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Original Article

Convolution for The Offset Linear Canonical Transform with Gaussian Weight and Its Application

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Abstract: This paper presents the convolution for the offset linear canonical transform (OLCT) with the Gaussian weight and its applications. The product theorem is also studied. In applications, some ways to design the filters in the OLCT domain as well as the multiplicative filter and the Gaussian filter are introduced.

Keywords: Reconstruction, Shannon theorem, convolution, filter, signal, offset linear canonical transform, fractional Fourier transform, Fourier transform.

1. Introduction

Throughout this paper we shall consider parameters $a,b,c,d,u_0,\omega_0 \in \mathbb{R}$ and i will be denoted the unit imaginary number. The Offset Linear Canonical Transform (OLCT) (see [1]) of a signal $f(t)$ with real parameters $A=(a,b,c,d,u_0,\omega_0)$, $(ad-bc=1)$ is defined as

$$F_A(u) := \mathbb{O}_A\{f(t)\}(u) := \begin{cases} \int_{\mathbb{R}} f(t)\mathcal{K}_A(u,t)dt, & b \neq 0 \\ \sqrt{d}\, e^{i\frac{cd}{2}(u-u_0)^2+i\omega_0 u} f\left(d(u-u_0)\right), & b=0 \end{cases}, \tag{1.1}$$

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where $\mathcal{K}_A(u,t) := K_A e^{i\left(\frac{d}{2b}u^2 - \frac{1}{b}tu + \frac{a}{2b}t^2 + \frac{(b\omega_0 - du_0)}{b}u + \frac{u_0}{b}t\right)}$, and $K_A = \frac{e^{\frac{idu_0^2}{2b}}}{\sqrt{2\pi bi}}$.

The inverse OLCT expression is given by

$$f(t) = \mathbb{O}_{A^{-1}}\{F_A(u)\}(t) = C\int_{\mathbb{R}} F_A(u)\mathcal{K}_{A^{-1}}(u,t)du, \tag{1.2}$$

where $A^{-1} = (d, -b, -c, a, b\omega_0 - du_0, cu_0 - a\omega_0)$, and

$$C = e^{i\frac{1}{2}\left(cdu_0^2 - 2adu_0\omega_0 + ab\omega_0^2\right)}. \tag{1.3}$$

In this paper, we only consider $b \neq 0$ since the OLCT becomes a chirp multiplication operation otherwise.

The OLCT is generalization of many operations, as follows: the Linear Canonical Transform (LCT), the Fractional Fourier Transform (FRFT), the Fourier Transform (FT). When $u_0 = \omega_0 = 0$, we back to the definition of the Linear Canonical Transform (see [2]).

The Fractional Fourier Transform (FRFT) (see [3]) is considered a special case of the OLCT when parameters A have the form $A = (\cos\theta, \sin\theta, -\sin\theta, \cos\theta, 0, 0)$. For any real angle θ, the FRFT is defined as

$$(\mathcal{F}_\theta f)(u) = \sqrt{\frac{1 - i\cot\theta}{2\pi}}\int_{\mathbb{R}} f(t)e^{i\left(\frac{\cot\theta}{2}u^2 - \frac{ut}{\sin\theta} + \frac{\cot\theta}{2}t^2\right)}dt, \qquad \sin\theta \neq 0. \tag{1.4}$$

When the angle $\theta = \frac{\pi}{2}$, the FRFT becomes the Fourier Transform (FT) (see [4]). In this paper, we will use the Fourier Transform and its inverse defined by

$$\Psi_{FT}(f(t))(u) := \int_{\mathbb{R}} f(t)e^{-iut}dt, \tag{1.5}$$

$$f(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \Psi_{FT}(f(t))(u)\cdot e^{iut}du, \tag{1.6}$$

respectively. If $f, h \in L^1(\mathbb{R})$, the classic Fourier convolution operation in the time domain is defined as

$$(f * h)(t) = \int_{\mathbb{R}} f(\tau)h(t-\tau)d\tau. \tag{1.7}$$

It is easy to see that

$$(f * h)(\lambda t) = \lambda f(\lambda t) * h(\lambda t), \quad \forall \lambda \in \mathbb{R}, \tag{1.8}$$

and

$$\Psi_{FT}\{(f * h)(t)\}(u) = \Psi_{FT}\{f(t)\}(u)\cdot\Psi_{FT}\{h(t)\}(u). \tag{1.9}$$

