3. The Hahn - decomposition for Submeasures in $\mathbb{R}^d$

3.1. Definition([4]). Let $S, T : \mathcal{B}(\mathbb{R}^d) \longrightarrow \mathbb{R}^+$ be submeasures in $\mathbb{R}^d$.

(a) The pair (S, T) is said to possess the *weak decomposition property*(WDP) if, for every $\alpha \in \mathbb{R}^+$, there exists a set $A_\alpha \in \mathcal{B}(\mathbb{R}^d)$ such that

$$\alpha T_{A_\alpha} \leqslant S_{A_\alpha} \text{ and } \alpha T_{A_\alpha^c} \geq S_{A_\alpha^c}.$$

(b) The pair (S, T) is said to possess the *strong decomposition property*(SDP) if, for every $\alpha \in \mathbb{R}^+$, there exists a set $A_\alpha \in \mathcal{B}(\mathbb{R}^d)$ such that the following conditions hold:

(i) For $A, B \in \mathcal{B}(\mathbb{R}^d)$; $B \subset A \subset A_\alpha$ implies

$$\alpha(T(A) - T(B)) \leqslant S(A) - S(B).$$

(ii) For $A \in \mathcal{B}(\mathbb{R}^d)$; $\alpha(T(A) - T(A \cap A_\alpha)) \geq S(A) - S(A \cap A_\alpha)$.

Observe that (SDP) implies (WDP) and if (S, T) possesses (WDP) then so does (T, S) (see [4]).

3.2. Definition. Let $T : \mathcal{B}(\mathbb{B}^d) \longrightarrow \mathbb{R}^+$ be a submeasure in $\mathbb{R}^d$. T is said to possess *stable property*(SP) if, for any sequense of Borel sets $\{A_n\} \subset \mathcal{B}(\mathbb{R}^d)$ satisfies $T(A_n) = 0$ for every n, then $T(\bigcup_{n=1}^{\infty}) = 0$. By $\mathcal{C}_0$ we denote the family of all submeasures in $\mathbb{R}^d$ which possess SP.

The following result is proved by Graf([4]) for the submeasures with the lower semi - continuous property. Here we will prove for the capacities in $\mathbb{R}^d$ which possess the SP. Note that the lower semi - continuity implies the $\sigma-$ subadditivity which implies the SP.

3.3. Proposition. *Assume that $S, T \in \mathcal{C}_0$. Then the following conditions are equivalent:*
(i) (S, T) has WDP.
(ii) There exists a Borel measurable function $f : \mathbb{R}^d \longrightarrow [0; +\infty]$ such that

$$\alpha T_{\{f \geq \alpha\}} \leqslant S_{\{f \geq \alpha\}} \text{ and } \alpha T_{\{f < \alpha\}} \geq S_{\{f < \alpha\}}, \tag{1}$$

for every $\alpha \in \mathbb{R}^+$.

Proof. (ii) $\Rightarrow$ (i). Let f be a Borel measurable function satisfying (1). For each $\alpha \in \mathbb{R}^+$, set

$$A_\alpha = \{f \geq \alpha\}.$$

Then we have

$$\alpha T_{A_\alpha} \leqslant S_{A_\alpha} \quad \text{and } \alpha T_{A_\alpha^c} \geq S_{A_\alpha^c}.$$

It means that (S, T) has the WDP.

(i) $\Rightarrow$ (ii). For each $\alpha \in \mathbb{R}^+$ let A_α be as in the definition of the WDP. A decreasing family $\{B_\alpha : \alpha \in \mathbb{R}^+\}$ is defined as follows.

$$B_0 = \mathbb{R}^d \quad ; \quad B_\alpha = \cap\{A_\beta \quad ; \ \beta \in \mathbb{Q}(\alpha)\} \text{ for every } \alpha > 0,$$

where $\mathbb{Q}(\alpha) = [0; \alpha) \cap \mathbb{Q}$ ($\mathbb{Q}$ denotes the set of all rational numbers). We define a function $f : \mathbb{R}^d \longrightarrow [0; +\infty]$ by Graf's formular :

$$f(x) = \sup\{\alpha; \ x \in B_\alpha\}.$$

We will show that

$$B_\alpha = \{f \geq \alpha\} \quad \text{for every } \alpha \in \mathbb{R}^+. \tag{2}$$

