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Original Article

The Expected Number of Extreme Discs

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Abstract: Given a finite set D of n planar discs whose centers are distributed randomly. We are interested in the expected number of extreme discs of the convex hull of D. We show that the expected number of extreme discs is at most $O(\log^2 n)$ for any distribution. This result can be used to derive expected complexity of convex hull algorithms.

Keywords: Convex hull, computational geometry, expected number.

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1. Introduction

Convex hull problem of a finite set of points or discs is one of the most extensively studied and well-understood in computational geometry because of its both theoretical and practical significance. The problem of finding convex hull has been around for about 50 years and its applications have contributed in many different areas such as computer graphics [1], image processing [2, 3], and pattern recognition [4],…. Besides, the convex hull problem is often used as a preprocessing step or as the most important intermediate sub-problem in solving other geometric problems [5] such as Voronoi diagrams constructing, triangulation computing, the farthest pairs problem [6],…. In order to solve the convex hull problem, one usually finds the extreme points or discs, respectively. In this paper we are interested in the number of extreme discs assuming that the centers of the given discs are randomly distributed.

Many algorithms finding the convex hull of a finite set of points have been proposed. It dated back to 1970 for the first publication on convex hull algorithm, which was called Gift-wrapping by Chand

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and Kapur [7]. Graham proposed in 1972 a slightly more sophisticated but much more efficient algorithm named Graham's scan for solving planar convex hull problem [8]. Another famous method for determining convex hull is the Quickhull algorithm, which was discovered independently in 1977 by Eddy [8] and in 1978 by Bykat [9]. The convex hull problem continues being an attractive problem with many other known algorithms such as incremental convex hull algorithm (by Kallay [10]), marriage-before-conquest (by Kirkpatrick and Seidel [11]), Chan's algorithm (by Chan [12]). Some of those algorithms are output-sensitive, i.e., their complexity depends on the number of extreme points. For a set of n finite points the number of extreme points can be as large as n. In 2004, Damerow and Sohler showed that number of extreme points in the average case is O(log$n$) [13]. From this it follows that Gift-wrapping and Quickhull algorithms have the average complexity of O($n$log$n$).

The problem of finding convex hull for a set of discs becomes more challenging. A natural way is to modify the convex hull algorithms for a finite set of points in order to apply them for the case of discs. In 1992, Rappaport proposed an O($n$log$n$) algorithm for solving the convex hull problem for discs applying the idea of the divide-and-conquer algorithm [14]. The monotone chain algorithm, which was published in 1995 by Devillers and Golin [15], can be considered as a modification of the incremental algorithm when the input discs are lexicographically sorted by their radius. In 1998, Chen et al. introduced a parallel method for finding the convex hull of a planar discs [16]. The Quickhull algorithm can also be modified for the case of discs [17]. Similarly to the case of points, the convex hull of a set D of n discs in the plane can be represented in an ordered sequence by a list CH(D) of extreme discs. However, different than the case of points, each disc can contribute more than one arcs to the boundary of the convex hull and hence may appear more than once in CH(D). That means the cardinality of CH(D) may be larger than the number of discs. In this paper, when we write the number of extreme discs we mean the cardinality of CH(D). In [14, 15] the authors show that this number can be at most (2n - 1). The question on the expected number of extreme discs when the centers of discs are randomly distributed has not been addressed and is the topic of our paper.

In this paper we consider a set $D=\{d_i(c_i,r_i),\ i=1,2,...,n\}$ of n planar discs, where $c_i(c_{ix},c_{iy})$ and $r_i\geq 0$ are the corresponding center and radius. Suppose that the centers are given randomly by an one-dimensional probability distribution function Δ. We show that the expected number of extreme discs is at most O(log^{2}n) for any distribution function.

The paper is structured as follows. Section 2 gives some definitions and geometrical notions that will be used in this paper. Section 3 considers the expected number of extreme discs of a disc set. Using this result, we discuss the expected complexity of algorithms computing convex hull of discs in Section 4.

2. Preliminaries

Throughout this paper, we focus on the problem of computing the number of extreme discs of a finite set of planar discs. For convenience of the reader, we recall in this section some necessary definitions.

Definition 1 (see [18]) Let $\mathcal{P}$ be a set of planar points. A point $p\in\mathcal{P}$ satisfying $p\notin \mathrm{conv}(\mathcal{P}\setminus\{p\})$ is called an *extreme point* of the $\mathrm{conv}\mathcal{P}$.

