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Bohl-Perron Theorem for Differential Algebraic Equations

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Abstract: This paper is concerned with the Bohl-Perron theorem for differential algebraic equations. We prove that the system $E(t)x'(t) = A(t)x(t),\ t \ge t_0$ is exponentially stable if and only if for any bounded input q, the equation

$$E(t)x'(t) = A(t)x(t) + q(t), \quad x(t_0) = 0, \quad t \ge t_0$$

has a bounded solution.

Keywords: Differential algebraic equation, asymptotic stability, input - output bounded function.

1. Introduction

In lots of applications there is a frequently arising question, namely how robust is a characteristic qualitative property of a system (e.g., the stability) when the system comes under the effect of uncertain perturbations. The designer wants to have operation systems working stably under small perturbation. Therefore, the investigation which conditions ensures robust stability play an important role both in theory and practice. On the other hand, to measure the robust stability, one proceed a test and expect that with rather good input, the output will satisfy some desired properties. For example, if the bounded input implies the boundedness of output then our system must be stable. The aim of this paper is to answer the above questions. We focus on studying the robust stability of time-varying systems of differential-algebraic equations (DAE-s) of the form

$$E(t)x'(t) = A(t)x(t), \quad t \ge t_0, \tag{1.1}$$

where $E(\cdot)$, $A(\cdot)$ are continuous matrix functions defined on $[0,\infty)$, valued in $\mathbb{R}^{d\times d}$. The leading term $E(t)$ is supposed to be singular for all $t \ge t_0$. If the system (1.1) is subjected to an outer force q, then it becomes

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$$E(t)x'(t) = A(t)x(t) + q(t), \quad t \ge t_0, \tag{1.2}$$

These systems occur in various applications, such as optimal control, electronic circuit simulation, multibody mechanics, etc., and they are described by a so-called differential algebraic systems with time varying, see [1, 2]. Therefore, it is worth to consider the stability such these system. To study that, the index notion, which plays a key role in the qualitative theory and in the numerical analysis of DAE-s, should be taken into consideration in the robust stability analysis, see [3, 4]. Many works are concerned to this problem such as [2, 5, 6]. In [7], Authors consider the Bohl-Perron type theorem for dynamic equation on time scales meanwhile in [8] Authors consider the stability under small perturbations for implicit difference equations. Our main goal of this paper is to develop these results by considering the relation between the stability of the system (1.1) and the analytic properties of outer force $q(t)$.

The paper is organized as follows. In the next section we recall some basic notions and preliminary results on the theory of linear DAE-s and deal with the solvability of DAE-s. In Section 3, we prove that if the system (1.1) is exponentially stable, then under small feedback perturbations $q(t) = B(t)x(t)$, the system (3.1) is still stable. In the last section, the famous Bohl-Perron theorem for linear equation is presented.

2. Preliminary

2.1. Some surveys on linear algebra

In this section, we survey some basic properties of linear algebra. Let (E, A) be a pencil of matrices. Suppose that rank $E = r$. Denote $S = \{x: \ Ax \in \operatorname{Im} E\}$ and let Q be a projector onto $\operatorname{Ker} E$.

Lemma 2.1 *The following assertions are equivalent*

a) $S \cap \operatorname{Ker} E = \{0\}$; (2.1)

b) *The matrix* $G = E - AQ$ *is nonsingular;* (2.2)

c) $\mathbb{R}^d = S \oplus \operatorname{Ker} E$; (2.3)

Proof see [9, Appendix 1].

Lemma 2.2 *Suppose that the matrix G is nonsingular. Then, there hold the following relations:*

a) $P = G^{-1}E$; *where* $P = I - Q$.