Indeed, $\alpha > 0$ then, by the definition, $x \in B_\alpha$ implies $f(x) \geq \alpha$. Conversely, if $x \in \{f \geq \alpha\}$ then, for every $\beta \in \mathbb{Q}(\alpha)$, there exists $\alpha' \in (\beta, \alpha) \cap \mathbb{Q}$ with $x \in B_{\alpha'}$. Thus we deduce $x \in A_\beta$. Since $\beta \in \mathbb{Q}(\alpha)$ is arbitrary we obtain

$$x \in \cap\{A_\beta; \beta \in \mathbb{Q}(\alpha)\} = B_\alpha.$$

Since $B_0 = \{f \geq 0\}$ our claim is verified. Because $B_\alpha \in \mathcal{B}(\mathbb{R}^d)$ for every $\alpha \in \mathbb{R}^+$ the function f defined above is also Borel measurable.

Next we will prove that

$$S(A_\alpha \cap A_\beta^c) = T(A_\alpha \cap A_\beta^c) = 0 \text{ for all } \alpha, \beta \in \mathbb{R}^+ \text{ with } \beta < \alpha. \tag{3}$$

From the definition of A_α and A_β we deduce

$$\alpha T(A_\alpha \cap A_\beta^c) \leqslant S(A_\alpha \cap A_\beta^c) \leqslant \beta T(A_\alpha \cap A_\beta^c).$$

This inequality implies

$$T(A_\alpha \cap A_\beta^c) = 0 = S(A_\alpha \cap A_\beta^c).$$

We claim that

$$\alpha T_{\{f \geq \alpha\}} \leqslant S_{\{f \geq \alpha\}} \text{ for every } \alpha \in \mathbb{R}^+. \tag{4}$$

If $\alpha = 0$ then there is nothing to show. For $\alpha > 0$, let $B \subset \{f \geq \alpha\}, B \in \mathcal{B}(\mathbb{R}^d)$. For every $\beta \in \mathbb{Q}(\alpha)$ we have $B \subset A_\beta$, therefore $\beta T(B) \leqslant S(B)$. Since $\beta \in \mathbb{Q}(\alpha)$ is arbitrary then

$$\alpha T(B) \leqslant S(B).$$

That means that (4) is proved.
To complete the proof of (i) $\Rightarrow$(ii) we will show that

$$\alpha T_{B_\alpha^c} \geq S_{B_\alpha^c} \text{ for every } \alpha \in \mathbb{R}^+. \tag{5}$$

If $\alpha = 0$ this inequality is satisfied by the definition of B_0. For $\alpha > 0$ let $B \in \mathcal{B}(\mathbb{R}^d)$ with $B \cap B_\alpha = \emptyset$ be arbitrary. We have

$$\begin{aligned} B &= \left[B \cap \left(\cup_{\beta \in \mathbb{Q}(\alpha)} A_\beta^c\right)\right] \bigcup \left[B \cap \left(\cup_{\beta \in \mathbb{Q}(\alpha)} A_\beta^c\right)^c\right] \\ &= \left[\cup_{\beta \in \mathbb{Q}(\alpha)} \left(B \cap A_\beta^c\right)\right] \bigcup \left[B \cap \left(\cap_{\beta \in \mathbb{Q}(\alpha)} A_\beta\right)\right] = \bigcup_{\beta \in \mathbb{Q}(\alpha)} \left(B \cap A_\beta^c\right). \end{aligned} \tag{6}$$

Because S possesses the SP and by (3), it follows

$$S\left(\cup_{\beta \in \mathbb{Q}(\alpha)} (B \cap A_\alpha \cap A_\beta^c)\right) \leqslant S\left(\cup_{\beta \in \mathbb{Q}(\alpha)} (A_\alpha \cap A_\beta^c)\right) = 0. \tag{7}$$

