THE APPROXIMATE CONTROLLABILITY FOR THE LINEAR SYSTEM DESCRIBED BY GENERALIZED INVERTIBLE OPERATORS

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Abstract. In this paper, we deal with the approximate controllability for a linear system described by generalized invertible operators in the infinite dimensional Hilbert spaces.

Keywords: Right invertible and generalized invertible operators, almost inverse operator, initial operator, right and left initial operators, initial value problem.

0. Introduction

The theory of right invertible operators was started with works of D. Przeworska-Rolewicz and then has been developed by M. Tasche, H. von Trotha, Z. Binderman and many other Mathematicians. By the appearance of this theory, the initial, boundary and mixed boundary value problems for the linear systems described by right invertible operators and generalized invertible operators were studied by many Mathematicians (see [4, 8]). Nguyen Dinh Quyet considered the controllability of linear system described by right invertible operators in the case when the resolving operator is invertible (see [10, 12, 13]). These results were generalized by A. Pogorzelec in the case of one-sized invertible resolving operarors (see [6, 8]) and by Nguyen Van Mau for the system described by generalized invertible operators (see [3, 4]). The above mentioned controllability is exactly controllable from one state to another. However, in infinite dimensional space, the exact controllability is not always realized. To overcome these restrictions, we define the so-called $F_1^{(r)}$-approximately controllable, in the sense of: "A system is approximately controllable if any state can be transfered to the neighbourhood of other state by an admissible control". In this paper, we consider the approximate controllability for the system $(LS)_0$ of the form (2.1)-(2.2) in infinite dimensional Hilbert space, with $\dim(\ker V) = +\infty$. The necessary and sufficient conditions for the linear system $(LS)_0$ to be approximately reachable, approximately controllable and exactly controllable are also found.

1. Preliminaries

Let X be a linear space over a field of scalars $\mathcal{F}$ ($\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$). Denote by $L(X)$ the set of all linear operators with domains and ranges belonging to X, and by $L_0(X)$ the set of all operators of $L(X)$ whose domain is X, i.e. $L_0(X) = \{A \in L(X) : \mathrm{dom}A = X\}$.

An operator $D \in L(X)$ is said to be *right invertible* if there exists an $R \in L_0(X)$ such that $RX \subset \mathrm{dom}D$ and $DR = I$ on $\mathrm{dom}R$ (where I is the identity operator), in this

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX case R is called a *right inverse* of D. The set of all right invertible operators of $L(X)$ will be denoted by $R(X)$. For a given $D \in R(X)$, we will denote by $\mathcal{R}_D$ the set of all right inverses of D, i.e. $\mathcal{R}_D = \{R \in L_0(X) : DR = I\}$.

An operator $F \in L_0(X)$ is said to be an *initial operator* for D corresponding to $R \in \mathcal{R}_D$ if $F^2 = F, FX = \ker D$ and $FR = 0$ on $\mathrm{dom} R$. The set of all initial operators for D will be denoted by $\mathcal{F}_D$.

Theorem 1.1. [8] *Suppose that $D \in R(X)$ and $R \in \mathcal{R}_D$. A necessary and sufficient condition for an operator $F \in L(X)$ to be an initial operator for D corresponding to R is that*

$$F = I - RD \quad on \quad \mathrm{dom} D. \tag{1.1}$$

Definition 1.1. [4, 5]

(i) An operator $V \in L(X)$ is said to be *generalized invertible* if there is an operator $W \in L(X)$ (called a generalized inverse of V) such that

$$\mathrm{Im} V \subset \mathrm{dom} W,\ \mathrm{Im} W \subset \mathrm{dom} V \quad \text{and} \quad VWV = V \text{ on } \mathrm{dom} V.$$

The set of all generalized invertible operators of $L(X)$ will be denoted by $W(X)$. For a given $V \in W(X)$, the set of all generalized inverses of V is denoted by $\mathcal{W}_V$.

(ii) If $V \in W(X), W \in \mathcal{W}_V$ and $WVW = W$ on $\mathrm{dom} W$, then W is called an *almost inverse* of V. The set of all almost inverse operators of V will be denoted by $\mathcal{W}_V^1$.

