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Cyclic Inequality Forms with Power 1/2,1/3

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Abstract: The purpose of this paper is to establish inequalities between two terms

$$F=\sum_{i=1}^{n}\sqrt{ax_i^2+bx_ix_{i+1}+cx_{i+1}^2+dx_i+ex_{i+1}+d};$$

$$G=\sum_{i=1}^{n}\sqrt[3]{ax_i^3+bx_i^2x_{i+1}+cx_ix_{i+1}^2+dx_{i+1}^3},$$

and $\sum_{i=1}^{n}x_i$ for a sequence of cyclic positive real numbers $(x_i)_{i=1}^{n+1}$ with $x_{n+1}=x_1$.

Keywords: Cyclic inequality, power 1/2,1/3.

1. Introduction

Let $x_1,x_2,...,x_n$ be sequence of positive number satisfying $x_1=x_n$. The cyclic inequalities of n variables have a form

$$S(x_1,x_2,...,x_n)\geq 0,$$

where *S* is a function of *n* variables, valued in **R**. In [2] author considers the function

$$f(x,y,z)=\frac{x}{y+z} \tag{1.1}$$

to establish a cyclic inequality

$$S(x_1,x_2,...,x_n)=\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+...+\frac{x_{n-1}}{x_n+x_1}+\frac{x_n}{x_1+x_2}-\frac{n}{2},$$

and conjectures that $S(x_1,x_2,...,x_n)\geq 0$. On the other hand, the well-known inequality

$$\sqrt{a^2+x^2}+\sqrt{b^2+y^2}+\sqrt{c^2+z^2}\geq\sqrt{(a+b+c)^2+(x+y+z)^2} \tag{1.2}$$

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is showed in [7, chapter V] with its corollary $\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\geq\sqrt{2}\left(a+b+c\right)$. Some inequalities are "similar" but they are in opposite side

$$\sqrt{a^2+5ab+4b^2}+\sqrt{b^2+5bc+4c^2}+\sqrt{c^2+5ca+4a^2}\leq 3\left(a+b+c\right) \quad (1.3)$$

$$\sqrt{a^2+bc}+\sqrt{b^2+ca}+\sqrt{c^2+ab}\leq\frac{3}{2}\left(a+b+c\right) \quad (1.4)$$

in [5, chapter 5].

In this paper, instead of using the function *f* in (1.1), we use the the functions (1.2), (1.3) and (1.4) to establish cyclic inequalities by comparing the functions

$$F\left(x_1,x_2,...,x_n\right)=\sum_{i=1}^{n}\sqrt{ax_i^2+bx_ix_{i+1}+cx_{i+1}^2} \text{ or}$$

$$G\left(x_1,x_2,...,x_n\right)=\sum_{i=1}^{n}\sqrt[3]{ax_i^3+bx_i^2x_{i+1}+cx_ix_{i+1}^2+dx_{i+1}^3}$$

with a function of $\sum_{i=1}^{n}x_i$, where $x_{n+1}=x_1$.

The main way to approach the problem is firstly to establish an inequality between $\sqrt{ax^2+bxy+cy^2}$ and αx+βy and then to give a condition on $\Delta=b^2-4ac$ for which we obtain the result

$$F=\sqrt{a+b+c}\geq\sum_{i=1}^{n}x_i \text{ or } F=\sqrt{a+b+c}\leq\sum_{i=1}^{n}x_i$$

depending on the sign of Δ. We also consider a general problem with the expression $A\left(x;y\right)=ax^2+bxy+cy^2+dx+ex+f$. By reducing it into a simpler form $A\left(x;y\right)=ax^2+bxy+cy^2+d$, we get the same inequality in case Δ>0. For general *n*, it is still an open problem.

For the expression *G*, by using the similar idea of proving *F*, we try to drive an expression

$$ax^3+bx^2y+cxy^2+dy^3=\left(\alpha x+\beta y\right)^3+\left(\gamma x+\delta y\right)\left(x-y\right)^2,$$

and depending on sign of γ,δ we have the comparison between *G* and $\sum x_i$. This paper is organised as follows. In section 2 we consider cyclic inequality with power $\frac{1}{2}$ by using the function $f\left(x,y\right)=ax^2+bxy+cy^2+dx+ey+f$. Section 3 deals with cyclic inequality with power $\frac{1}{3}$ with the function $f\left(x,y\right)=ax^3+bx^2y+cxy^2+dy^3$.