From (6), (7) and $S \in \mathcal{C}_0$ we get

$$\begin{aligned} S(B) &\leqslant S(B \cap A_\alpha) + S(B \cap A_\alpha^c) \\ &= S\left[\left(\cup_{\beta \in \mathbb{Q}(\alpha)} (B \cap A_\beta^c)\right) \cap A_\alpha\right] + S(B \cap A_\alpha^c) \\ &= S\left(\cup_{\beta \in \mathbb{Q}(\alpha)} (B \cap A_\alpha \cap A_\beta^c)\right) + S(B \cap A_\alpha^c) \\ &= S(B \cap A_\alpha^c) \leqslant \alpha T(B \cap A_\alpha^c) \leqslant \alpha T(B). \end{aligned}$$

(5) is proved.

3.4. Definition. Assume that $S, T \in \mathcal{C}_0$.

(a) If (S, T) has the WDP then every Borel measurable function $f : \mathbb{R}^d \longrightarrow [0, +\infty]$ such that (1) is satisfied is called a *decomposion function* of (S, T).

(b) Two Borel measurable functions $f, g : \mathbb{R}^d \longrightarrow [0, +\infty]$ are called $T-$ *equivalent* if $T(\{f \neq g\}) = 0$.

Then we have

3.5. Proposition. *Let $S, T \in \mathcal{C}_0$ and (S,T) has the WDP. Then any two decomposition functions of (S,T) are $S-$ and $T-$ equivalent.*

Proof. Let $f, g : \mathbb{R}^d \longrightarrow [0, +\infty]$ be decomposition functions of (S,T). For $p, q \in \mathbb{Q}^+ = \mathbb{Q} \cap \mathbb{R}^+$ with $p < q$ we define

$$A_{p,q} = \{f < p\} \cap \{g \geq q\}.$$

It is clear that

$$\{f < g\} = \cup\{A_{p,q} \ ; \ p, q \in \mathbb{Q}^+, p < q\}.$$

By Proposition 3.3 we have

$$qT(A_{p,q}) \leqslant S(A_{p,q}) \leqslant pT(A_{p,q}) \text{ for } p, q \in \mathbb{Q}^+, p < q.$$

It follows that

$$T(A_{p,q}) = 0 = S(A_{p,q}) \text{ for } p, q \in \mathbb{Q}^+, p < q.$$

Since S, T possess the SP, we get

$$S(\{f < g\}) = S\left(\cup\{A_{p,q} \ ; \ p, q \in \mathbb{Q}^+, p < q\}\right) = 0,$$

and

$$T(\{f < g\}) = T\left(\cup\{A_{p,q} \ ; \ p, q \in \mathbb{Q}^+, p < q\}\right) = 0.$$

Exchanging the role of f and g leads to

$$S(\{g < f\}) = T(\{g < f\}) = 0.$$

Therefore,

$$S(\{f \neq g\}) \leqslant S(\{f < g\}) + S(\{g < f\}) = 0,$$

and

$$T(\{f \neq g\}) \leqslant T(\{f < g\}) + T(\{g < f\}) = 0.$$

Hence f, g are $S-$ and $T-$ equivalent.

The following proposition is proved by Graf for the set - functions with the monotone property and the finite subadditive property so it is true for the submeasures in $\mathbb{R}^d$.

3.6. Proposition. *Let $S, T : \mathcal{B}(\mathbb{R}^d) \longrightarrow \mathbb{R}^+$ be submeasures in $\mathbb{R}^d$. Then (S,T) has the SDP if only if the following holds :*

(i) (S,T) has the WDP.

(ii) For every $\alpha \in \mathbb{R}^+$, for every $A \in \mathcal{B}(\mathbb{R}^d)$ with $\alpha T_A \leqslant S_A$, and for every $B \in \mathcal{B}(\mathbb{R}^d)$ with $B \subset A$ the inequality

$$\alpha(T(A) - T(B)) \leqslant S(A) - S(B)$$

is satisfied.

