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Original Article

A Note on Infinite Type Germs of a Real Hypersurface in $\mathbb{C}^2$

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Abstract: The purpose of this article is to show that there exists a smooth real hypersurface germ (M, p) of D'Angelo infinite type in $\mathbb{C}^2$ such that it does not admit any (singular) holomorphic curve that has infinite order contact with M at p.

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1. Introduction

Let (M, p) be a germ at p of a real smooth hypersurface in $\mathbb{C}^n$ and let r be a local defining function for M near p. *The normalized order of contact* of the curve γ with M at p is defined by

$$\tau(M,\gamma,p) := \frac{\nu(r\circ\gamma)}{\nu(\gamma)},$$

Where $\gamma(0)=p$ and $\nu(\gamma)$ is the vanishing order of $\gamma(t)-\gamma(0)$ at $t=0$, $\nu(r\circ\gamma)$ is the vanishing order of $r\circ\gamma(t)$ at $t=0$. The *D'Angelo type* of M at p is defined by

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$$\tau(M,p)=\sup_{\gamma}\tau(M,\gamma,p)=\sup_{\gamma}\frac{\nu(r\circ\gamma)}{\nu(\gamma)},$$

where the supremum is taken over all germs $\gamma:\Delta\to\mathbb{C}^n$ of non-constant holomorphic curves with $\gamma(0)=p$. Here and in what follows, $\Delta_\varepsilon=\{z\in\mathbb{C}:|z|<\varepsilon\}$ $(\varepsilon>0)$ and $\Delta:=\Delta_1$. We say that p is of *D'Angelo finite type* if $\tau(M,p)<+\infty$ and of *D'Angelo infinite type* if otherwise.

Throughout the paper, we assume that (M,p) is of D'Angelo infinite type. Then, there exists a sequence of non-constant holomorphic curves γ_n such that $\frac{\nu(r\circ\gamma_n)}{\nu(\gamma_n)}\to+\infty$ as $n\to\infty$. It is natural to ask whether there exists a variety that has infinite order contact with (M,p). This question pertains to the regularity issue of $\bar{\partial}$-Neumann problems over pseudoconvex domains (see [1, 2, 3, 4], and the references therein).

If (M,p) is real-analytic, then by using the ideal theoretic method L. Lempert and J. P. D'Angelo [5, 6] showed that M contains a nontrivial holomorphic curve γ_∞ passing through p. For a germ of a real analytic hypersurface in $\mathbb{C}^3$, we refer the interested reader to [7] for a proof of this result by using a geometric construction.

For the case when (M,p) is a real smooth hypersurface in $\mathbb{C}^n$, J. E. Fornæss, L. Lee and Y. Zhang [8] proved that if $\tau(M,p)=+\infty$, then there exists a formal complex curve in the hypersurface M through p. However, Kang-Tae Kim and V. T. Ninh [9, Proposition 4] asserted independently that there is a formal curve $\varphi(t)=\left(-\sum_{j=1}^{\infty}a_jt^j,t\right)$ which has infinite order contact with M at p for the case $M\subset\mathbb{C}^2$.

In [9], Kang-Tae Kim and V. T. Ninh pointed out that in general there is no such a regular holomorphic curve γ_∞. We ensure that this result still holds even for singular holomorphic curve γ_∞. Namely, our aim is to prove the following theorem.

Theorem 1. There exists a hypersurface germ $(M,0)$ in $\mathbb{C}^2$ with $\tau(M,0)=+\infty$ that does not admit any (singular) holomorphic curve that has infinite order contact with M at 0.

We now briefly sketch the idea of proof of Theorem 1. As in the proof of Example 2 in [9], we construct a certain sequence of smooth functions $\{f_n\}\subset C_0^\infty(\mathbb{C})$ with $\operatorname{supp}(f_n)$ tending to $\{0\}$ such that f_n is harmonic in a sufficiently small disc in $\operatorname{supp}(f_n)$ for each $n\in\mathbb{N}^*$. Moreover, the series $\sum_{n=1}^{\infty}f_n$ converges uniformly on $\mathbb{C}$ to a smooth function $f(z)$. Then the desired hypersurface M can be defined by

$$M = \{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Re}(z_1) + f(z_2) = 0\},$$

which finishes the proof of Theorem 1.

In this paper, we only deal with a smooth real hypersurface in $\mathbb{C}^2$. However, the statement of Theorem 1 remains valid even for higher-dimensional hypersurfaces.