- We also have the Young's inequality (see [5]). If $f \in L^p(\mathbb{R})$, $h \in L^q(\mathbb{R})$, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, $(p, q, r \geq 1)$. Then the following inequality holds

$$\|f * h\|_r \leq C_1\|f\|_p \cdot \|h\|_q, \tag{1.10}$$

where C_1 is a positive constant.

Now we will exemplify some basic properties of $\mathbb{O}_A$ (see [6]).

Suppose $f \in L^1(\mathbb{R})$, and $\tau, \eta \in \mathbb{R}$, we have

- Time shift:

$$\mathbb{O}_A\{f(t-\tau)\} = e^{-i\left(\frac{ac}{2}\tau^2 - c\tau(u-u_0) - a\tau\omega_0\right)} F_A(u-a\tau).$$

- Modulation:

$$\mathbb{O}_A\{e^{i\eta t} f(t)\}(u) = e^{-i\left(\frac{bd}{2}\eta^2 - d\eta(u-u_0) - b\eta\omega_0\right)} F_A(u-b\eta).$$

- Time shift/modulation:

$$\mathbb{O}_A\{e^{\eta y} f(t-\tau)\} = e^{-i\left(\frac{ac}{2}\tau^2 + bc\tau\eta + \frac{bd}{2}\eta^2\right)} e^{i(c\tau+d\eta)(u-u_0)} e^{i(a\tau+b\eta)\omega_0} F_A(u-a\tau-b\eta).$$

- $\mathbb{O}_A$ is a linear, continuous and one-to-one map from the Schwartz space $\mathcal{S}$ onto $\mathcal{S}$ (whose inverse is obviously also continuous).

Let $C_0(\mathbb{R})$ be the Banach space of all continuous functions on $\mathbb{R}$ that vanish at infinity and being endowed with the supremum norm $\|\cdot\|_\infty$, and let $\|f\|_1 := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} |f(t)|dt$ be the norm in $L^1(\mathbb{R})$.

- (Riemann-Lebesgue type lemma for the OLCT). If $f \in L^1(\mathbb{R})$, then $\mathbb{O}_A f \in C_0(\mathbb{R})$, and

$$\|\mathbb{O}_A f\|_\infty \le \frac{1}{\sqrt{|b|}}\|f\|_1.$$

- (Plancherel type theorem for the OLCT). Let f be a complex-valued function in the space $L^2(\mathbb{R})$ and let

$$\mathbb{O}_A f(u,k) := \int_{|t|<k} \mathcal{K}_A(u,t) f(t) dt.$$

Then, as $k \to \infty$, $\mathbb{O}_A f(u,k)$ converges strongly (over $\mathbb{R}$) to a function, say $\mathbb{O}_A f \in L^2(\mathbb{R})$, and, reciprocally,

$$f(u,k) := C\int_{|t|<k} \mathcal{K}_{A^{-1}}(u,t) \mathbb{O}_A f(t) dt$$

converges strongly to $f(u)$, where C is the same as in (1.3).