Definition 1.2. [4]

(i) An operator $F^{(r)} \in L(X)$ is said to be a *right initial operator* of $V \in W(X)$ corresponding to $W \in \mathcal{W}_V^1$ if $(F^{(r)})^2 = F^{(r)}$, $\mathrm{Im} F^{(r)} = \ker V$, $\mathrm{dom} F^{(r)} = \mathrm{dom} V$ and $F^{(r)} W = 0$ on $\mathrm{dom} W$.

(ii) An operator $F^{(l)} \in L_0(X)$ is said to be a *left initial operator* of $V \in W(X)$ corresponding to $W \in \mathcal{W}_V^1$ if $(F^{(l)})^2 = F^{(l)}, F^{(l)} X = \ker W$ and $F^{(l)} V = 0$ on $\mathrm{dom} V$.

The set of all right and left initial operators of $V \in W(X)$ are denoted by $\mathcal{F}_V^{(r)}$ and $\mathcal{F}_V^{(l)}$, respectively.

Lemma 1.1. [4] *Let $V \in W(X)$ and $W \in \mathcal{W}_V$. Then*

$$\mathrm{dom} V = WV(\mathrm{dom} V) \oplus \ker V. \tag{1.2}$$

Theorem 1.2. [4] *Let $V \in W(X)$ and let $W \in \mathcal{W}_V^1$.*

(i) A necessary and sufficient condition for an operator $F^{(r)} \in L(X)$ to be a right initial operator of V corresponding to W is that $F^{(r)} = I - WV$ on $\mathrm{dom} V$.

(ii) A necessary and sufficient condition for an operator $F^{(l)} \in L_0(X)$ to be a left initial operator of V corresponding to W is that $F^{(l)} = I - VW$ on $\mathrm{dom} W$.

Theorem 1.3. [14] *Let X, Y, Z be the infinite dimensional Hilbert spaces. Suppose that $F \in L(X, Z)$ and $T \in L(Y, Z)$. Then two following conditions are equivalent*

(i) $\mathrm{Im}F \subset \mathrm{Im}T$,

(ii) There exists $c > 0$ such that $\|T^*f\| \geqslant c\|F^*f\|$ for all $f \in Z^*$ (where Z^* is the conjugate space of Z).

Theorem 1.4. (The separation theorem) *Suppose that M and N are convex sets in the Banach space X and $M \cap N = \emptyset$.*

(i) If $\mathrm{int}M \neq \emptyset$ then there exists a $x^* \in X^*, x^* \neq 0$, $\lambda \in \mathbb{R}$ such that

$$\langle x^*, x\rangle \leqslant \lambda \leqslant \langle x^*, y\rangle, \quad \text{for every} \quad x \in M,\ y \in N\,.$$

(ii) If M is a compact set in X, N is a closed set then there exists $x^* \in X^*, x^* \neq 0$, $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\langle x^*, x\rangle \leqslant \lambda_1 < \lambda_2 \leqslant \langle x^*, y\rangle, \quad \text{for every} \quad x \in M,\ y \in N\,.$$

The theory of right invertible, generalized invertible operators and their applications can be seen in [4, 8]. The proof of Theorems 1.3 and Theorem 1.4 can be found in [2, 14].

2. Approximate controllability

Let X and U be infinite dimensional Hilbert spaces over the same field of scalars $\mathcal{F}$ ($\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$). Suppose that $V \in W(X)$, with $\dim(\ker V) = +\infty$; $F^{(r)}$ and $F^{(l)}$ are right and left initial operators of V corresponding to $W \in \mathcal{W}_V^1$, respectively; $A \in L_0(X)$, and $B \in L_0(U, X)$.

Consider the linear system $(LS)_0$ of the form:

$$Vx = Ax + Bu\,,\ u \in U\,,\ BU \subset (V - A)\mathrm{dom}V \tag{2.1}$$

$$F^{(r)}x = x_0\,, \quad x_0 \in \ker V\,. \tag{2.2}$$

The spaces X and U are called the *space of states* and the *space of controls,* respectively. So that, elements $x \in X$ and $u \in U$ are called *states* and *controls,* respectively. The element $x_0 \in \ker V$ is said to be an *initial state.* A pair $(x_0, u) \in (\ker V) \times U$ is called an *input.* If the system (2.1)-(2.2) has solution $x = G(x_0, u)$ then this solution is called *output* corresponding to input (x_0, u).