2. Proof of Theorem 1

Proof of Theorem 1. The proof of this theorem proceeds along the same lines as that of Example 2 in [9]. For the convenience of the reader, we shall provide some crucial arguments given in [9]. First of all, let $\{M_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be a sequence of real numbers such that $|M_n| > 2n^{n\gamma_n+2}$, $n \in \mathbb{N}^*$, where $\{\gamma_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$ with $\gamma_n \to +\infty$ as $n \to \infty$. Let $\{\alpha_n\} \subset \mathbb{R}^+$ be a strictly decreasing sequence of positive numbers with $\alpha_n \to 0$ as $n \to \infty$ such that, for each $n \in \mathbb{N}^*$, there exists a holomorphic function g_n on Δ_{α_n} satisfying that $\nu(g_n) = n$ and

$$g_n^{(j)}(0) = \begin{cases} M_n & if\ j = n, \\ 0 & if\ n \nmid j. \end{cases}$$

For instance, for every $n \in \mathbb{N}^*$, we define $g_n(z) := \dfrac{1}{(\alpha_n)^n - z^n} - \dfrac{1}{(\alpha_n)^n}$, where $\alpha_n := \dfrac{1}{M_n^{\frac{1}{2n}}}$ and $M_n > 2n^{n^2+2}$ (see [9, Example 2].

For each $n = 1, 2, \ldots$, denote by $\tilde{f}_n(z)$ the C^∞-smooth function on $\mathbb{C}$ such that

$$\tilde{f}_n(z) = \begin{cases} \mathrm{Re}(g_n(z)) & if\ |z| < \alpha_{n+1}, \\ 0 & if\ |z| > \alpha_n. \end{cases}$$

Then, one can see that and $\nu(\tilde{f}_n) = n$ and

$$\frac{\partial^j \tilde{f}_n}{\partial z^j}(0) = \begin{cases} \dfrac{M_n}{2} & if\ j = n, \\ 0 & if\ n \nmid j. \end{cases} \tag{1}$$

Now let $\{\lambda_n\}$ be an increasing sequence of positive numbers such that

$$\lambda_n \geq \max\left\{1, \left\| \frac{\partial^{k+l} \tilde{f}_n}{\partial z^k \partial \bar{z}^l} \right\|_\infty : k, l \in \mathbb{N}, k + l \leq n \right\},$$

where $\|\cdot\|_\infty$ represents the supremum norm. Let us define a function f_n by setting $f_n(z) := \frac{1}{n^2\lambda_n^n}\tilde{f}_n(\lambda_n z)$ for each $n \in \mathbb{N}^*$. Then, by the repeated use of the chain rule, we obtain

$$\frac{\partial^k f_n}{\partial z^k}(z) = \frac{1}{n^2\lambda_n^{n-k}}\frac{\partial^k \tilde{f}_n}{\partial z^k}(\lambda_n z),\ \mathrm{k} = 0,1,\ldots .$$

This together with (1) implies that

$$\frac{\partial^k f_n}{\partial z^k}(0) = \begin{cases} \frac{M_n}{2n^2} & if\ k = n, \\ 0 & if\ n \nmid k. \end{cases}$$

Let us define a function f by setting $f(z) := \sum_{n=1}^{\infty} f_n(z)$. Then, for every $k, j \in \mathbb{N}$, a direct computation shows that

$$\begin{aligned}\sum_{n=1}^{\infty}\left\|\frac{\partial^{k+l} f_n}{\partial z^k \partial \overline{z}^l}(z)\right\|_\infty &\le \sum_{n=1}^{k+l}\frac{1}{n^2\lambda_n^{n-k-l}}\left\|\frac{\partial^{k+l}\tilde{f}_n}{\partial z^k \partial \overline{z}^l}(z)\right\|_\infty + \sum_{n=k+l+1}^{\infty}\frac{1}{n^2\lambda_n^{n-k-l-1}}\frac{\left\|\frac{\partial^{k+l}\tilde{f}_n}{\partial z^k \partial \overline{z}^l}(z)\right\|_\infty}{\lambda_n} \\ &\le \sum_{n=1}^{k+l}\frac{1}{n^2\lambda_n^{n-k-l}}\left\|\frac{\partial^{k+l}\tilde{f}_n}{\partial z^k \partial \overline{z}^l}(z)\right\|_\infty + \sum_{n=k+l+1}^{\infty}\frac{1}{n^2} \\ &< +\infty.\end{aligned}$$

Hence, this ensures that $f \in C^\infty(\mathbb{C})$.