b) $-G^{-1}AQ = Q$;

c) $\tilde{Q} = -QG^{-1}A$ *is the projector onto* $\operatorname{Ker} E$ *along S;*

d) *QG−1 does not depend on the choice of Q.*

Proof See [9, Appendix 1]. □

2.2. Solvability of implicit linear dynamic equations

We consider the linear differential-algebraic system

$$E(t)x'(t) = A(t)x(t) + q(t), \quad t \ge t_0, \tag{2.4}$$

where *E, A* are continuous matrix functions as in Section 1 and q is a continuous function defined on $[0, \infty)$, valued in $\mathbb{R}^n$. Suppose that $Ker\, E(t)$ is smooth in the sense there exists an continuously differential projector $Q(t)$ onto $Ker\,E(t)$, i.e., $Q \in C^1(0,\infty,\mathbb{R}^{n\times n})$ and Q^2=Q, Im Q(t)=Ker E(t) for all $t \geq 0$. Set $P = I - Q$, then $P(t)$ is a projector along $Ker\,E(t)$. With these notations, the system (2.4) can be rewritten into the form

$$E(t)(Px)'(t) = \overline{A}(t)x(t) + q(t), \quad t \geq t_0, \tag{2.5}$$

where $\overline{A} = A + EP$. Define $G = E - \overline{A}Q$.

Definition 2.3 (see also [6, Section 1.2]) The DAE (2.4) is said to be index-1 tractable if $G(t)$ is invertible for almost every $t \in [0,\infty)$.

Note that by Lemma 2, the index-1 property does not depend on the choice of the projectors Q, see [6, 5].

Now let (2.4) be index-1. Taking into account the equalities $G^{-1}E = P,\ G^{-1}A = -Q + G^{-1}AP,$ and multiplying both sides of (2.2) with PG^{-1}, QG^{-1} respectively, we obtain

$$\begin{cases} (Px)' = (P' + PG^{-1}\overline{A})Px + PG^{-1}q \\ Qx = QG^{-1}\overline{A}Px + QG^{-1}q. \end{cases}$$

Thus, the system is decomposed into two parts: a differential part and an algebraic one. Hence, it is clear that we need only to address the initial value condition to the differential components. Denote $u = Px$, the differential part becomes

$$u' = (P' + PG^{-1}\overline{A})u + PG^{-1}q \tag{2.6}$$

Multiplying both sides of (3.3) with Q yields $Qu' = QP'u \ \Rightarrow (Qu)' = Q'(Qu)$. Hence, the equation (3.3) has the invariant property in the sense that every solution starting in $\operatorname{Im} P(t_0)$ remains in $\operatorname{Im} P(t)$ for all t.

We consider the homogeneous case $q(t) = 0$ and construct the Cauchy operator generated by (2.4). Let $\Phi_0(t,s)$ denote the Cauchy operator generated by the equation (3.3), i.e.,

$$\begin{cases} \dfrac{d}{dt}\Phi_0(t,s) = \left(P' + PG^{-1}\overline{A}\right)\Phi_0(t,s) \\ \Phi_0(s,s) = I. \end{cases}$$

Then, the Cauchy operator generated by system (2.4) is defined by

$$\begin{cases} E\dfrac{d}{dt}\Phi(t,s) = A\Phi(t,s) \\ P(s)\left(\Phi(s,s) - I\right) = 0 \end{cases}$$

and it can be given as follows:

$$\Phi(t,s) = \left(I + QG^{-1}\overline{A}\right)\Phi_0(t,s)P(s) = \tilde{P}(t)\Phi_0(t,s)P(s),$$

where $\tilde{Q}(t)$ is the canonical operator defined by (2.3) in Lemma 2 and $\tilde{P}(t) = I - \tilde{Q}(t)$ is a projector on $S(t)$ along $\tilde{Q}(t)$. By the arguments used in [6, Section 1.2], the unique solution of the initial value problem for (2.4) with the initial condition

$$P(t_0)\big(x(t_0)-x_0\big)=0, \qquad t \ge t_0, \tag{2.7}$$

can be given by the constant-variation formula

$$x(t)=\Phi_0(t,t_0)P(t_0)x_0+\int_{t_0}^{t}\Phi(t,\rho)PG^{-1}q(\rho)d\rho+QG^{-1}q(t). \tag{2.8}$$

3. Stability under small perturbations

Consider the perturbation under the form $q(t)=B(t)x(t)$ where $B(t)$ is a matrix function. Then, the equation (2.4) becomes

$$E(t)x'(t)=\big(A(t)+B(t)\big)x(t), \qquad t \ge t_0, \tag{3.1}$$

The equation (3.1) is an index -1 tractable if and only if the matrix $\overline{G}=E-(\overline{A}+B)Q$ is invertible. It is easy to see that $\overline{G}=(I-BQG^{-1})G.$ Therefore, the invertibility of G is equivalent to the invertibility of $I-BQG^{-1}$. It is seen by Lemma 2 that the invertibility of $I-BQG^{-1}$ does not depend on the choice of Q.