(iii) For every $\alpha \in \mathbb{R}^+, A \in \mathcal{B}(\mathbb{R}^d)$, and $B \in \mathcal{B}(\mathbb{R}^d)$ with $B \subset A, \alpha T_B \leqslant S_B$, $\alpha T_{A\setminus B} \geq S_{A\setminus B}$ the inequality

$$\alpha(T(A) - T(B)) \geq S(A) - S(B)$$

is satisfied.

The following example is a pair of submeasures which possesses the WDP but does not have the SDP.

3.7. Example. Let $S, T : \mathcal{B}(\mathbb{R}) \longrightarrow \mathbb{R}^+$ be set - functions are defined by

$$S(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{if } 1 \notin A \neq \emptyset \\ 2 & \text{if } 1 \in A \end{cases}$$

and

$$T(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 2 & \text{if } [0,1] \subseteq A \\ 1 & \text{otherwise.} \end{cases}$$

Then S, T have the following property:

(i) $S, T \in \mathcal{C}_0$.

(ii) T is not lower semi - continuous.

(iii) (S, T) has the WDP, but

(iv) (S, T) does not have the SDP.

Proof. (i). It is easy to see that S, T satisfy the conditions (1.) - (4.) in the definition of submeasues and Definition 3.2.

(ii). For each $n \in \mathbb{N}$ we defined

$$A_n = [0; 1 - \frac{1}{n}] \cup \{1\}.$$

Then $\{A_n\}_{n=1}^{\infty}$ is a increasing sequence of Borel sets in $\mathbb{R}$ and $\cup_{n=1}^{\infty} A_n = [0,1]$. Morever $T(A_n) = 1$ for every n. Therefore

$$T(\cup_{n=1}^{\infty} A_n) = 2 > 1 = \lim_{n\to\infty} T(A_n).$$

Thus T is a capacity in $\mathbb{R}$ but is not capacity in the sence of Graf.

(iii). Define a Borel measurable function $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ by

$$f(x) = \begin{cases} 1 & \text{if } x \neq 1 \\ 2 & \text{if } x = 1. \end{cases}$$

We will show that f satisfies (1). We consider following cases:

Case 1. $\alpha \leqslant 1$. Note that $T(A) \leqslant S(A)$ for every $A \in \mathcal{B}(\mathbb{R})$ and $\{f < \alpha\} = \emptyset$, which implies (1) holds.

Case 2 . $1 < \alpha \leqslant 2$. We have

$$\{f \geq \alpha\} = \{1\} \;\; ; \;\; \{f < \alpha\} = \mathbb{R} \setminus \{1\}.$$

Hence, (1) is satisfied.

Case 3 . $\alpha > 2$. Note that

$$\{f \geq \alpha\} = \emptyset \;\; ; \;\; \{f < \alpha\} = \mathbb{R}.$$

Then we have

$$\alpha T(A) \geq 2.1 \geq S(A) \text{ for every } A \in \mathcal{B}(\mathbb{R}), A \neq \emptyset.$$

Hence, (1) is satisfied in this case.

By Proposition 3.3, (S, T) has the WDP.

(iv). For $\alpha = \frac{1}{2}$, $\alpha T_A \leqslant S_A$ for every $A \in \mathcal{B}(\mathbb{R})$. Let $A = [0, 1]$, $B = \{1\}$. Then $S(A) = S(B) = T(A) = 2$, $T(B) = 1$. Hence

$$\alpha(T(A) - T(B)) = \frac{1}{2}(2 - 1) > 0 = S(A) - S(B).$$

It means that the condition (ii) in Proposition 3.6 is violated for (S, T). Therefore, (S, T) does not have the SDP.

3.8. Definition. Let $S, T : \mathcal{B}(\mathbb{R}^d) \longrightarrow \mathbb{R}^+$ be capacities in $\mathbb{R}^d$. S is said to be *absolutely continuous* with respect to T and write $S \ll T$ if, for every $A \in \mathcal{B}(\mathbb{R}^d), T(A) = 0$ implies $S(A) = 0$.