- (Parseval type identity for the OLCT). For any $f, h \in L^2(\mathbb{R})$ the following identity holds

$$\langle \mathbb{O}_A f, \mathbb{O}_A h\rangle = \langle f, h\rangle,$$

where $\langle\cdot,\cdot\rangle$ is denoting the usual inner product in $L^2(\mathbb{R})$. In the special case when $h = f$, it holds

$$\|\mathbb{O}_A f\|_2 = \|f\|_2.$$

For convenience, we denote $\kappa = (du_0 - b\omega_0)(2-\sqrt{2})$, $\mathcal{E}_A(t) = e^{i\left(\frac{a}{2b}t^2 + \frac{u_0}{b}t\right)}$, $\overline{f}(t) = \mathcal{E}_A(t) f(t)$, and the Gaussian function $\mathcal{G}(t) = \frac{1}{b\sqrt{\pi}} e^{-\frac{1}{2b^2}(t-\kappa)^2}$. The OLCT (1.1) becomes

$$F_A(u) = \mathbb{O}_A\{f(t)\}(u) = K_A e^{i\left(\frac{d}{2b}u^2 + \frac{(b\omega_0 - du_0)}{b}u\right)} \int_{\mathbb{R}} \overline{f}(t) e^{-\frac{iut}{b}} dt. \tag{1.11}$$

There are many different types of convolutions for the OLCT. Most of them have the weight functions in the form $e^{i(\lambda u^2 + \mu u)}$ (see [1]). In [7], some convolutions for the FRFT with the Hermite weights in the form $e^{i\lambda u^2}\psi_n(u)$, and the Gaussian weight in the form $e^{i\lambda u^2}e^{-\frac{1}{2}u^2}$, are also obtained. In this paper, we focus on studying the convolution for the OLCT with the Gaussian weight in the form $e^{-\frac{1}{2}u^2}$, and its applications.

The paper is divided into two sections and organized as follows. In the next section, we provide the convolution for the OLCT with the Gaussian weight function and study its product theorem. Some special cases of this convolution are also deduced.

2. Convolution for the OLCT with the Gaussian weight function and product theorem

Definition 2.1. Let $f, h \in L^1(\mathbb{R})$, the convolution for the OLCT of two signals $f(t)$ and $h(t)$ with the Gaussian weight function $\mathcal{G}(t)$ is defined by

$$\left(f \underset{\mathcal{G}}{\otimes} h\right)(t) = K_A(\mathcal{E}_A(t))^{-1}\left(\overline{f} * \overline{h} * \mathcal{G}\right)\left(\sqrt{2}t\right). \tag{2.1}$$

It easily seen that if $f, h \in L^1(\mathbb{R})$ then $\left(f \underset{\mathcal{G}}{\otimes} h\right)(t) \in L^1(\mathbb{R})$. Moreover, $\left\| f \underset{\mathcal{G}}{\otimes} h \right\|_1 \le C_2 \|f\|_1 \cdot \|h\|_1$, where C_2 is a positive constant.

Theorem 2.1. Assume that $f, h \in L^1(\mathbb{R})$, $z(t) = \left(f \underset{\mathcal{G}}{\otimes} h\right)(t)$ and $F_A(u)$, $Z_A(u)$, $H_A(u)$ denote the OLCT of the signals $f(t)$, $z(t)$, $h(t)$ with a set of parameters A, respectively. The factorization following identity is fulfilled

$$Z_A(u) = e^{-\frac{1}{4}u^2} \cdot F_A\left(\frac{u}{\sqrt{2}}\right) \cdot H_A\left(\frac{u}{\sqrt{2}}\right).$$

Moreover, if $e^{-\frac{1}{4}u^2} \cdot F_A\left(\frac{u}{\sqrt{2}}\right) \cdot H_A\left(\frac{u}{\sqrt{2}}\right) \in \mathbb{O}_A\left(L^1(\mathbb{R})\right)$ then

$$z(t) = \mathbb{O}_{A^{-1}}\left\{e^{-\frac{1}{4}u^2} \cdot F_A\left(\frac{u}{\sqrt{2}}\right) \cdot H_A\left(\frac{u}{\sqrt{2}}\right)\right\}(t). \tag{2.2}$$