Definition 3.1

1. The DAE (2.4) is said to be stable if for any $\varepsilon>0$ and $t_1>t_0$, there exists a positive constant $\delta=\delta(\varepsilon)$ such that if $\|\tilde{P}(t_1)x_1\|<\delta$ implies $\|x(t)\|<\varepsilon$ for all $t>t_1$, where $x(\cdot)$ is the solution of (2.4) satisfying $\|\tilde{P}(t_1)\big(x(t_1)-x_1\big)\|=0.$

The DAE (2.4) is uniformly stable if it is stable and the above δ is independent of t_1.

2. The DAE (2.4) is said to be exponential stable if there exist the positive numbers $M>0, \alpha>0$ such that

$$\|x(t)\| \le Me^{-\alpha(t-s)}\|\tilde{P}(s)x(s)\|, \qquad t \ge s, t, s \ge t_0.$$

Following the classical way, we see that exponential stabilily and uniformly stability of differential algebraic equation are characterized in term of its transition operator as the follows:

Theorem 3.2

1. The DAE (2.4) *is uniformly stable if and only if there exist the positive numbers* $M_0>0$ *such that*

$$\|\Phi(t,s)\| \le M_0, \qquad t \ge s, t, s \ge t_0.$$

2. The DAE (2.4) *is exponentially stable if and only if there exist the positive numbers* $M>0,\ \alpha>0$ *such that*

$$\|\Phi(t,s)\| \le Me^{-\alpha(t-s)}, \qquad t \ge s, \quad t,s \ge t_0. \tag{3.2}$$

Proof See [4]. □

For the uniform stability, we have the following result.

Theorem 3.3 *Assume that the equation* (2.4) *is index-1, uniformly stable and satisfies*

1. *The matrices* $I-\tilde{Q}(t)\tilde{G}^{-1}(t)B(t)$ *are invertible* $\forall t_1 \geq t_0$ *and* $\sup_{t\geq t_0}\left\|I-\tilde{Q}(t)\tilde{G}^{-1}(t)B(t)\right\| = c < \infty.$
2. *The integrals* $\int_{t_0}^{\infty}\left\|\tilde{P}(t)\tilde{G}^{-1}(t)B(t)\right\| dt = N < \infty.$

Then, the system (3.1) *is uniformly stable, i.e., there exists a constant* $M_1 > 0$ *such that the solution* x($\cdot$) *of* (3.1) *satisfies*

$$\|x(t)\| \leq M_1\|x(s)\|, \quad \forall t > s \geq t_0.$$

Proof By using the constant-variation formula (2.8), for all $t > s \geq t_0$, we have

$$x(t) = \Phi(t,s)\tilde{P}(s)x_s + \int_s^t \Phi(t,\rho)\tilde{P}\tilde{G}^{-1}B(\rho)x(\rho)d\rho + \tilde{Q}\tilde{G}^{-1}B(t)x(t)$$

$$\Rightarrow \left(I-\tilde{Q}\tilde{G}^{-1}B(t)\right)x(t) = \Phi(t,s)\tilde{P}(s)x_s + \int_s^t \Phi(t,\rho)\tilde{P}\tilde{G}^{-1}B(\rho)x(\rho)d\rho.$$

Therefore,

$$\|x(t)\| \leq \left\|\left(I-\tilde{Q}\tilde{G}^{-1}B(t)\right)^{-1}\right\|\left(\left\|\Phi(t,s)\tilde{P}(s)x_s\right\| + \int_s^t\left\|\Phi(t,\rho)\tilde{P}\tilde{G}^{-1}B(\rho)x(\rho)\right\|d\rho\right). \tag{3.3}$$

By virtue of uniform stability of the equation (2.4), there exists the numbers $M_o > 0$ such that

$$\left\|\Phi(t,s)\tilde{P}(s)\right\| = \|\Phi(t,s)\| \leq M_0, \quad t \geq s \geq t_0.$$