Note that, the inclusion $BU \subset (V - A)\mathrm{dom}V$ holds. If the resolving operator $I - WA$ is invertible then the initial value problem (2.1)-(2.2) is well-posed for an arbitrarily fixed pair $(x_0, u) \in (\ker V) \times U$, and its unique solution is given by (see [Mbou])

$$G(x_0, u) = E_A(WBu + x_0)\,, \text{ where } E_A = (I - WA)^{-1}\,. \tag{2.3}$$

Write

$$\mathrm{Rang}_{U,x_0}G = \bigcup_{u \in U} G(x_0, u)\,, \quad x_0 \in \ker V\,. \tag{2.4}$$

Clearly, $\mathrm{Rang}_{U,x_0} G$ is the set of all solutions of (2.1)-(2.2) for arbitrarily fixed initial state $x_0 \in \ker V$. This is reachable set from the initial state x_0 by means of controls $u \in U$.

Definition. Let the linear system $(LS)_0$ of the form (2.1) – (2.2) be given. Suppose that $G(x_0, u)$ is defined by (2.3).

(i) A state $x \in X$ is said to be *approximately reachable* from the initial state $x_0 \in \ker V$ if for any $\varepsilon > 0$ there exists a control $u \in U$ such that $\|x - G(x_0, u)\| < \varepsilon$.

(ii) The linear system $(LS)_0$ is said to be *approximately reachable* from the initial state $x_0 \in \ker V$ if

$$\overline{\mathrm{Rang}_{U,x_0} G} = X .$$

Theorem. *The linear system $(LS)_0$ is approximately reachable from zero if and only if the identity*

$$B^* W^* E_A^* h = 0, \quad \text{it implies} \quad h = 0 . \tag{2.5}$$

Proof. By Definition 2.1 the system $(LS)_0$ is approximately reachable from zero if

$$\overline{E_A W B U} = X . \tag{2.6}$$

According to Theorem 1.4, the condition (2.6) is equivalent to the thing that if $h \in X^*$ such that

$$\langle h, x \rangle = 0 , \forall x \in \overline{E_A W B U}, \quad \text{it follows} \quad h = 0 . \tag{2.7}$$

Since $E_A W B U$ is a subspace of X, (2.7) holds if and only if that

$$\langle h, x \rangle = 0 , \forall x \in E_A W B U \quad \text{then} \quad h = 0 ,$$

or

$$\langle h, E_A W B u \rangle = 0 , \forall u \in U, \quad \text{it implies} \quad h = 0 .$$

This is equivalent to that if

$$\langle B^* W^* E_A^* h, u \rangle = 0 , \forall u \in U \quad \text{then} \quad h = 0 .$$

This implies that if

$$B^* W^* E_A^* h = 0 \quad \text{then} \quad h = 0 .$$

Conversely, if the condition (2.5) is satisfied then (2.7) is also satisfied, and therefore we obtain (2.6).

Definition 2.2. [4] Let the linear system $(LS)_0$ of the form (2.1) – (2.2) be given and let $F_1^{(r)} \in \mathcal{F}_V^{(r)}$ be an arbitrary right initial operator for V.

(i) A state $x_1 \in \ker V$ is said to be $F_1^{(r)}$*-reachable from the initial state* $x_0 \in \ker V$ if there exists a control $u \in U$ such that $x_1 = F_1^{(r)} G(x_0, u)$. The state x_1 is called a final state.

(ii) The system $(LS)_0$ is said to be $F_1^{(r)}$-*controllable* if for every initial state$x_0 \in \ker V$,

$$F_1^{(r)}(\text{Rang}_{U,x_0} G) = \ker V\,.$$

(iii) The system $(LS)_0$ is said to be $F_1^{(r)}$-*controllable* to $x_1 \in \ker V$ if

$$x_1 \in F_1^{(r)}(\text{Rang}_{U,x_0} G)\,,$$

for every initial state $x_0 \in \ker V$.

Definition 2.3. Let the linear system $(LS)_0$ of the form (2.1) – (2.2) be given. Suppose that $F_1^{(r)} \in \mathcal{F}_V^{(r)}$ is an arbitrary right initial operator for V.