Proof. Based on classic Fourier convolution (1.7), the convolution (2.1) can be expressed as

$$\left(f \underset{\mathcal{G}}{\otimes} h\right)(t) = K_A(\mathcal{E}_A(t))^{-1} \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{f}(u)\overline{h}(v)\mathcal{G}\left(\sqrt{2}t - u - v\right) du dv. \tag{2.3}$$

Since (1.11), we realize that

$$e^{-\frac{1}{2}u^2} \cdot F_A(u) \cdot H_A(u) = e^{-\frac{1}{2}u^2} K_A^2 e^{i\left(\frac{d}{b}u^2 + \frac{2(b\omega_0 - du_0)}{b}u\right)} \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{f}(\tau)\overline{h}(v) e^{-\frac{iu\tau}{b}} e^{-\frac{iuv}{b}} d\tau dv$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}t^2 + iut} dt K_A^2 e^{i\left(\frac{d}{b}u^2 + \frac{2(b\omega_0 - du_0)}{b}u\right)} \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{f}(\tau)\overline{h}(v) e^{-\frac{iu\tau}{b}} e^{-\frac{iuv}{b}} d\tau dv$$

$$= \frac{K_A^2}{\sqrt{2\pi}} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\left(\frac{d}{b}u^2 - \frac{1}{b}(\tau + v + \kappa - bt)u + \frac{(b\omega_0 - du_0)}{b}(\sqrt{2}u)\right)} \overline{f}(\tau)\overline{h}(v) e^{-\frac{1}{2}t^2} d\tau dv dt .$$

By making $s = \tau + v + \kappa - bt$, we obtain

$$e^{-\frac{1}{2}u^2} F_A(u) H_A(u)$$

$$= K_A \int_{\mathbb{R}} e^{i\left(\frac{d}{2b}(u\sqrt{2})^2 - \frac{1}{b}(u\sqrt{2})\left(\frac{s}{\sqrt{2}}\right) + \frac{(b\omega_0 - du_0)}{b}(\sqrt{2}u)\right)} \mathcal{E}_A\left(\frac{s}{\sqrt{2}}\right) \left\{ \frac{K_A\left(\mathcal{E}_A\left(\frac{s}{\sqrt{2}}\right)\right)^{-1}}{b\sqrt{2\pi}} \times \right.$$

$$\left. \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{f}(\tau)\overline{h}(v) e^{-\frac{1}{2b^2}(s - \tau - v - \kappa)^2} d\tau dv \right\} ds$$

$$= \int_{\mathbb{R}} \mathcal{K}_A\left(u\sqrt{2}, \frac{s}{\sqrt{2}}\right) \left\{ \frac{K_A\left(\mathcal{E}_A\left(\frac{s}{\sqrt{2}}\right)\right)^{-1}}{b\sqrt{\pi}} \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{f}(\tau)\overline{h}(v)\mathcal{G}(s - \tau - v) d\tau dv \right\} d\left(\frac{s}{\sqrt{2}}\right)$$

$$= \int_{\mathbb{R}} \mathcal{K}_A\left(u\sqrt{2}, \frac{s}{\sqrt{2}}\right) \left(f \underset{\mathcal{G}}{\otimes} h\right)\left(\frac{s}{\sqrt{2}}\right) d\left(\frac{s}{\sqrt{2}}\right)$$

$$= \mathbb{O}_A\left\{\left(f \underset{\mathcal{G}}{\otimes} h\right)\right\}\left(u\sqrt{2}\right).$$

The proof is completed. □

Remark 2.1. Furthermore, using the fomula (1.8) the convolution (2.1) can also be rewritten as

$$\left(f \underset{\mathcal{G}}{\otimes} h\right)(t) = 2K_A\left(\mathcal{E}_A(t)\right)^{-1}\left(\overline{f}\left(\sqrt{2}t\right) * \overline{h}\left(\sqrt{2}t\right) * \mathcal{G}\left(\sqrt{2}t\right)\right). \tag{2.4}$$