3.9. Proposition. *Let $S, T \in \mathcal{C}_0$ with (S, T) has WDP. Morever let $f : \mathbb{R}^d \longrightarrow [0, +\infty]$ be a decomposition function of (S, T). Then the following conditions are equivalent:*

(i) $S \ll T$.

(ii) $\forall A \in \mathcal{B}(\mathbb{R}^d); S(A) = 0 \Leftrightarrow \int_A f dT = 0$.

Proof. Clearly (ii) implies (i). To prove the converse let $A \in \mathcal{B}(\mathbb{R}^d)$ be arbitrary.

If $S(A) = 0$ then we deduce that, for every $\alpha \in \mathbb{R}^+$,

$$\alpha T(A \cap \{f \geq \alpha\}) \leqslant S(A \cap \{f \geq \alpha\}) \leqslant S(A) = 0.$$

This last inequality implies $T(A \cap \{f \geq \alpha\}) = 0$ for all $\alpha \in (0, +\infty)$. This, in turn, leads to

$$\int_A f dT = \int_0^\infty T(A \cap \{f \geq \alpha\}) d\alpha = 0.$$

If $\int_A f dT = 0$ then, by the definition of the integral

$$T(A \cap \{f \geq \alpha\}) = 0$$

for all $\alpha \in (0, +\infty)$. Since $S \ll T$ this implies

$$S(A \cap \{f \geq \alpha\}) = 0$$

for all $\alpha \in (0, +\infty)$, hence, by S has the SP,

$$S(A \cap \{f > 0\}) = S(\cup_{\alpha \in \mathbb{Q}^+_*} (A \cap \{f \geq \alpha\})) = 0,$$

where $\mathbb{Q}_*^+ = \mathbb{Q} \cap (0, +\infty)$.

For all $\alpha \in (0, +\infty)$, since $A \cap \{f = 0\} \subset \{f < \alpha\}$ for all $\alpha > 0$ we have

$$\alpha T(A \cap \{f = 0\}) \geq S(A \cap \{f = 0\}).$$

This implies

$$S(A \cap \{f = 0\}).$$

Hence we obtain

$$S(A) \leqslant S(A \cap \{f = 0\}) + S(A \cap \{f > 0\}) = 0.$$

4. The Radon - Nikodym derivatives for Submeasures in $\mathbb{R}^d$

4.1. Definition ([6]). Let S, T be monotone set functions on $\mathbb{R}^d$. We say that (S, T) has *Radon - Nikodym property*(RNP) if there exists a Borel measurable function $f : \mathbb{R}^d \longrightarrow \mathbb{R}^+$ such that

$$S(A) = \int_A f dT \quad \text{for every } \ A \in \mathcal{B}(\mathbb{R}^d).$$

Then the function f is called the Radon - Nikodym derivative of S with respect to T and written as $f = dS/dT$.

As in the case of measures, we see that if $T(A) = 0$, then $S(A) = 0$. However, unlike the situation for measures, this condition of $S \ll T$ is only a *necessary condition* for S to admite a Radon - Nikodym derivative with respect to T. Depending upon additional properties of S and T, sufficient conditions can be found. For example, suppose that S and T both belong to the class of capacities μ of the following type (Graf, [4]):

(a) $\mu(\emptyset) = 0$;
(b) $\mu(A) \leqslant \mu(B)$ for any $A, B \in \mathcal{B}(\mathbb{R}^d)$ with $A \subset B$;
(c) $\mu(A \cup B) \leqslant \mu(A) + \mu(B)$ for any $A, B \in \mathcal{B}(\mathbb{R}^d)$; and
(d) $\mu(\cup_{n=1}^{\infty} A_n) = \lim\limits_{n\to\infty} \mu(A_n)$ for any increasing sequence $\{A_n\} \subset \mathcal{B}(\mathbb{R}^d)$.

Then as shown by Graf (1980), a necessary and sufficient condition for S to admit a Radon - Nikodym derivative with respect to T is (S, T) has the SDP and $S \ll T$.