Since $\sup_{t\geq t_0}\left\|I-\tilde{Q}(t)\tilde{G}^{-1}(t)B(t)\right\| = c$,

$$\|x(t)\| \leq cM_0\|x_s\| + cM_0\int_s^t\left\|\tilde{P}\tilde{G}^{-1}B(\rho)\right\|\|x(\rho)\|d\rho.$$

By using Gronwall-Bellman inequality, we get

$$\|x(t)\| \leq cM_0\exp\left\{\int_s^t\left\|\tilde{P}\tilde{G}^{-1}B(\rho)\right\|\|x(\rho)\|d\rho\right\} \leq cM_0e^N\|x(s)\|.$$

Put $M_1 = cM_0e^N$, we obtain the proof. □

Theorem 3.4 *If the equation* (2.4) *is index-1, exponential stable and satisfies, the matrices* $I-\tilde{Q}(t)\tilde{G}^{-1}(t)B(t)$ *are invertible,* $\forall t > t_0$ *and*

$$\sup_{t\geq t_0}\left\|I-\tilde{Q}(t)\tilde{G}^{-1}(t)B(t)\right\| = c < \infty, \qquad \limsup_{t\to\infty}\left\|\tilde{P}(t)\tilde{G}^{-1}(t)B(t)\right\| = \delta < \frac{\alpha}{cM},$$

where α, M *is defined by Definition 1. Then, there exist constants K >0 and* α_1 *such that*

$$\|x(t)\| \leq Ke^{-\alpha_1(t-s)}\|x(s)\|, \quad \forall t \geq s \geq t_0.$$

for every solution x(·) of (3.1). *That is, the perturbed equation* (3.1) *preserves the exponential stability.*

Proof. Let ε_0 be a positive number such that $\varepsilon_0 < \dfrac{\alpha - \delta cM}{cM}$. Then, follow the second assumption, there exists $T_0 > t$ such that

$$\left\|\tilde{P}(t)\tilde{G}^{-1}(t)B(t)\right\| = \delta + \varepsilon_0, \quad \forall t > T_0. \tag{3.4}$$

By the continuity of the solutions of (3.1) on the initial condition we can find a constant M_{T_0} (where M_{T_0} depends only on T_0) such that

$$\|x(t)\| \le M_{T_0}\|x(s)\|, \quad \text{for all} \quad t_0 \le s \le t \le T_0. \tag{3.5}$$

First, we consider the case $t > T_0 > s \ge t_0$. Then, follow the estimations (3.3), (3.4) and (3.5), we get

$$\begin{aligned}
\|x(t)\| &\le \left\|\left(I - \tilde{Q}\tilde{G}^{-1}B(t)\right)^{-1}\right\|\left(\left\|\Phi(t,T_0)\tilde{P}(T_0)x_{T_0}\right\| + \int_{T_0}^{t}\left\|\Phi(t,\rho)\tilde{P}\tilde{G}^{-1}B(\rho)x(\rho)\right\|d\rho\right)\\
&\le cM\left(e^{-\alpha(t-T_0)}\|x(T_0)\| + \int_{T_0}^{t} e^{-\alpha(t-\rho)}\left\|\tilde{P}\tilde{G}^{-1}B(\rho)\right\|\|x(\rho)\|d\rho\right)\\
&\le cMe^{-\alpha(t-T_0)}\|x(T_0)\| + cM\left(\delta+\varepsilon_0\right)\int_{T_0}^{t} e^{-\alpha(t-\rho)}\|x(\rho)\|d\rho.
\end{aligned}$$

Multiplying both sides of the above inequality with $e^{\alpha t}$ yields

$$e^{\alpha t}\|x(t)\| \le cMe^{\alpha T_0}\left\|x_{T_0}\right\| + cM\left(\delta+\varepsilon_0\right)\int_{T_0}^{t} e^{\alpha\rho}\|x(\rho)\|d\rho.$$

By using Gronwall-Bellman inequality, we obtain

$$\|x(t)\| \le cMe^{\alpha T_0}\|x(T_0)\|e^{cM(\delta+\varepsilon_0)(t-T_0)}.$$

Therefore,

$$\|x(t)\| \le cMe^{\alpha T_0}\|x(T_0)\|e^{c\left(M(\delta+\varepsilon_0)-\alpha\right)(t-T_0)}.$$