Remark 2.2. In particular, if we chosse $\overline{h}(t) = \delta(t)$, where $\delta(t)$ is the Dirac delta function, we then have

$$\left(f \underset{\mathcal{G}}{\otimes} \mathcal{E}_A^{-1}\delta\right)(t) = K_A\left(\mathcal{E}_A(t)\right)^{-1}\left(\overline{f} * \mathcal{G}\right)\left(\sqrt{2}t\right) = \sqrt{2}K_A\left(\mathcal{E}_A(t)\right)^{-1}\left(\overline{f}\left(\sqrt{2}t\right) * \mathcal{G}\left(\sqrt{2}t\right)\right). \tag{2.5}$$

3. Applications

3.1. The Gaussian filter in the OLCT domain. The Gaussian filter is of importance in the signal processing. In this subsection, based on the remark 2, the Gaussian filter in the OLCT domain will introduced.

The output signal $r_{out}(t)$ can be expressed as following

$$r_{out}(t) = \sqrt{2}K_A\left(\mathcal{E}_A(t)\right)^{-1}\left(\overline{r}_{in}\left(\sqrt{2}t\right) * \mathcal{G}\left(\sqrt{2}t\right)\right). \tag{3.1}$$

The method to achieve the multiplicative filter in the OLCT domain through the convolution (2.1) is shown in Fig 1.

In this following example, our objective is using the proposed filters to restore an observed signal $r_{in}(t) = y(t) + n(t)$ where $y(t), n(t)$ denote the desired signal and the additive noise, respectively.

Example 3.1. *Let* $A = \left(-\frac{2}{7}, \frac{1}{7}, -1, -3, 0, 0\right)$, $r_{in}(t) = e^{-\frac{21}{2}t^2} \cdot \sin(1.5t) + e^{i(t+20)^2}$, $y(t) = e^{-\frac{21}{2}t^2} \cdot \sin(1.5t)$, $n(t) = e^{i(t+20)^2}$ *, and the Gaussian function* $\mathcal{G}(t) = \frac{7}{\sqrt{\pi}} e^{-\frac{49}{2}t^2}$.

$$r_{out}(t) = \sqrt{\frac{7}{\pi i}} e^{it^2}\left(\overline{r}_{in}\left(\sqrt{2}t\right) * \mathcal{G}\left(\sqrt{2}t\right)\right).$$

Then the results of Gaussian filter is given in Fig. 2

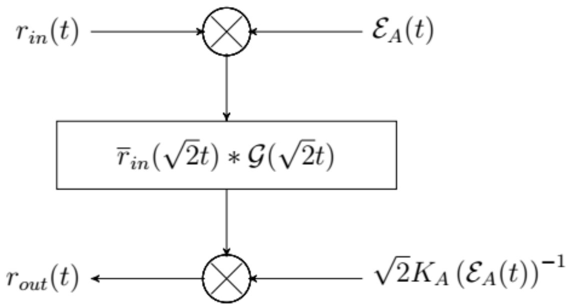

Figure 1. The method to achieve Gaussian filter in the OLCT domain.

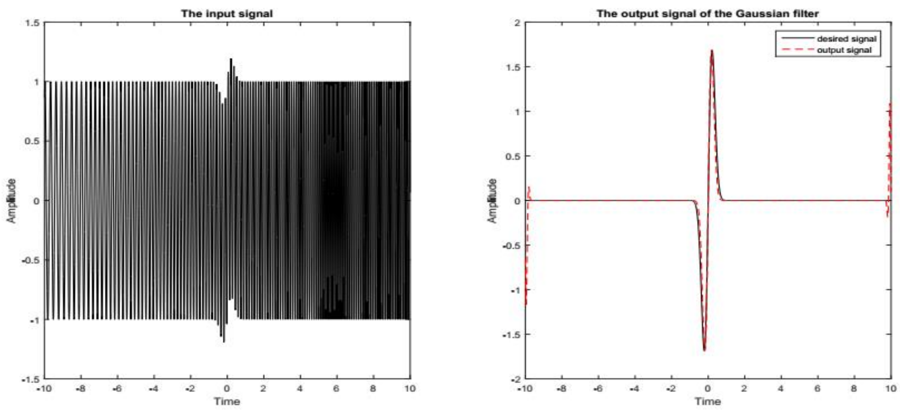

Figure 2. Results of Gaussian filter achieve by using the convolution (2.1).