Let $\{A_t : t \in \mathbb{R}^+\}$ be a family of Borel sets in $\mathbb{R}^d$. A decreasing family $\{B_t : t \in \mathbb{R}^d\} \subset \mathcal{B}(\mathbb{R}^d)$ is defined as follows :

$$B_0 = \mathbb{R}^d \ \ ; \ \ B_t = \cap_{q \in \mathbb{Q}(t)} A_q \quad \text{for each } t > 0,$$

where $\mathbb{Q}(t) = [0; t) \cap \mathbb{Q}$. Since $\mathbb{Q}(t)$ is countable, it follows that B_t is a Borel set for every $t \in \mathbb{R}^+$. We define a function $f : \mathbb{R}^d \longrightarrow \mathbb{R}^+$ by the following formula:

$$f(x) = \begin{cases} 0 & \text{if } \ x \in \cap_{t \in \mathbb{Q}^+} B_t \\ \sup\{t; x \in B_t\}, & \text{otherwise.} \end{cases} \tag{8}$$

It is easy to see that

$$B_t = \{f \geq t\} \cup B_\infty \text{ and } \{f < t\} = B_t^c \cup B_\infty, \text{ for every} t \in (0, +\infty), \tag{9}$$

where $B_\infty = \cap_{t \in \mathbb{Q}^+} B_t$. Consequently, f is a Borel measurable function.

Now we prove the sufficient condition of the Radon - Nikodym Theorem for submeasures possessing SP in $\mathbb{R}^d$.

4.2. Theorem. *Assumse that $S, T \in \mathcal{C}_0$. Then (S,T) has the RNP if there exists a family $\{A_t; t \in \mathbb{R}^+\} \subset \mathcal{B}(\mathbb{R}^d)$ satisfying following conditions:*

(i) $\lim_{n\to\infty} S(B_n) = \lim_{n\to\infty} T(B_n) = 0$;

(ii) $S(B_t \setminus A_t) = T(B_t \setminus A_t) = 0$ *for every* $t \in \mathbb{R}^+$;

(iii) For any $s, t \in \mathbb{R}^+$ *with* $s < t$ *and* $A \in \mathcal{B}(\mathbb{R}^d)$,

$$s[T(A \cap A_s) - T(A \cap A_t)] \leqslant S(A \cap A_s) - S(A \cap A_t) \leqslant t[T(A \cap A_s) - T(A \cap A_t)].$$

Proof. The first we establish some relations between families $\{A_t\}, \{B_t\}$ and $\{f \geq t\}$.

Claim 1. For every $A \in \mathcal{B}(\mathbb{R}^d)$ we have

$$S(A) = \lim_{n\to\infty} S(A \cap B_n^c) = \lim_{n\to\infty} S(A \cap \{f < n\})$$

and

$$T(A) = \lim_{n\to\infty} T(A \cap B_n^c) = \lim_{n\to\infty} T(A \cap \{f < n\}).$$

Proof. For any $n \in \mathbb{N}$, we get

$$S(A \cap B_n^c) \leqslant S(A) \leqslant S(A \cap B_n) + S(A \cap B_n^c)$$

$$\leqslant S(B_n) + S(A \cap B_n^c),$$

hence by (i) we have

$$S(A) = \lim_{n\to\infty} S(A \cap B_n^c).$$

From (9) and $S(B_\infty) \leqslant \lim_{n\to\infty} S(B_n) = 0$ we deduce

$$S(A \cap \{f < n\}) = S[A \cap (B_n^c \cup B_\infty)] = S(A \cap B_n^c),$$

and the result follows. Similarly,

$$T(A) = \lim_{n\to\infty} T(A \cap B_n^c) = \lim_{n\to\infty} T(A \cap \{f < n\}).$$

Claim 2. $S(A_t \setminus B_t) = T(A_t \setminus B_t) = 0$ for every $t \in \mathbb{R}^+$.