Paying attention on (3.5) obtains

$$\|x(t)\| \le cMe^{\alpha T_0}M_{T_0}\|x(s)\|e^{c\left(M(\delta+\varepsilon_0)-\alpha\right)(t-T_0)}.$$

Thus,

$$\|x(t)\| \le cMe^{\alpha T_0}M_{T_0}\|x(s)\|e^{-\alpha_1(t-T_0)} = K_1\|x(s)\|e^{-\alpha_1(t-T_0)},$$

where $K_1 = cMe^{\alpha T_0}M_{T_0}$ and $-\alpha_1 = cM(\delta+\varepsilon_0) - \alpha < 0$.

In the case $t > s \ge T_0$, it follows from the estimate $\left\|\tilde{P}(t)\tilde{G}^{-1}(t)B(t)\right\| \le \delta + \varepsilon_0$ holds for all $\rho \ge s$. Similarly, we have

$$\|x(t)\| \le c\left(Me^{-\alpha(t-s)}\|x(s)\| + \int_{s}^{t} Me^{-\alpha(t-\rho)}\left\|\tilde{P}\tilde{G}^{-1}B(\rho)\right\|\|x(\rho)\|d\rho\right),$$

and

$$e^{\alpha t}\|x(t)\| \le cMe^{\alpha s}\|x(s)\| + cM(\delta+\varepsilon_0)\int_s^t e^{\alpha\rho}\|x(\rho)\|d\rho,$$

which implies

$$e^{\alpha t}\|x(t)\| \le \ cMe^{\alpha s}\|x(s)\|e^{(cM(\delta+\varepsilon_0))(t-s)},$$

$$\Rightarrow\ \ \|x(t)\| \ \le\ cMe^{\alpha s}\|x(s)\|e^{(cM(\delta+\varepsilon_0)-\alpha)(t-s)} = cMe^{-\alpha_1(t-s)}\|x(s)\|.$$

For the remaining case $t_0 \le s \le t \le T_0$, with $\alpha_1 > 0$ defined above, we have

$$\|x(t)\| \le M_{T_0}\|x(s)\| = M_{T_0}e^{T_0}e^{-T_0}\|x(s)\| \le M_{T_0}e^{T_0}e^{-(t-s)}\|x(s)\|.$$

Put $K = \max\left\{K_1, cM, M_{T_0}e^{T_0}\right\}$, we get $\|x(t)\| \le Ke^{-(t-s)}\|x(s)\|$. The proof is completed. □

4. Bohl-Perron Theorem for differential algebraic equations

The main aim of this section is to prove the Bohl-Perron's Theorem for linear differential algebraic equation. That is we investigate the relation between the exponential stability of DAE (1.1) and the boundedness of solutions of nonhomogeneous equation (2.4).

In solving the equation (2.4) we see that the function q is split into two components $PG^{-1}q$ and $QG^{-1}q$. Therefore, we consider q as an element of

$$L(t_0) = \left\{ q \in C\left([0,\infty], \mathbb{R}^d\right) :\ \sup_{t\ge t_0}\left\|\tilde{Q}(t)\tilde{G}^{-1}(t)q(t)\right\| < \infty \ \text{ and } \sup_{t\ge t_0}\left\|\tilde{P}(t)\tilde{G}^{-1}(t)q(t)\right\| < \infty \right\}.$$

It is easy to see that $L(t_0)$ is a Banach space with the norm

$$\|q\| = \sup_{t\ge t_0}\left(\left\|\tilde{Q}(t)\tilde{G}^{-1}(t)q(t)\right\| + \ \left\|\tilde{P}(t)\tilde{G}^{-1}(t)q(t)\right\|\right).$$

