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A Simple Proof for a Theorem of Nagel and Schenzel

Duong Thi Huong*

Department of Mathematics, Thang Long University, Hanoi, Vietnam

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Abstract: Nagel-Schenzel's isomorphism that has many applications was proved by using spectral sequence theory. In this short note, we present a simple proof for the theorem of Nagel and Schenzel.

Keywords: Local cohomology, filter regular sequence.

1. Introduction

Throughout this paper, let R be a commutative Noetherian ring, M a finitely genrated R-module and I an ideal of R. Local cohomology $H_I^i(M)$ introduced by Grothendieck, is an important tool in both algebraic geometry and commutative algebra (cf. [2]). Moreover, the notion of I-filter regular sequences on M is an useful technique in study local cohomology. In [4] Nagel and Schenzel proved the following theorem (see also [1]).

Theorem 1.1. *Let I be an ideal of a Noetherian ring R and M a finitely generated R-module. Let $x_1, \dots, x_n$ an I-filter regular sequence of M. Then we have*

$$H_I^i(M) \cong \begin{cases} H^i_{(x_1,\dots,x_t)}(M) & if\ i < t \\ H_I^{i-t}(H^i_{(x_1,\dots,x_t)}(M)) & if\ i \geq t. \end{cases}$$

The most important case of Theorem 1.1 is $i = t$, and $H_I^t(M) \cong H_I^0(H^t_{(x_1,\dots,x_t)}(M))$ is a submodule of $H^t_{(x_1,\dots,x_t)}(M)$. Recently, many applications of this fact have been found [3,5]. It should be noted that Nagel-Schenzel's theorem was proved by using spectral sequence theory. The aim of this short note is to give a simple proof for Theorem 1.1 based on standard argument on local cohomology [2].

2. Proofs

Firstly, we recall the notion of I-filter regular sequence on M.

*Corresponding author. Tel.: 84-983602625.
Email: duonghuongtlu@gmail.com

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Definition 2.1. Let M be a finitely generated module over a local ring $(R, \mathfrak{m}, \mathrm{k})$ and let $x_1, \dots, x_t \in I$ be a sequence of elements of R. Then we say that $x_1, \dots, x_t$ is a *I-filter regular sequence* on M if the following conditions hold:

$$\mathrm{Supp}(((x_1, \dots, x_{i-1})M : x_i)/(x_1, \dots, x_{i-1})M \subseteq V(I)$$

for all $i = 1, \dots, t$, where $V(I)$ denotes the set of prime ideals containing I. This condition is equivalent to $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in Ass_R M/(x_1, \dots, x_{i-1})M \setminus V(I)$ and for all $i = 1, \dots, t$.

Remark 2.2. It should be noted that for any $t \geq 1$ we always can choose a I-filter regular sequence $x_1, \dots, x_t$ on M. Indeed, by the prime avoidance lemma we can choose $x_1 \in I$ and $x_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in Ass_R R \setminus V(I)$. For $i > 1$ assume that we have $x_1, \dots, x_{i-1}$, then we choose $x_i \in I$ and $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in Ass_R R/(x_1, \dots, x_{i-1}) \setminus V(I)$ by the prime avoidance lemma again. For more details, see [1, Section 2].

The I-filter regular sequence can be seen as a generalization of the well-known notion of regular sequence (cf. [4, Proposition 2.2]).

Lemma 2.3. A sequence $x_1, \dots, x_t \in I$ is an I-filter regular sequence on M if and only if for all $\mathfrak{p} \in \mathrm{Spec}(R) \setminus V(I)$, and for all $i \leq t$ such that $x_1, \dots, x_i \in \mathfrak{p}$ we have $\frac{x_1}{1}, \dots, \frac{x_t}{1}$ is an $M_\mathfrak{p}$-sequence.

Corollary 2.4. Let $x_1, \dots, x_t \in I$ be an I-filter regular sequence on M. Then $H^i_{(x_1, \dots, x_t)}(M)$ is I-torsion for all $i < t$.

Proof. For each $\mathfrak{p} \in \mathrm{Spec}(R) \setminus V(I)$ we have either $(x_1, \dots x_t)R_\mathfrak{p} = R_\mathfrak{p}$ or $x_1, \dots, x_t$ is an $M_\mathfrak{p}$-regular sequence by Lemma 2.3. For the first case we have

$$\left(H^i_{(x_1, \dots, x_t)}(M)\right)_\mathfrak{p} \cong H^i_{(x_1, \dots, x_t)R_\mathfrak{p}}(M_\mathfrak{p}) = 0$$

for all $i \geq 0$. For the second case we have

$$\left(H^i_{(x_1, \dots, x_t)}(M)\right)_\mathfrak{p} \cong H^i_{(x_1, \dots, x_t)R_\mathfrak{p}}(M_\mathfrak{p}) = 0$$

for all $i < t$ by the Grothendieck vanishing theorem [2, Theorem 6.2.7]. Therefore we have $\left(H^i_{(x_1, \dots, x_t)}(M)\right)_\mathfrak{p} \cong 0$ for all $i < t$ and for all $\mathfrak{p} \in \mathrm{Spec}(R) \setminus V(I)$. So $H^i_{(x_1, \dots, x_t)}(M)$ is I-torsion for all $i < t$. □

It is well-known that local cohomology $H^i_{(x_1, \dots, x_t)}(M)$ agrees with the i-th cohomology of the Čech complex with respect to the sequence $x_1, \dots, x_t$

$$0 \to M \xrightarrow{d^0} \bigoplus_i M_{x_i} \xrightarrow{d^1} \bigoplus_{i<j} M_{x_i x_j} \xrightarrow{d^2} \cdots \xrightarrow{d^{t-1}} M_{x_1 \dots x_t} \to 0 \quad (\star)$$

The following simple fact is the crucial key for our proof.