Proof. Let $A = A_t \setminus A_s$ in (iii), then we obtain

$$-sT(A_t \setminus A_s) \leqslant -S(A_t \setminus A_s) \leqslant -tT(A_t \setminus A_s)$$

for any $s, t \in \mathbb{R}^+$ with $s < t$. It brings about

$$T(A_t \setminus A_s) = S(A_t \setminus A_s) = 0. \tag{10}$$

for any $s, t \in \mathbb{R}^+$ with $s < t$. Observe that

$$A_t \setminus B_t = A_t \setminus (\cap_{q \in \mathbb{Q}(t)} A_q) = \cup_{q \in \mathbb{Q}(t)} (A_t \setminus A_q).$$

Since $q < t$, we obtain from (10) and $S, T \in \mathcal{C}_0$,

$$S(A_t \setminus B_t) = T(A_t \setminus B_t) = 0 \quad \text{for every } t \in \mathbb{R}^+.$$

Claim 3. For any $t \in \mathbb{R}^+$ and $A \in \mathcal{B}(\mathbb{R}^d)$,

$$S(A \cap A_t) = S(A \cap \{f \geq t\}) \quad \text{and} \quad T(A \cap A_t) = T(A \cap \{f \geq t\}).$$

Proof. Note that

$$A \cap A_t = [A \cap (A_t \setminus B_t)] \cup (A \cap A_t \cap B_t).$$

By Claim 2, we obtain $S(A \cap A_t) = S(A \cap A_t \cap B_t)$. Similarly we can obtain $S(A \cap B_t) = S(A \cap A_t \cap B_t)$. Thus, with (9) and $S(B_\infty) = 0$ we get

$$\begin{aligned} S(A \cap A_t) = S(A \cap B_t) &= S[A \cap (\{f \geq t\} \cup B_\infty)] \\ &= S[(A \cap \{f \geq t\}) \cup (A \cap B_\infty)] = S(A \cap \{f \geq t\}). \end{aligned}$$

A similar reasoning is applied to T.

Claim 4. For any $s, t \in \mathbb{R}^+$ with $s < t$ and $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned} s[T(A \cap \{f \geq s\}) - T(A \cap \{f \geq t\})] &\leqslant S(A \cap \{f \geq s\}) - S(A \cap \{f \geq t\}) \\ &\leqslant t[T(A \cap \{f \geq s\}) - T(A \cap \{f \geq t\})]. \end{aligned}$$

Proof. By (iii) and Claim 3,

$$\begin{aligned} s[T(A \cap \{f \geq s\}) - T(A \cap \{f \geq t\})] &= s[T(A \cap A_s) - T(A \cap A_t)] \\ &\leqslant S(A \cap A_s) - S(A \cap A_t) \\ &= S(A \cap \{f \geq s\}) - S(A \cap \{f \geq t\}). \end{aligned}$$

Similarly we can obtain

$$S(A \cap \{f \geq s\}) - S(A \cap \{f \geq t\}) \leqslant t[T(A \cap \{f \geq s\}) - T(A \cap \{f \geq t\})].$$

Now we are able to complete the proof of our main result.

Claim 5. For $f : \mathbb{R}^d \longrightarrow \mathbb{R}^+$ be given in (8) we have

$$S(A) = \int_A f dT \quad \text{for every } A \in \mathcal{B}(\mathbb{R}^d).$$

Proof. Let $A(n) := A \cap \{f < n\}$ for every $A \in \mathcal{B}(\mathbb{R}^d), n \in \mathbb{N}$. Then as $A(n) \subset \{f < n\}$,

$$\int_0^n T(A(n) \cap \{f \geq t\})dt = \int_0^\infty T(A(n) \cap \{f \geq t\})dt. \tag{11}$$

We first prove that

$$S(A(n)) = \int_0^\infty T(A(n) \cap \{f \geq t\})dt \quad \text{for every } n \in \mathbb{N}.$$