3.2. The multiplicative filter in the OLCT domain. In this subsection, $r_{in}(t)$ and $r_{out}(t)$ are denoted as the input signal and output signal, respectively.

The output signal of OLCT can be obtained as following

$$r_{out}(t) = \left(r_{in} \underset{\mathcal{G}}{\otimes} h\right)(t) = \mathbb{O}_{A^{-1}}\left\{e^{-\frac{1}{4}u^2} \cdot \mathbb{O}_A\{r_{in}(t)\}\left(\frac{u}{\sqrt{2}}\right) \cdot H_A\left(\frac{u}{\sqrt{2}}\right)\right\}(t). \quad (3.2)$$

Let $\tilde{H}_A(u) = e^{-\frac{1}{2}u^2} H_A(u)$, since the OLCT-frequency spectrum is usually interested only in the region $[u_1, u_2]$, then the filter impulse response $h(t)$ can be selected such that $\tilde{H}_A(u)$ is constant over $[u_1, u_2]$, and zero or rapid decay outside that region. In paticular, we then have

$$r_{out}(t) = \mathbb{O}_{A^{-1}}\left\{\mathbb{O}_A\{r_{in}(t)\}\left(\frac{u}{\sqrt{2}}\right)\right\}(t), \quad u \in \left[u_1\sqrt{2}, u_2\sqrt{2}\right].$$

Moreover, $H_A(u)$ can also be chosen equal the constant over $[u_1, u_2]$, and zero outside that region. Thus, we can get

$$r_{out}(t) = \mathbb{O}_{A^{-1}}\left\{e^{-\frac{1}{4}u^2} \cdot \mathbb{O}_A\{r_{in}(t)\}\left(\frac{u}{\sqrt{2}}\right)\right\}(t), \quad u \in \left[u_1\sqrt{2}, u_2\sqrt{2}\right].$$

By denoting $E(u) = e^{-\frac{1}{4}u^2} \cdot \mathbb{O}_A\{r_{in}(t)\}\left(\frac{u}{\sqrt{2}}\right) \cdot H_A\left(\frac{u}{\sqrt{2}}\right)$, the realization method is given by Fig.3. Therefore, when the OLCT becomes the LCT or the FRFT, it is easy to implement in the designing of multiplicative filters through the product in the OLCT domain (see [2]).

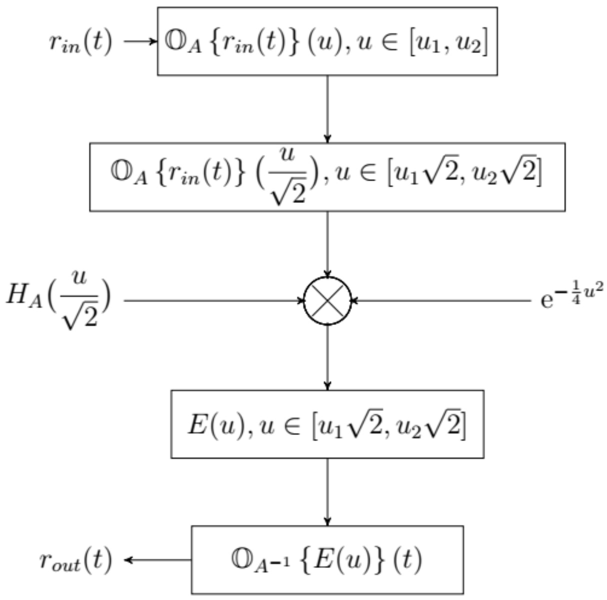

Figure 3. The method to achive multiplicative filter in the OLCT domain.

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