2. Cyclic inequality with power $\frac{1}{2}$

First at all, we establish an inequality for $\sqrt{ax^2+bxy+cy^2+d}$. To start with, we will find the condition such that the expression $\sqrt{ax^2+bxy+cy^2+d}$ is defined for all $x,y\geq 0$.

Put $A(x;y)=ax^2+bxy+cy^2+d$ and $\Delta=b^2-4ac$.

Lemma 2.1 $A(x;y)\geq 0$ for all x,y≥0 if and only if one of two following conditions holds

i) a,c,b,d≥0;

ii) a,c,d≥0,b<0 and Δ≤0.

Proof. Since $A(0,0)=d$, it follows that $d\geq 0$. Writing $A(x,y)=y^2(at^2+bt+c)+d$ with $t=\frac{x}{y}$ yields $at^2+bt+c\geq 0$ for all $t\geq 0$. Indeed, if there is t_0 such that $at_0^2+bt_0+c<0$ then we let $y\to\infty$ to get a contradiction. The case $a=0$ is trivial, so let $a\neq 0$. From the property $at^2+bt+c\geq 0$ for all $t\geq 0$ it follows $a,c\geq 0$. Hence, if $b\geq 0$ we have *i*). Otherwise, If $b<0$, we use the expression $A(x;y)=y^2a\left(\left(t+\frac{b}{2a}\right)^2-\frac{\Delta}{4a^2}\right)$ then with $t=-\frac{b}{2a}>0$ we see that $A(x;y)\geq 0$ is equivalent to $\Delta\leq 0$. Thus we have *ii*). The proof is complete. □

Remark 2.2 *In both conditions we have a+b+c≥0. If a+b+c=0 then the condition becomes* $a=c=-\frac{b}{2}=0.$

For the sake of simplicity, from now on we assume $a>0$. Firstly, we consider the case $d=0$.

Lemma 2.3 *Let a,b,c be real numbers such that a+b+c>0. There exist α,β,γ such that α+β>0 and*

$$ax^2+bxy+cy^2=(\alpha x+\beta y)^2+\gamma(x-y)^2.$$

Proof. Developing the desired equality and comparing coefficients of both sides we have the equation

$$\begin{cases}\alpha^2+\beta=a\\ \beta^2+\gamma=c\\ 2\alpha\beta-2\gamma=b.\end{cases} \tag{2.1}$$

Summing these equations, we get $(\alpha+\beta)^2=a+b+c$. Thus, $\alpha+\beta=\sqrt{a+b+c}$. Now, subtracting the first and the second equation in (2.1) we have $\alpha^2-\beta^2=a-c$ which follows $\alpha-\beta=\frac{a-c}{\sqrt{a+b+c}}$.

Hence

$$\alpha=\frac{2a+b}{2\sqrt{a+b+c}},\beta=\frac{2c+b}{2\sqrt{a+b+c}}.$$

Substituting α,β into the last equation in (2.1) yields

$$\gamma = \frac{4ac - b^2}{4(a+b+c)}.$$

We have the proof. □

Theorem 2.4 *Let a,b,c be real numbers such that a,c,a+b+c>0. For any sequence of real positive numbers* $x_1, x_2, ..., x_{n+1}$ *with* $x_{n+1} = x_1$ *we have*

i) if Δ≤0 then $F(x_1, x_2, ..., x_n) \geq \sqrt{a+b+c}\sum_{i=1}^{n} x_i;$

ii) if Δ>0,a,b,c≥0 then $F(x_1, x_2, ..., x_n) \leq \sqrt{a+b+c}\sum_{i=1}^{n} x_i.$

The equality occurs if and only if $x_1 = x_2 = ... = \xi_v$.