Lemma 4.1 *If for every function $q(.) \in L(t_0)$, the solution $x(.,t_0)$ of the Cauchy problem* (2.4) *with the initial condition $P(t_0)x(t_0) = 0$ is bounded, then there is a constant k such that for all $t \ge t_0$,*

$$\sup_{t\ge t_0}\|x(t,t_0)\| \le k\|q\|. \tag{4.1}$$

Proof By assumption, for any $q(.) \in L(t_0)$, the solution $x(t)$ associated to q of the Cauchy problem (2.4) with the initial condition $P(t_0)x(t_0) = 0$ is bounded on $[t_0,\infty)$. Therefore, if we define a family of operators $\{V_t\}_{t\ge t_0}$ as following:

$$V_t:\ \mathrm{L}(t_0) \ \to\ \mathbb{R}_d$$

$$\mathrm{q} \quad \mapsto\ V_t(q) = x(t,t_0).$$

From the assumption of Lemma, we have $\sup_{t\ge t_0}\|V_t q\| \le \infty$ for any $q \in L(t_0)$. Using Uniform Boundedness Principle, there exists a constant k >0, independent of t such that

$$\sup_{t\ge t_0}\|x(t,t_0)\| = k\|V_t q\| \le k\|q\| \ \text{ for all } \ t \ge t_0.$$

The lemma is proved. □

Lemma 4.2 *Assume that the solution of the Cauchy problem* (2.4) *associated with every q in* $L(t_0)$ *and the initial condition* $P(t_0)x(t_0)=0$ *is bounded. Let* $t_1 \geq t_0$, *then there exists a constant k such that*

$$\sup_{t\geq t_1}\|x(t,t_1)\| \leq k\|\tilde{q}\| \quad \text{for all } t \geq t_1.$$

where $x(t,t_1)$ *is the solution of* (2.4) *associated with q e in* $L(t_1)$ *and the initial condition* $P(t_1)x(t_1)=0$.

Proof Let $\tilde{q}$ be arbitrary function in $L(t_1)$. By variation of constants formula, the solution of the Cauchy problem (2.4) with the initial condition $P(t_1)x(t_1)=0$ corresponding to $\tilde{q}$ is of the form

$$x(t,t_1)=\int_{t_1}^{t}\Phi(t,\rho)\tilde{P}\tilde{G}^{-1}\tilde{q}(\rho)d\rho+\tilde{Q}\tilde{G}^{-1}\tilde{q}(t). \tag{4.2}$$

Define $\overline{q}$ in $L(t_0)$ as follow: if $t<t_1$ then $\overline{q}(t)=0$, else $\overline{q}(t)=\tilde{q}(t)$. It is easy to see that the funtion

$$z(t)=x(t,t_1) \ \text{ if } \ t\geq t_1, \qquad z(t)=0 \ \text{ if } \ t<t_1,$$

is the solution of the Cauchy problem (2.4) associated with $q=\overline{q}$ and initial condition $P(t_0)x(t_0)=0$. By Lemma 3,

$$\sup_{t\geq t_1}\|x(t,t_1)\|=\sup_{t\geq t_0}\|z(t)\|\leq k\|\overline{q}\|=k\|\tilde{q}\| \ .$$

The proof is complete. □

Theorem 4.3 *All the solutions of the Cauchy problem (2.4) with the initial condition* $P(t_0)x(t_0)=0$, *associated with an arbitrary q in* $L(t_0)$ *are bounded, if and only if the index-1 DAE (1.1) is exponent-tially stable.*

Proof The proof contains two parts.

Necessity. First, we prove that if all the solutions of the equation (2.4) with the initial condition $P(t_0)x(t_0)=0,$ associated with q in $L(t_0)$, are bounded then the DAE(1.1) is exponentially stable. With an arbitrary $t_1\geq t_0$, let $\chi(t)=\|\Phi(t,t_1)\|,\ t\geq t_1$ and $t\geq x_{t_1}\in\mathbb{R}^d$ such that $P(t_1)x(t_1)=0$. Then, for any $a\in\mathbb{R}^d$, we consider the function

$$q(t)=\frac{E(t)\Phi(t,t_1)a}{\chi(t)},\ t\geq t_1.$$

It is obvious that

$$\left\|\tilde{P}(t)\tilde{G}^{-1}(t)q(t)\right\|=\left\|\tilde{P}(t)\tilde{G}^{-1}(t)\frac{E(t)\Phi(t,t_1)a}{\chi(t)}\right\|=\left\|\tilde{P}(t)\frac{\Phi(t,t_1)a}{\chi(t)}\right\|\leq\|a\|,$$

$$\left\|\tilde{Q}(t)\tilde{G}^{-1}(t)q(t)\right\|=\left\|\tilde{Q}(t)\tilde{G}^{-1}(t)\frac{E(t)\Phi(t,t_1)a}{\chi(t)}\right\|=0.$$