Next, let us fix a sequence of prime numbers $\{p_n\}_{n=1}^{\infty}$ with $p_n \to +\infty$ as $n \to +\infty$. Then it is easy to see that

$$\begin{aligned}\frac{\partial^{p_n}}{\partial z^{p_n}} f(0) &= \sum_{k=2}^{\infty}\frac{\partial^{p_n}}{\partial z^{p_n}} f_k(0) \\ &= \sum_{j=2}^{p_n-1}\frac{\partial^{p_n}}{\partial z^{p_n}} f_j(0) + \frac{\partial^{p_n}}{\partial z^{p_n}} f_{p_n}(0) + \sum_{j=p_n+1}^{\infty}\frac{\partial^{p_n}}{\partial z^{p_n}} f_j(0) \\ &= \frac{Mp_n}{2p_n^2}.\end{aligned}$$

We now define a hypersurface germ M at $(0,0)$ by setting

$$M = \left\{(z_1, z_2) \in \mathbb{C}^2 : \rho = \mathrm{Re}\, z_1 + f(z_2) = 0\right\}.$$

We shall show that $\tau(M,0)=+\infty$. To do this, for each $N\geq 2$, consider a holomorphic curve $\gamma_N=(z_1,z_2)$ defined on $\left\{t\in\mathbb{C}:|t|<\frac{\alpha_{N+1}}{\lambda_N}\right\}$ by

$$z_1(t)=-\sum_{n=1}^{N}\frac{1}{n^2\lambda_n^n}g_n(\lambda_n t);\ z_2(t)=t.$$

Then, we have $\rho\circ\gamma_N(t)=\sum_{n=N+1}^{\infty}f_n(t)$. Furthermore, since $\nu(f_n)=n$ for $n=1,2,\ldots$, it follows that $\nu(\rho\circ\gamma_N)=N+1$, and hence $\tau(M,0)=+\infty$.

We finally prove that there does not exist a (singular) holomorphic curve $\gamma_\infty:(\mathbb{C},0)\to(\mathbb{C}^2,0)$, such that $\nu(\rho\circ\gamma_\infty)=+\infty$. Note that, by a change of variables, we can assume that such a (singular) holomorphic curve γ_∞ is represented by a parametrization $\gamma_\infty(t)=(h(t),t^m)$ for some positive integer m, where h is a holomorphic function on a neighborhood of the origin in $\mathbb{C}$. Indeed, suppose otherwise that such a holomorphic curve exists. Then $\rho\circ\gamma_\infty(t)=\mathrm{Re}\big(h(t)+f(t^m)\big)=o(t^\infty)$ and thus

$$\begin{aligned}0=\frac{\partial^{p_nm}}{\partial z^{p_nm}}\bigg|_{z=0}\big(\mathrm{Re}\,h(z)+f(z^m)\big)&=\frac{1}{2}\frac{\partial^{p_nm}}{\partial z^{p_nm}}h(0)+\frac{\partial^{p_nm}}{\partial z^{p_nm}}\bigg|_{z=0}f(z^m)\\&=\frac{1}{2}\frac{\partial^{p_nm}}{\partial z^{p_nm}}h(0)+(m!)^{p_n}\frac{\partial^{p_n}}{\partial z^{p_n}}\bigg|_{z=0}f(z)\\&=\frac{1}{2}\frac{\partial^{p_nm}}{\partial z^{p_nm}}h(0)+(m!)^{p_n}\frac{Mp_n}{2p_n^2}.\end{aligned}$$

Consequently, $h^{(p_nm)}(0)=-(m!)^{p_n}\frac{Mp_n}{2p_n^2}$, and moreover, since $\left(\frac{n}{3}\right)^n\leq n!\leq\left(\frac{n+1}{2}\right)^n$ and $|M_n|\geq 2n^{n\gamma_n+2}$ we have

$$\sqrt[p_nm]{\frac{\left|h^{(p_nm)}(0)\right|}{(p_nm)!}}=\sqrt[p_nm]{\frac{(m!)^{p_n}M_{p_n}}{(p_nm)!2p_n^2}}\geq\frac{(m!)^{\frac{1}{m}}p_n^{\frac{\gamma p_n}{m}}}{\frac{p_nm+1}{2}}\geq\frac{2mp_n^{\frac{\gamma p_n}{m}}}{3(p_nm+1)}\geq\frac{1}{3}p_n^{\frac{\gamma p_n}{m}-1}.$$

Therefore, we obtain

$$\limsup_{N\to\infty}\sqrt[N]{\frac{\left|h^N(0)\right|}{N!}}\geq\limsup_{p_n\to\infty}\sqrt[p_nm]{\frac{\left|h^{p_nm}(0)\right|}{(p_nm)!}}=\lim_{p_n\to\infty}\frac{1}{3}p_n^{\frac{\gamma p_n}{m}-1}=+\infty.$$

This implies that the Taylor series of $h(z)$ at 0 has radius of convergence 0, which is absurd since h is holomorphic in a neighborhood of the origin. Hence, the proof is complete.

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