Lemma 2.5. Let $x \in I$ be any element of R. Then $H^i_I(M_x) = 0$ $for\ all$ $i \geq 0$.

Proof. Obiviously the multiplication map $M_x \xrightarrow{x} M_x$ is an isomorphism. It induces isomorphism maps $H^i_I(M_x) \xrightarrow{x} H^i_I(M_x)$ for all $i \geq 0$. But $H^i_I(M_x)$ is I- torsion, so it (x)-torsion since $x \in I$. Therefore $H^i_I(M_x) = 0$ for all $i \geq 0$. □

We are ready to prove the theorem of Nagel and Schenzel.

Proof of Theorem 1.1. We set $(\underline{x})$ the ideal generated by $x_1, \ldots, x_t$. Let C_j the j -th chain of Čech complex $(\star)$ and set $L_j := \mathrm{Im}\, \mathrm{d}^{j-1}$ and $K_j := \mathrm{Ker}\, \mathrm{d}^{j}$ for all $j \geq 1$. We split the Čech complex $(\star)$ into short exact sequences $0 \to H^0_{(\underline{x})}(M) \to M \to L_1 \to 0 \qquad (A_0)$

$$0 \to L_1 \to K_1 \to H^1_{(\underline{x})}(M) \to 0 \qquad (B_1)$$

$$0 \to K_1 \to C_1 \to L_2 \to 0 \qquad (A_1)$$

$$\ldots$$

$$0 \to L_j \to K_j \to H^j_{(\underline{x})}(M) \to 0 \qquad (B_j)$$

$$0 \to K_j \to C_j \to L_{j+1} \to 0 \qquad (A_j)$$

$$\ldots$$

$$0 \to L_{t-1} \to K_{t-1} \to H^{t-1}_{(\underline{x})}(M) \to 0 \quad (B_{t-1})$$

$$0 \to K_{t-1} \to C_{t-1} \to L_t \to 0 \qquad (A_{t-1})$$

$$0 \to L_t \to M_{x_1 \ldots x_t} \to H^t_{(\underline{x})}(M) \to 0. \quad (B_t)$$

By Lemma 2.3 we have $H^i_I(C_j) = 0$ for all $i \geq 0$ and all $j \geq 1$ Since L_j and K_j are submodules of C_j for all $j \geq 1$ we have $H^0_I(L_j) \cong H^0_I(K_j) = 0$ for all $j \geq 1$. We also note that $H^j_{(\underline{x})}(M)$ is I-torsion for all $j < t$ by Corollary 2.4, so $H^0_I\left(H^j_{(\underline{x})}(M)\right) = H^j_{(\underline{x})}(M)$ and $H^i_I\left(H^j_{(\underline{x})}(M)\right) \cong 0$ for all $j < t$ and for all $i \geq 1$.

Now applying the functor $H^i_I(-)$ to the short exact sequence (A_0) and using the above observations we have

$$H^0_I(M) \cong H^0_{(\underline{x})}(M)$$

and

$$H^i_I(M) \cong H^i_I(L_1) \quad (1)$$

for all $i \geq 1$.

For each $j = 1, \ldots, t-1$, applying the local cohomology functor $H^i_I(-)$ to the short exact sequence (A_j) we have $H^1_I(K_j) = 0$ and the isomorphism

$$H^i_I(L_{j+1}) \cong H^{i+1}_I(K_j) \qquad (C_j)$$

for all $i \geq 1$. Furthermore, if we apply $H^i_I(-)$ for the short exact sequence (B_j), then we get the short exact sequence

$$0 \to H^j_{(\underline{x})}(M) \to H^1_I(L_j) \to H^1_I(K_j) \to 0,$$

and the isomorphism

$$H^i_I(L_j) \cong H^i_I(K_j) \qquad (D_j)$$

for all $i \geq 2$. Note that $H^1_I(K_j) = 0$ as above, so

$$H^j_{(\underline{x})}(M) \cong H^1_I(L_j). \qquad (2)$$

We next show that $H_I^i(M) \cong H_{(\underline{x})}^i(M)$ for all $i = 1, \dots, t-1$. Indeed, using isomorphisms $(1), (2), (C_j)$ and (D_j) consecutively, we have

$$H_I^i(M) \overset{(1)}{\cong} H_I^i(L_1) \overset{(D_1)}{\cong} H_I^i(K_1) \overset{(C_1)}{\cong} H_I^{i-1}(L_2) \overset{(D_2)}{\cong} \cdots \overset{(C_{i-1})}{\cong} H_I^1(L_i) \overset{(2)}{\cong} H_{(\underline{x})}^i(M).$$

Therefore, we have showed the first case of Nagel-Schenzel's isomorphism $H_I^i(M) \cong H_{(\underline{x})}^i(M)$ for all $i = 0, \dots, t-1$. Finally, for $i \geq t$ by similar arguments we have

$$H_I^i(M) \overset{(1)}{\cong} H_I^i(L_1) \overset{(D_1)}{\cong} H_I^i(K_1) \overset{(C_1)}{\cong} H_I^{i-1}(L_2) \overset{(D_2)}{\cong} \cdots \overset{(C_{t-1})}{\cong} H_I^{i-t+1}(L_t).$$

On the other hand, by applying the functor $H_I^i(-)$ to the short exact sequence (B_t) we have $H_I^{i-t}\left(H_{(\underline{x})}^t(M)\right) \cong H_I^{i-t+1}(L_t)$

for all $i \geq t$. Thus $H_I^i(M) \cong H_I^{i-t}(H_{(\underline{x})}^t(M))$ for all $i \geq t$, and we finish the proof. □

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