Let $0 = t_0 < t_1 < \cdots < t_k = n$,

$$s_k = \sum_{i=1}^{k}(t_i - t_{i-1})T(A(n) \cap \{f \geq t_i\}),$$

and

$$S_k = \sum_{i=1}^{k}(t_i - t_{i-1})T(A(n) \cap \{f \geq t_{i-1}\}).$$

Then we have

$$s_k \leqslant \int_0^n T(A(n) \cap \{f \geq t\})dt \leqslant S_k.$$

Note that as $\max\{t_i - t_{i-1}; i = 1, \cdots k\} \to 0$,

$$s_k \to \int_0^n T(A(n) \cap \{f \geq t\})dt, \quad S_k \to \int_0^n T(A(n) \cap \{f \geq t\})dt. \tag{12}$$

Using the first inquality in Claim 4 and the fact that $S(A(n) \cap \{f \geq t_k\}) = T(A(n) \cap \{f \geq t_k\}) = 0$, we obtain

$$\begin{aligned}
s_k &= \sum_{i=1}^{k}(t_i - t_{i-1})T(A(n) \cap \{f \geq t_i\}) \\
&= \sum_{i=1}^{k} t_i T\left(A(n) \cap \{f \geq t_i\}\right) - \sum_{i=1}^{k} t_{i-1} T\left(A(n) \cap \{f \geq t_i\}\right) \\
&= \sum_{i=1}^{k-1} t_i \left[T(A(n) \cap \{f \geq t_i\}) - T(A(n) \cap \{f \geq t_{i+1}\})\right] \\
&\leqslant \sum_{i=1}^{k-1} \left[S(A(n) \cap \{f \geq t_i\}) - S(A(n) \cap \{f \geq t_{i+1}\})\right] \\
&= S(A(n) \cap \{f \geq t_1\}) \leqslant S(A(n)).
\end{aligned}$$

Similarly, using the second inequality in Claim 4, $S_k \geq S(A(n))$. Therefore

$$s_k \leqslant S(A(n)) \leqslant S_k.$$

Combining (11) and (12), we obtain

$$S(A(n)) = \int_0^n T(A(n) \cap \{f \geq t\})dt = \int_{A(n)} f dT.$$

By Claim 1 we have

$$S(A) = \lim_{n \to \infty} S(A(n)) = \int_0^\infty T(A \cap \{f \geq t\})dt = \int_A f dT.$$

Consequently, we reach the conclution of the Theorem 4.2.

Remark. 1. Observe that conditions (i) - (iii) imply $S \ll T$. In fact, assumse that $T(A) = 0$. Then from Claim 4,

$$S(A \cap \{f \geq s\}) - S(A \cap \{f \geq t\}) = 0 \text{ for every } s, t \in \mathbb{R}^+ \text{ with } s < t.$$

Let $t = n \to \infty$, we obtain from (i),

$$S(A \cap \{f \geq s) = \lim_{n \to \infty} S(A \cap \{f \geq n\}) \leqslant \lim_{n \to \infty} S(A \cap B_n) = 0,$$

for every $s \in \mathbb{R}^+$.
It follows that $S(A \cap \{f \geq 0\}) = S(A) = 0$.

2. (S, T) has the RNP implies (ii) and (iii) (see, [6]).

The following example is a pair (S, T) of submeasures in $\mathbb{R}^d$ which satisfies the conditions of Theorem 4.2, and hence (S, T) has the RNP.

4.3. Example. Let T be defined as in the Example 3.7. Define $S : \mathcal{B}(\mathbb{R}^d) \longrightarrow \mathbb{R}^+$ by

$$S(A) = \begin{cases} 0 & \text{if } A \cap [0,1] = \emptyset \\ 1 - \inf\{x : x \in A \cap [0,1]\} & \text{if } A \cap [0,1] \neq \emptyset. \end{cases}$$

Then $S, T \in \mathcal{C}_0$ (T is not capacity in the sence of Graf) and (S, T) has the RNP.

Proof. It is easy to see that $S, T \in \mathcal{C}_\prime$. Define a family $\{A_t; t \in \mathbb{R}^+$ by

$$A_t = \begin{cases} [0, 1-t] & \text{if } t \leqslant 1 \\ \emptyset & \text{if } t > 1. \end{cases}$$

Then the family $\{A_t; t \in \mathbb{R}^+\}$ satisfies conditions (i) - (iii) of Theorem 4.2.

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