(i) The system $(LS)_0$ is said to be $F_1^{(r)}$-*approximately reachable* from a initial state $x_0 \in \ker V$ if

$$\overline{F_1^{(r)}(\text{Rang}_{U,x_0} G)} = \ker V\,.$$

(ii) The system $(LS)_0$ is said to be $F_1^{(r)}$-*approximately controllable* if for any initial state $x_0 \in \ker V$, the following identity yelds

$$\overline{F_1^{(r)}(\text{Rang}_{U,x_0} G)} = \ker V.$$

(iii) The system $(LS)_0$ is said to be $F_1^{(r)}$-*approximately controllable* to $x_1 \in \ker V$ if

$$x_1 \in \overline{F_1^{(r)}(\text{Rang}_{U,x_0} G)}\,,$$

for every initial state $x_0 \in \ker V$.

Lemma 2.1. *Let the linear system $(LS)_0$ of the form (2.1) – (2.2) be given and let $F_1^{(r)} \in \mathcal{F}_V$ be an arbitrary initial operator. Suppose that the system $(LS)_0$ is $F_1^{(r)}$-approximately controllable to zero and*

$$F_1^{(r)} E_A(\ker V) = \ker V\,. \tag{2.8}$$

Then the final state $x_1 \in \ker V$ is $F_1^{(r)}$-approximately reachable from zero.

Proof. By the assumption, $0 \in \overline{F_1^{(r)}(\text{Rang}_{U,x_0} G)}$, for all $x_0 \in \ker V$. Therefore, for every $x_0 \in \ker V$ and $\varepsilon > 0$, there exists a control $u_0 \in U$ such that

$$\|F_1^{(r)} E_A(WBu_0 + x_0)\| < \varepsilon\,. \tag{2.9}$$

Condition (2.8) implies that for any $x_1 \in \ker V$ there exists $x_2 \in \ker V$ such that

$$F_1^{(r)} E_A x_2 = -x_1\,.$$

The last equality and (2.9) together imply that for every $x_1 \in \ker V$ and $\varepsilon > 0$, there exists a control $u_1 \in U$ such that

$$\|F_1^{(r)} E_A WBu_1 - x_1\| < \varepsilon\,.$$

It means that the final state x_1 is $F_1^{(r)}$-approximately reachable from zero.

Theorem 2.2. *Suppose that all assumptions of Lemma 2.1 are satisfied. Then the system $(LS)_0$ is $F_1^{(r)}$-approximately controllable.*

Proof. According to the assumption, for any $x_0 \in \ker V$ and $\varepsilon > 0$, there exists a control $u_0 \in U$ such that

$$\|F_1^{(r)} E_A(WBu_0 + x_0)\| < \frac{\varepsilon}{2}\,. \tag{2.10}$$

By Lemma 2.1, for an $x_1 \in \ker V$ there exists $u_1 \in U$ such that

$$\|F_1^{(r)} E_A WBu_1 - x_1\| < \frac{\varepsilon}{2}\,. \tag{2.11}$$

From (2.10) and (2.11) it follows that for $x_0, x_1 \in \ker V$ and $\varepsilon > 0$, there exists a control $u = u_0 + u_1 \in U$ such that

$$\begin{aligned}\|F_1^{(r)} E_A(WBu + x_0) - x_1\| &= \|F_1^{(r)} E_A[WB(u_0 + u_1) + x_0] - x_1\| \\ &\leqslant \|F_1^{(r)} E_A(WBu_0 + x_0)\| + \|F_1^{(r)} E_A WBu_1 - x_1\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.\end{aligned}$$

The arbitrariness of $x_0, x_1 \in \ker V$ and $\varepsilon > 0$ implies

$$\overline{F_1^{(r)}(\mathrm{Rang}_{U,x_0} G)} = \ker V\,.$$

Theorem 2.3. *Let the linear system $(LS)_0$ be given and let $F_1^{(r)} \in \mathcal{F}_V$ be an arbitrary initial operator. Then the system $(LS)_0$ is $F_1^{(r)}$-approximately controllable if and only if it is $F_1^{(r)}$-approximately controllable to every element $y' \in F_1^{(r)} E_A WV(\mathrm{dom} V)$.*