Proof. Applying Lemma

$$\sqrt{ax_i^2 + bx_i x_{i+1} + cx_{i+1}^2} = \sqrt{(\alpha x_i + \beta x_{i+1})^2 + \gamma (x_i - x_{i+1})^2},$$

for $i = 1,2,...,n, x_{n+1} = x_1$.

i) It is clear that Δ≤0 is equivalent to γ≥0 or $\sqrt{ax^2 + bxy + cy^2} \geq |\alpha x + \beta y|$. Therefore

$$F(x_1, x_2, ..., x_n) \geq \sum_{i=1}^{n} |\alpha x_i + \beta x_{i+1}| \geq \left|\sum_{i=1}^{n} (\alpha + \beta) x_i\right|$$

$$\geq \sqrt{a+b+c}\sum_{i=1}^{n} x_i.$$

We have the first conclusion.

ii) The second is easy since α,β≥0,γ<0. □

Now, we study the problem in a more general context. Firstly, we consider the following proposition:

Proposition 2.5. *The relation*

$$ax^2 + bxy + cy^2 + dx + ey + f \geq 0$$

holds for all real values of x and y if and only if one of the following items i), ii), iii) or iv) is satisfied

i) $a>0, \begin{vmatrix} 2a & b \\ b & 2c \end{vmatrix} > 0, \begin{vmatrix} 2a & b & d \\ b & 2c & e \\ d & e & 2f \end{vmatrix} > 0$

ii) $a>0, \begin{vmatrix} 2a & b \\ b & 2c \end{vmatrix} = 0, \begin{vmatrix} 2a & b \\ d & e \end{vmatrix} = 0, \begin{vmatrix} 2a & d \\ d & 2f \end{vmatrix} > 0,$

iii) $a=b=d=0, c>0, \begin{vmatrix} 2c & e \\ e & 2f \end{vmatrix} > 0,$

iv) $a=b=c=d=e=0, f>0.$

Proof. See [6], chapter 11. □

Denote

$$A(x;y) = ax^2 + bxy + cy^2 + dx + ey + f, a \neq 0,\ \Delta = b^2 - 4ac.$$

By changing variables $x=X+m, y=Y+n$ then $A(x;y) = aX^2 + bXY + cY^2 + DX + EY + F$ where $D=2am+bn+d, E=2cn+bm+e, F= am^2 + bmn + cn^2 + dm + en + f$. It depends on $\Delta\neq 0$ or $\Delta=0$ we can choose m,n such that $D=E=0$ and $A(x;y)$ transforms into one of the two types:

i) $A_1(x;y) = ax^2 + bxy + cy^2 + d$;

ii) $A_2(x;y) = ax^2 + by + c$.

It is not difficult to give some conditions on a,b,c,d,e,f such that $A(x;y) = 0$ for all $x,y\geq 0$. However, these conditions are very complicated. So, for simple, from now on, we only consider the cases of $A_1(x;y), A_2(x;y) \geq 0$ for all $x,y\geq 0$.

For the case *i*: $A(x;y) = ax^2 + bxy + cy^2 + d$. We now consider the case $d>0$. As in Lemma 1, to make sure $A(x;y) \geq 0$ for all $x,y\geq 0$ we need conditions: $a,b,c,d\geq 0$ or $a,c,d \geq 0, b<0, \Delta = b^2 - 4ac \leq 0.$ Without loosing of generality we can assume $d=1$. Then,

$$F(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} \sqrt{ax_i^2 + bx_i x_{i+1} + cx_{i+1}^2 + 1}, (x_{n+1} = x_1).$$

Theorem 2.6 Let a,b,c be real numbers such that $a,c,d, a+b+c>0, \Delta = b^2 - 4ac \leq 0.$ For any real positive number $x_1, x_2, ..., x_n$ we have

$$F(x_1, x_2, ..., x_n) \geq \sqrt{(a+b+c)\left(\sum_{i=1}^{n} x_i\right)^2 + n^2}.$$

The equality occurs if and only if $x_1 = x_2 = ... = x_n$.

Proof. Since $\Delta\leq 0$, $\gamma\geq 0$. Applying Lemma 2 we have $\sqrt{ax_i^2 + bx_i x_{i+1} + cx_{i+1}^2 + 1} \geq \sqrt{(\alpha x_i + \beta x_{i+1})^2 + 1}.$ Therefore,

$$\begin{aligned} F &\geq \sum_{i=1}^{n} \sqrt{(\alpha x_i + \beta x_{i+1})^2 + 1} \\ &\geq \sqrt{\left(\sum_{i=1}^{n} \alpha x_i + \beta x_{i+1}\right)^2 + (1 + ... + 1)^2} \\ &= \sqrt{(\alpha+\beta)^2 \left(\sum_{i=1}^{n} x_i\right)^2 + n^2} \\ &= \sqrt{(a+b+c)\left(\sum_{i=1}^{n} x_i\right)^2 + n^2}. \end{aligned}$$

And we have the conclution. □

Now, consider the case Δ>0, to estimate *F*, we need some more conditions on *a*,*b*,*c* and it becomes very complicate if *n*≥4. For *n*=2,3 we have results:

Proposition 2.7 *i) If a,c>0,b≥a+c then*

$$F_2(x;y)\le\sqrt{(a+b+c)(x+y)^2+4}.$$

ii) If a,c>0,b≥2a+2c then

$$F_3(x;y;z)\le\sqrt{(a+b+c)(x+y+z)^2+9}.$$

The equality holds iff x=y in i), or x=y=z in ii).