Let $D=\{d_1,d_2,...,d_n\}$ be a set of n discs in the plane with $d_i=(c_i,r_i),\ i=1,2,...,n$, where $c_i(c_{ix},c_{iy})$ and $r_i\geq 0$ are the corresponding center and radius. Let conv$D$ be the convex hull of D, which is the smallest convex region containing all of the discs. Its boundary $\partial\,\mathrm{conv}D$ consists of a

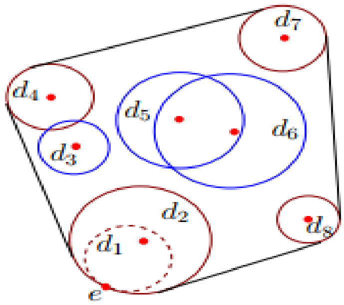

Figure 1. Extreme discs.

sequence of arcs and tangent lines connecting consecutive arcs. Assume that the set D does not have two coincident discs. We will denote by ∂d the boundary of a discs d.

Definition 2 A disc d in D is called an *extreme disc* of conv$D$ if its boundary ∂d passes through an extreme point of conv$D$ and the disc d is not inside another disc in D.

In Figure 1, d_2, d_4, d_7, d_8 are extreme discs. The disc d_1 is not an extreme disc because it lies inside the disc d_2.

The convex hull of D can be represented in different ways. We represent it according to Rappaport's representation [16] storing extreme disks of D in an ordered sequence by a list CH(D), that is, $\mathrm{CH}(D)=\{d_1, d_2, ..., d_h, d_{h+1}\}$, where $d_1 = d_{h+1}$, such that d_t and d_{t+1} contribute two consecutive arcs on the boundary $\partial\,\mathrm{conv}D$ of conv$D$ for $t=1,2,...,h.$. Note that, an extreme disc may appear more than once in CH(D), so the list CH(D) may contain two elements d_i and d_j having different indices $i \neq j$ but they are the same disc $d_i = d_j$. In Figure 2, the set D has seven discs with $\mathrm{CH}(D)=\{d_1, d_4, d_2, d_4, d_7, d_4, d_3, d_1\}$, where d_1, d_2, d_3, d_4 are extreme discs and d_4 appears three times in CH(D).

Note that the number of arcs on the boundary of the convex is equal to the number of extreme discs in CH(D). We also use the phrase “the number of extreme discs of D” to mean “the number of extreme discs in CH(D)”.

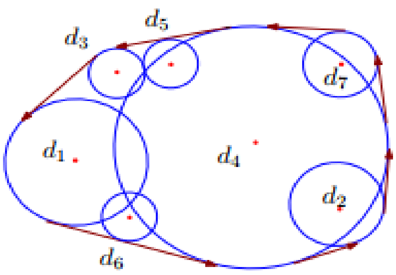

Figure 2. The convex hull of discs.

3. The expected number of extreme discs

In this section we will derive an upper bound on the expected number of extreme discs of D assuming that the centers of discs are randomly given by a probability distribution function Δ. Denote $\mathcal{C}$ as the set of the centers. We will prove that the expected number of extreme discs is at most $\mathrm{O}(\log^2 n)$ for any distribution function Δ.

As we already discussed before, an extreme disc may appear more than once in CH(D). The total number of extreme discs is however bounded by 2n-1.

Lemma 1 (see [14, 15]) Let D be a set of n discs in $\mathbb{R}^2$. Then the number of extreme discs of D is at most 2n-1, that is, $|\mathrm{CH}(D)| \leq 2n-1$.

In order to prove our main result, we need the following two lemmas.

Lemma 2 (see [13]) Let be $\mathcal{P}$ a set of n points in $\mathbb{R}^2$ chosen according to any probability distribution Δ. Then the probability for $p \in \mathcal{P}$ being an extreme point of $\mathcal{P}$ is bounded by the following inequality

$$\mathbb{P}_p^{\mathcal{P}} \leq 4\frac{\log n}{n}.$$

For simplicity of notation, suppose that the discs in D are sorted by decreasing radius with ties being broken arbitrarily $r_1 \geq r_2 \geq ... \geq r_n$. Let D_i be the set of the first i discs and $\mathcal{C}_i$ be the set of centers of discs in D_i. The basic idea of the algorithm in [15] is to construct step by step CH(D_i) for $i = 1, 2, ..., n.$ It is shown in that paper that while going from CH(D_i) to CH(D_{i+1}) the number of arcs of the convex hull increases by at most 2.

Lemma 3 (see [15]) *We have*

$$f(D_{i+1}) \leq f(D_i) + 2,$$

where f(D_i) and f(D_{i+1}) are the number of arcs of convD_i and convD_{i+1} respectively.