Proof. By $F_1^{(r)} E_A WV(\mathrm{dom} V) \subset \ker V$, the necessary condition is easy to be obtained. To prove the sufficient condition, we prove the equality

$$F_1^{(r)} E_A[WV(\mathrm{dom} V) \oplus \ker V] = \ker V\,. \tag{2.12}$$

Indeed, since $(I - WA)(\mathrm{dom} V) \subset \mathrm{dom} V = WV(\mathrm{dom} V) \oplus \ker V$ (by Lemma 1.1 and the properties of the generalized invertible operators [Mcon, Mbou, Mal]), there exists a set $E \subset \mathrm{dom} V$ and $Z \subset \ker V$ such that

$$WVE \oplus Z = (I - WA)(\mathrm{dom} V)\,.$$

This implies $E_A(WVE \oplus Z) = E_A(I - WA)(\mathrm{dom} V) = \mathrm{dom} V$. Thus, we have

$$\begin{aligned}\ker V = F_1^{(r)}(\mathrm{dom} V) &= F_1^{(r)} E_A(WVE \oplus Z) \\ &\subset F_1^{(r)} E_A[WV(\mathrm{dom} V) \oplus \ker V] \\ &\subset \ker V\,.\end{aligned}$$

Therefore, the formula (2.12) holds.

Suppose that the system $(LS)_0$ is $F_1^{(r)}$-approximately controllable to $y' = F_1^{(r)} E_A W V$ $y \in \mathrm{dom}V$, i.e. for every $y \in \mathrm{dom}V$ and arbitrary $\varepsilon > 0$ there exists a control $u_0 \in U$ such that

$$\|F_1^{(r)} E_A(WBu_0 + x_0) - F_1^{(r)} E_A W V y\| < \frac{\varepsilon}{2} .$$

That is

$$\|F_1^{(r)} E_A(WBu_0 + x_0 + x_2) - F_1^{(r)} E_A(WVy + x_2)\| < \frac{\varepsilon}{2} . \tag{2.13}$$

where $x_2 \in \mathrm{ker}V$ is arbitrary.

By the formula (2.12), for every $x_1 \in \ker V$, there exists $y_1 \in \mathrm{dom}V$ and $x_2' \in \mathrm{ker}V$ such that

$$x_1 = F_1^{(r)} E_A(WVy_1 + x_2') .$$

This equality and (2.13) together imply

$$\|F_1^{(r)} E_A(WBu_0' + x_0 + x_2') - x_1\| < \frac{\varepsilon}{2} . \tag{2.14}$$

On the other hand, from $0 \in F_1^{(r)} E_A W V \mathrm{dom}V$ and the assumptions, it follows that $(LS)_0$ is $F_1^{(r)}$-approximately controllable to zero, i.e.

$$0 \in \overline{F_1^{(r)}(\mathrm{Rang}_{U,x_0} G)} , \quad \text{for arbitrary } x_0 \in \mathrm{ker}V .$$

Thus, for the element $x_2' \in \mathrm{ker}V$ there exists $u_1 \in U$ such that

$$\|F_1^{(r)} E_A(WBu_1 - x_2')\| < \frac{\varepsilon}{2} . \tag{2.15}$$

Using (2.14) and (2.15 then for $x_0, x_1 \in \mathrm{ker}V$ and $\varepsilon > 0$ there exist $u = u_0' + u_1 \in U$ so that

$$\begin{aligned}
\|F_1^{(r)} E_A(WBu + x_0) - x_1\| &= \|F_1^{(r)} E_A[WB(u_0' + u_1) + x_0] - x_1\| \\
&= \|F_1^{(r)} E_A(WBu_0' + x_0 + x_2') - x_1 + F_1^{(r)} E_A(WBu_1 - x_2')\| \\
&\leqslant \|F_1^{(r)} E_A(WBu_0' + x_0 + x_2') - x_1\| + \|F_1^{(r)} E_A(WBu_1 - x_2')\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .
\end{aligned}$$

Thus,

$$\overline{F_1^{(r)}(\mathrm{Rang}_{U,x_0} G)} = \mathrm{ker}V .$$

Theorem 2.4. *Let a linear system $(LS)_0$ of the form (2.1) – (2.2) be given and let $F_1^{(r)} \in \mathcal{F}_V^{(r)}$. Then the system $(LS)_0$ is $F_1^{(r)}$-approximately reachable from zero if and only if*

$$B^* W^* E_A^* (F_1^{(r)})^* h = 0 \quad \textit{implies} \quad h = 0 . \tag{2.16}$$