Proof. *i*) We have

$$F_2(x;y)=\sqrt{(\alpha x+\beta y)^2+\gamma(x-y)^2+1}+\sqrt{(\alpha y+\beta x)^2+\gamma(y-x)^2+1}$$
$$\le\sqrt{2\left((\alpha x+\beta y)^2+(\alpha y+\beta x)^2+2\gamma(x-y)^2+2\right)}.$$

Moreover,

$$(\alpha+\beta)^2(x+y)^2-2(\alpha x+\beta y)^2-2(\beta x+\alpha y)^2-4\gamma(x-y)^2$$
$$=(x-y)^2\left(-\alpha^2+2\alpha\beta-\beta^2-4\gamma\right)=(x-y)^2(-a+b-c)\ge 0$$

since *b*≥*a*+*c*. Hence,

$$F_2(x;y)\le\sqrt{(\alpha+\beta)^2(x+y)^2+4}=\sqrt{(a+b+c)(x+y)^2+4}.$$

ii) Similarly,

$$F_2(x;y)\le\sqrt{3\left((\alpha x+\beta y)^2+(\alpha y+\beta z)^2+(\alpha z+\beta x)^2+\gamma K+3\right)}.$$

where $K=(x-y)^2+(y-z)^2+(z-x)^2\ge 0.$ Noting that

$$(\alpha+\beta)^2(x+y+z)^2-3(\alpha x+\beta y)^2-3(\alpha y+\beta z)^2-3(\alpha z+\beta x)^2-3\gamma K$$
$$\left(-\alpha^2+\alpha\beta-\beta^2-3\gamma\right)K=\left(-a+\frac{b}{2}-c\right)K\ge 0.$$

And from this we have the proof. □

We continue with the case ii) of the general problem: $A(x;y)=ax^2+by+c.$ In this case, we need condition: *a*,*b*,*c*≥0. If *b*=0, the problem have been considered, For the case *a*=0, it is easy to show that

Proposition 2.8 For b,c≥0 we have

$$\sqrt{b\sum_{i=1}^{n}x_i+c}\le\sum_{i=1}^{n}\sqrt{bx_i+c}\le\sqrt{nb\sum_{i=1}^{n}x_i+n^2c}.$$

Thus we need only considering the case *a*,*b*>0:

Theorem 2.9 *i)* $F_2(x;y)\ge\sqrt{a(x+y)^2+2b(x+y)+4c}$ *if* $b^2-4ac\le 0$.

ii) $F_n \geq \sqrt{\frac{a}{n}\left(\sum_{i=1}^{n} x_i\right)^2 + b\sum_{i=1}^{n} x_i + c} + (n-1)\sqrt{c}.$

Proof. By changing variables again, we can assume that *a*=*b*=1. Then

i) By squaring

$$\left(\sqrt{x^2+y+c}+\sqrt{y^2+x+c}\right)^2-\left((x+y)^2+2(x+y)+4c\right)$$
$$=2\sqrt{y^2+c+x}\sqrt{x^2+c+y}-x-y-2c-2xy$$

and

$$\left(2\sqrt{y^2+c+x}\sqrt{x^2+c+y}\right)^2-(x+y+2c+2xy)^2$$
$$=(x-y)^2(4c+4x+4y-1)\geq 0.$$

we obtain the result.