Thus, $q\in L(t_0)$ and $\|q\|=\sup_{t\geq t_0}\left(\left\|\tilde{Q}(t)\tilde{G}^{-1}(t)q(t)\right\|+\ \left\|\tilde{P}(t)\tilde{G}^{-1}(t)q(t)\right\|\right)\leq\|a\|.$ Moreover,

$$x(t,t_1) = \Phi(t,t_1)\tilde{P}(t_1)x_{t_1} + \int_{t_1}^{t} \Phi(t,\tau)\tilde{P}\tilde{G}^{-1}\tilde{q}(\tau)d\tau + \tilde{Q}(t)\tilde{G}^{-1}(t)\tilde{q}(t)$$

$$= \int_{t_1}^{t} \Phi(t,\tau)\tilde{P}(\tau)\frac{\Phi(\tau,t_1)a}{\chi(\tau)}d\tau = \int_{t_1}^{t} \frac{\Phi(t,t_1)a}{\chi(\tau)}d\tau .$$

Put $\Psi(t) = \int_{t_1}^{t} \frac{1}{\chi(\tau)}d\tau$, we have

$$x(t) = \Phi(t,t_1)\Psi(t)a. \tag{4.3}$$

From Lemma 3, we obtain $\|x(t)\| = \|\Phi(t,t_1)\Psi(t)a\| = \|\Phi(t,t_1)a\|\Psi(t) \le k\|q\| \le k\|a\|$, which implies $\|\Phi(t,t_1)\| \le \frac{k}{\Psi(t)}$. Thus,

$$\Psi'(t) \ge \frac{\Psi(t)}{k} \quad \text{or} \quad \Psi(t) \le k\Psi'(t).$$

Then, with a fixed number c such that $c \ge t_1$, we have $\Psi(t) \le \Psi(c)e^{\frac{1}{k}(t-c)}$. Therefore,

$$\|\Phi(t,t_1)\| \le \frac{k}{\Psi(c)}e^{-\frac{1}{k}(t-c)} = \frac{ke^{-\frac{1}{k}(c-t_1)}}{\Psi(c)}e^{-\frac{1}{k}(t-t_1)}.$$

Setting $\alpha = \frac{1}{k}$, $N_1 = \frac{ke^{-\frac{1}{k}(c-t_1)}}{\Psi(c)}$ and $K_1 = \max\left\{N_1, \max_{t_1 \le t \le c}\frac{\|\Phi(t,t_1)\|}{e^{-\alpha(t-t_1)}}\right\}$, we obtain the estimate

$$\|\Phi(t,t_1)\| \le K_1 e^{-\alpha(t-t_1)}, \quad \text{for all} \quad t \ge t_1.$$

Sufficiency. To complete the proof, we will show that if (1.1) is exponentially stable then all solutions of the Cauchy problem (2.4) with the initial condition $P(t_0)x(t_0) = 0$, associated with q in $L(t_0)$ are bounded. Let $q \in L(t_0)$, suppose that $\sup_{t\ge t_0}\left\|\tilde{P}(t)\tilde{G}^{-1}(t)q(t)\right\| = C_1$, $\sup_{t\ge t_0}\left\|\tilde{Q}(t)\tilde{G}^{-1}(t)q(t)\right\| = C_2$. Using again the formula (2.8) we have

$$\|x(t)\| \le \int_{t_0}^{t}\left\|\Phi(t,\rho)e^{-\alpha(t-\rho)}\tilde{P}\tilde{G}^{-1}q(\rho)\right\|d\rho + \left\|\tilde{Q}\tilde{G}^{-1}q(t)\right\| \le MC_1\left(1-e^{-\alpha(t-t_0)}\right) + C_2.$$

Thus, the solutions of (2.4) associated with q are bounded. The proof is complete. □

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