Proof. Suppose that the system $(LS)_0$ is $F_1^{(r)}$-approximately reachable from zero. We then have

$$\overline{F_1^{(r)}(\text{Rang}_{U,0}G)} = \ker V\,.$$

It means

$$\overline{F_1^{(r)}E_AWBU} = \ker V\,. \tag{2.17}$$

According to Theorem 1.3, the equality (2.17) holds if and only if for $h \in (\ker V)^*$ so that

$$\langle h, x\rangle = 0\,,\ \forall x \in \overline{F_1^{(r)}E_AWBU}, \quad \text{it follows that} \quad h = 0\,. \tag{2.18}$$

Because $F_1^{(r)}E_ARBU$ is a subspace of $\ker V$, the condition (2.18) is equivalent to

$$\langle h, x\rangle = 0\,,\ \forall x \in F_1^{(r)}E_AWBU \Rightarrow h = 0\,,$$

or equivalently

$$\langle h, F_1^{(r)}E_AWBu\rangle = 0\,,\ \forall u \in U \Rightarrow h = 0\,.$$

It is satisfied if and only if

$$\langle B^*W^*E_A^*(F_1^{(r)})^*h, u\rangle = 0\,,\ \forall u \in U \Rightarrow h = 0\,. \tag{2.19}$$

Hence, the condition (2.19) means that $B^*W^*E_A^*(F_1^{(r)})^*h = 0$ implies $h = 0$.

Conversely, if (2.16) is satisfied then (2.19) holds. This implies (2.17) and therefore we obtain

$$\overline{F_1^{(r)}(\text{Rang}_{U,0}G)} = \ker V\,.$$

Theorem 2.5. *A necessary and sufficient condition for the linear system $(LS)_0$ to be $F_1^{(r)}$-controllable is that there exists a real number $\alpha > 0$ such that*

$$\|B^*W^*E_A^*(F_1^{(r)})^*f\| \geq \alpha\|f\|, \quad \textit{for all} \quad f \in (\ker V)^*\,. \tag{2.20}$$

Proof. Necessity. Suppose that the system $(LS)_0$ is $F_1^{(r)}$-controllable, we have

$$F_1^{(r)}(\text{Rang}_{U,x_0}G) = \ker V\,, \quad \text{for every} \quad x_0 \in \ker V\,.$$

It follows that $F_1^{(r)}E_AWBU = \ker V$. By Theorem 1.3, there exists a real number $\alpha > 0$ such that

$$\|(F_1^{(r)}E_AWB)^*f\| \geq \alpha\|f\|\,, \quad \text{for all} \quad f \in (\ker V)^*\,,$$

i.e. the condition (2.20) holds.

Sufficiency. Suppose that the condition (2.20) is satisfied. By using Theorem 1.3, we obtain

$$F_1^{(r)}E_AWBU \supseteq \ker V$$

Moreover, $F_1^{(r)}E_AWBU \subseteq \ker V$. Since $F_1^{(r)}$ is a right initial operator for V. Consequently, we have $F_1^{(r)}E_AWBU = \ker V$. This implies

$$F_1^{(r)}(\text{Rang}_{U,x_0}G) = \ker V\,, \quad \text{for } x_0 \in \ker V\,.$$

Theorem 2.6. *The linear system $(LS)_0$ is $F_1^{(r)}$-controllable to zero if and only if there exists $\beta > 0$ such that*

$$\|B^* W^* E_A^* (F_1^{(r)})^* f\| \geq \beta \|E_A^* (F_1^{(r)})^* f\|, \quad \text{for every} \quad f \in (\ker V)^* . \tag{2.21}$$

Proof. Suppose that the system $(LS)_0$ is $F_1^{(r)}$-controllable to zero. We then have

$$0 \in F_1^{(r)}(\mathrm{Rang}_{U,x_0} G), \quad \text{for all} \quad x_0 \in \ker V .$$