ii) We will show that $\sqrt{a}+\sqrt{b}\geq\sqrt{c}+\sqrt{a+b-c}$ for any non-negative number *a*,*b*,*c* with $c=min(a,b,c)$. Indeed, twice squaring reduces the inequality to $(a-c)(b-c)\geq 0.$ From this inequality we have

$$\begin{aligned}
F &= \sum_{i=1}^{n}\sqrt{x_i^2+x_{i+1}+c}\\
&\geq \sqrt{c}+\sqrt{x_1^2+x_2^2+x_2+x_3+c}+\sum_{i=3}^{n}\sqrt{x_i^2+x_{i+1}+c}\\
&\geq \ldots\\
&\geq (n-1)\sqrt{c}+\sqrt{\sum_{i=1}^{n}x_i^2+\sum_{i=1}^{n}x_i+c}\\
&\geq (n-1)\sqrt{c}+\sqrt{\frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)^2+\sum_{i=1}^{n}x_i+c}.
\end{aligned}$$

We have the proof. □

3. Cyclic inequality with power $\frac{1}{3}$

Let us consider the expression $f(x,y)=ax^3+bx^2y+cxy^2+dy^3$. In case of *a*+*b*+*c*+*d*=0, we know that there exists α,β,γ such that $ax^3+bx^2y+cxy^2+dy^3=(x-y)\left(ax^2+(a+b)xy-dy^2\right)$. However, if *a*+*b*+*c*+*d*≠0 we have

Lemma 3.1 *For a,b,c,d,x,y are real numbers such that a+b+c+d≠0, there exist unique α,β,γ,δ such that the equality*

$$ax^3+bx^2y+cxy^2+dy^3=(\alpha x+\beta y)^3+(\gamma x+\delta y)(x-y)^2 \tag{3.1}$$

holds.

Proof. The right hand side of (3.1) is equal to

$$(\alpha x+\beta y)^3+\left(\delta x^3+(\delta-2\gamma)x^2y+(\gamma-2\delta)xy^2+\delta y^3\right).$$

Hence, by comparing the coefficients of both sides the equality of (3.1) we need to show that the following system has a unique solution

$$\begin{cases} \alpha^3+\gamma=a \\ 3\alpha^2\beta+\delta-2\gamma=b \\ 3\alpha\beta^2+\gamma-2\delta=c \\ \beta^3+\delta=d. \end{cases} \tag{3.2}$$

Indeed, by eliminating γ,δ from the equations in (3.2) we easily get

$$\begin{cases} \alpha+\beta=\sqrt[3]{a+b+c+d} \\ (2\alpha-\beta)(\alpha+\beta)^2=b+2a-d \\ (2\beta-\alpha)(\alpha+\beta)^2=c+2d-a. \end{cases}$$

So we have

$$\begin{cases} \alpha=\dfrac{1}{3}\left(D+\dfrac{b+2a-d}{D^2}\right)=\dfrac{3a+2b+c}{3D^2} \\ \beta=\dfrac{1}{3}\left(D+\dfrac{c+2d-a}{D^2}\right)=\dfrac{b+2c+3d}{3D^2} \end{cases}$$

where, for simplicity, we put $D=\sqrt[3]{a+b+c+d}$. From the firts and fourth equations of (3.2) we have

$$\begin{cases} \gamma=a-\left(\dfrac{3a+2b+c}{3D^2}\right)^3; \\ \delta=d-\left(\dfrac{b+2c+3d}{3D^2}\right)^3. \end{cases} \tag{3.3}$$

The lemma is proved. □

We now want to establish a cyclic inequality between the function

$$G=\sum_{k=1}^{n}\sqrt[3]{ax_k^3+bx_k^2x_{k+1}+cx_kx_{k+1}^2+dx_{k+1}^3},$$

and the sum $\sum_{i=1}^{n}x_i$ for non-negative numbers $x_1,x_2,...,x_n$ with $x_{n+1}=x_1$.

Theorem 3.2 *Suppose for a,b,c,d be real numbers whose sum is different from 0, and* $27a(a+b+c+d)^2\geq(3a+2b+c)^3, 27d(a+b+c+d)^2\geq(b+2c+3d)^3$ *then*

$$G \geq \sqrt[3]{a+b+c+d}\sum_{k=1}^{n} x_k$$

for all $x_1, x_2, ..., x_n \geq 0$ *with* $x_n = x_1$. We have the reverse inequality

$$G \leq \sqrt[3]{a+b+c+d}\sum_{k=1}^{n} x_k$$

if

$$27a\left(a+b+c+d\right)^2 \leq \left(3a+2b+c\right)^3, 27d\left(a+b+c+d\right)^2 \leq \left(b+2c+3d\right)^3.$$

Proof. The proof can be implied directly from the Lemma 3 by noting that with these assumptions we have $\gamma,\delta \geq 0$. □

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