Combining the above two lemmas we get our main theorem.

Theorem 1 Let D be the set of n discs with the centers are chosen according to any probability distribution Δ. Then expected number of extreme discs of D is O($\log^2$n).

Proof For simplicity of notation we also assume that the discs in the set D are arranged in non-increasing order of the radius $r_1 \geq r_2 \geq ... \geq r_n$. Let $D_i = \{d_1, d_2, ..., d_i\}$ be the set of first i discs of D, $\mathcal{C} = \{c_1, c_2, ..., c_n\}$ be the set of centers of discs in D_i, and $f(D_i)$ and $\mathbb{E}f(D_i)$ are the number of arcs and expected number of arcs of convD_i, respectively.

The disc d_{i+1} has the smallest radius among all disc in the set D_{i+1}. Therefore, the necessary condition for d_{i+1} to be an extreme disc of D_{i+1} is that its center c_{i+1} must be an extreme point of the set $\mathcal{C}_{i+1}$. According to Lemma 1, the probability for c_{i+1} being an extreme point of the set $\mathcal{C}_{i+1}$ satisfies

$$\mathbb{P}_{c_{i+1}}^{\mathcal{C}_{i+1}} \leq 4\frac{\log(\mathrm{i}+1)}{i+1}.$$

Hence the probability for d_{i+1} being an extreme disc of D_{i+1} is bounded above by

$$\mathbb{P}_{d_{i+1}}^{D_{i+1}} \leq 4\frac{\log(i+1)}{i+1}.$$

According to Lemma 3, by adding the disc d_{i+1} to D_i and calculating convD_{i+1}, the number of arcs increases by at most 2, i.e.,

$$f(D_{i+1}) \leq f(D_i) + 2.$$

Obviously, if d_{i+1} is not an extreme disc of D_{i+1} then the number of arcs of convD_{i+1} is equal to the one of convD_i. Only if d_{i+1} is an extreme disc of D_{i+1}, then the number of arcs of convD_{i+1} may increase compared to the one of convD_i. Therefore we have

$$\mathbb{E}f(D_{i+1}) \leq \mathbb{E}f(D_i) + 2\mathbb{P}_{d_{i+1}}^{D_{i+1}}. \tag{1}$$

Note that $f(D) = f(D_n)$ and $f(D_0) = 0$. Summing both side of the inequality (1) over $i = 1, 2, ..., n-1$ and eliminating the same terms on both side yields

$$\begin{aligned}\mathbb{E}f(D) &\leq 2\sum_{i=0}^{n-1}\mathbb{P}_{d_{i+1}}^{D_{i+1}} \\ &\leq 2\sum_{i=0}^{n-1}4\left(\frac{\log(i+1)}{i+1}\right) \\ &\leq 8\log n\sum_{i=1}^{n}\frac{1}{i} \\ &= \mathrm{O}\left(\log^2 n\right).\end{aligned}$$

Since the number of arcs *f(D)* of conv$D$ is equal to the number of extreme discs in CH(D), our theorem is proven.

4. On the complexity of algorithms computing convex hull of discs

Recall that several convex hull algorithms are output-sensitive, i.e., their computational complexity depends on the number of extreme points. For example, Gift-wrapping algorithm [7] and Quickhull algorithm [19] have worst case complexity of O(nh), while ultimate planar convex hull algorithm [11] and Chan's algorithm [12] have worst case complexity of O($n$log$h$), where n is the number of points in the original set and h is the number of extreme points. Since the expected number of extreme points is O(log$n$) [13], we automatically get the O($n$log$n$) expected complexity of Gift-wrapping algorithm and Quickhull algorithm and O($n$loglog$n$) of the ultimate planar convex hull algorithm and Chan's algorithm.

Similarly, the number of extreme discs of a disc set can be used to evaluate the computational complexity of convex hull algorithms for discs. As it is shown in Section 3 that the expected number of extreme discs is at most O($\log^2 n$), any convex hull algorithm for discs with a worst case complexity of O(nh), where n is the number of discs and h is the number of extreme discs, has the expected computational complexity of at most O($n\log^2 n$). The Quickhull algorithm for discs [17] is an example of algorithms of that type.

5. Conclusion

In this paper we prove that the expected number of extreme discs of a set D of n discs is at most O($\log^2 n$). Consequently, the Quickhull algorithm for discs has an expected complexity of O($n\log^2 n$).

There is still a gap compared to the expected number of O(log*n*) for the case of points and it is a topic of future research.

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