Therefore, for arbitrary $x_0 \in \ker V$, there exists $u \in U$ such that

$$F_1^{(r)} E_A (WBu + x_0) = 0 .$$

It implies that for $x_0' \in \ker W$, there exists $u' \in U$ such that $F_1^{(r)} E_A x_0' = F_1 E_A W B u'$. Thus, $F_1^{(r)} E_A (\ker V) \subseteq F_1^{(r)} E_A W B U$. Using Theorem 1.3, there exists $\beta > 0$ such that

$$\|(F_1 E_A W B)^* f\| \geq \beta \|(F_1 E_A)^* f\| , \quad \text{for all} \quad f \in (\ker V)^* .$$

Conversely, suppose that (2.21) is satisfied. By Theorem 1.3, it is concluded that

$$F_1^{(r)} E_A (\ker V) \subseteq F_1^{(r)} E_A W B U .$$

Hence, for every $x_0 \in \ker V$, there exists $u \in U$ such that

$$F_1^{(r)} E_A (WBu + x_0) = 0 ,$$

i.e. the system $(LS)_0$ is $F_1^{(r)}$-controllable to zero.

Example. Let $X = C[-1,1]$ be a space of all continuous functions defined on the closed interval $[-1,1]$, $D = \dfrac{d}{dt}$ is a right invertible operator in $L(X)$, $\mathrm{dom}D = C^1[-1,1]$. The operator $R = \int_0^t$ is a right inverse of D. The initial operator for D corresponding to R is defined as follows: $(Fx)(t) = (I - RD)x(t) = x(0)$, for $x \in \mathrm{dom}D$ (see [Mcon]).

Let $(Px)(t) = \frac{1}{2}(x(t) + x(-t))$, $Q = I - P$, $X^+ = PX$, $X^- = QX$, i.e. $X = X^+ \oplus X^-$. Put $V = PD, W = RP$ we then have $VWV = V$ on $\mathrm{dom}V$ and $WVW = W$ on $\mathrm{dom}W$. Thus, $V \in W(X)$ and $W \in \mathcal{W}_V^1$. By Theorem 1.2, the operators $F^{(r)}$ and $F^{(l)}$ are right and left initial operators for V corresponding to W, respectively, which are defined by the following formulae

$$(F^{(r)}x)(t) = [(I - RPPD)x](t) = \frac{1}{2}(x(t) + x(-t)) = (Px)(t) ,$$
$$(F^{(l)}x)(t) = [(I - PDRP)x](t) = [(I - P)x](t) = (Qx)(t) .$$

Now consider the linear system:

$$Vx = \alpha I x + Bu \quad , \quad u \in U = X^+ \tag{2.22}$$
$$F^{(r)}x = x_0 , \quad x_0 \in \ker V , \tag{2.23}$$

where $B \in L_0(X^+)$, I is the identity operator and α is a given real number.

So we have is completely proved that ker$V$ consists all even differentiable functions defined on $[-1, 1]$ and the problem (2.22)-(2.23) is equivalent to

$$(I - \alpha RP)x = RPBu + x_0\,. \tag{2.24}$$

Since $(I-\alpha RP)(I+\alpha RP) = (I+\alpha RP)(I-\alpha RP) = I-\alpha^2 RPRP = I-\alpha^2 R^2 QP = I$ (by $QP = 0$), for arbitrarily fixed $u \in U$ and $x_0 \in$ ker V, the problem (2.22)-(2.23) has a unique solution

$$x = G(x_0, u) = E_A(RPBu + x_0)\,,\ E_A = (I + \alpha RP)\,. \tag{2.25}$$

From $RPRP = 0$ it follows that

$$(I + \alpha RP)(RPBU + x_0) = RPBU \oplus \{(I + \alpha RP)x_0\}\,. \tag{2.26}$$

The conditions (2.25) and (2.26) imply (see [Mbou])

$$\mathrm{Rang}_{U,x_0} G = RPBU \oplus \{(I + \alpha RP)x_0\}\,. \tag{2.27}$$

Thus, the system (2.22)-(2.23) is $F_1^{(r)}$-approximately controllable for a right initial operator $F_1^{(r)}$ of V if and only if

$$\overline{F_1^{(r)}(RPBU \oplus \{(I + \alpha RP)x_0\})} = \mathrm{ker}V\,,$$

for every initial state $x_0 \in \mathrm{ker}V$.

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