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Original Article

Nonlinear Dynamic Analysis of Rectangular FGM Plates of Variable Thickness Subjected to Mechanical Load

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Abstract: This paper establishes the governing equations of rectangular plates of variable thickness subjected to mechanical load by using the classical plate theory, the geometrical nonlinearity in von Karman-Donnell sense. Solutions to the problem are derived according to Galerkin method. Nonlinear dynamic responses, critical dynamic loads are obtained by using Runge-Kutta method and the Budiansky–Roth criterion. The effect of volume-fraction index k and some geometric factors are considered and numerical results are presented.

Keywords: Dynamic responses, nonlinear vibration, rectangular FGM plate, variable thickness.

1. Introduction

Rectangular FGM plates are used extensively in spacecraft, nuclear reactors or defense industry and in civil engineering, v.v. Today, analysis of vibration and dynamic stability of FGM plate structures has been studied by many authors.

Firstly, for dynamic problems of constant thickness plate structures, Ungbhakorn et al. [1] investigated thermo-elastic vibration of FGM plates with distributed patch mass based on the third-order shear deformation theory and Energy method. Talha et al. [2] analyzed free vibration of FGM plates by using HSDT and finite element method. Duc et al. used the Galerkin method and the higher-order shear deformable plate theory to study the post-buckling of thick symmetric functionally graded plates resting on

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elastic foundations under thermomechanical loads [3] and buckling and post-buckling of thick functionally graded plates subjected to in-plane compressive [4]. Bich et al. [5] examined nonlinear post-buckling of eccentrically stiffened functionally graded plates and shallow shells based on the classical shell theory and the smeared stiffeners technique. Hebali et al. [6] and Mahi et al. [7] studied static and free-vibration of FGM plates under mechanical load based on hyperbolic shear deformation theory. Benferhat et al. [8] used Hamilton's principle and higher-order shear deformation theory to study vibration of FG plates resting on elastic foundation. R. Kandasamy et al. [9] used FSDT and finite element method to investigate free vibration and thermal buckling behavior of moderately thick FGM plates in thermal environments.

Secondly, for dynamic problems of variable thickness plate structures, E. Efraim et al. [10] based on the FSDT to study vibration of variable thickness thick annular isotropic and FGM plates. S. H. Hosseini-Hashemi et al. [11] based on the classical plate theory and differential quadrature method (DQM) to deal with free vibration problem of radially FG circular and annular sectorial thin plates with variable thickness resting on elastic foundations. M. Shariyat and M. M. Alipou [12] studied vibration of variable thickness two-directional FGM circular plates resting on elastic foundations by using power series. V. Tajeddini et al. [13] employed linear elastic theory and Ritz method to investigate 3D free vibration of thick circular FG plates with variable thickness. F. Tornabene et al. [14] examined natural frequencies of FGM sandwich shells with variable thickness by using HSDT and local generalized differential quadrature method. A. H. Sofiyev [15] used Ritz method to study buckling of continuously varying thickness orthotropic composite truncated conical shell under mechanical load. A. R. Akbari and S. A. Ahmadi [16] analyzed buckling of a FG thick cylinder shells with variable thickness under mechanical load by using DQM. P. T. Thang et al. [17] investigated effects of variable thickness and imperfection on nonlinear buckling of S-FGM cylindrical panels subjected to mechanical load based on the classical shell theory and using Galerkin method. These authors also investigated effect of variable thickness on buckling and post-buckling behavior of S-FGM plates resting on elastic medium [18].

In conclusion, according to the above review reveals and author's knowledge, there were many studies on FGM plate and shell structures but has no publication on nonlinear vibration and dynamic stability of FGM rectangular plate with variable thickness under mechanical load. In this paper, we investigate nonlinear vibration and dynamic stability of rectangular plates with variable thickness subjected to mechanical load. The governing equations are established based on the classical plate theory. Nonlinear dynamic responses are obtained by using Galerkin method and Runge-Kutta method. Critical dynamic loads are obtained by using the Budiansky–Roth criterion.

2. Governing equations

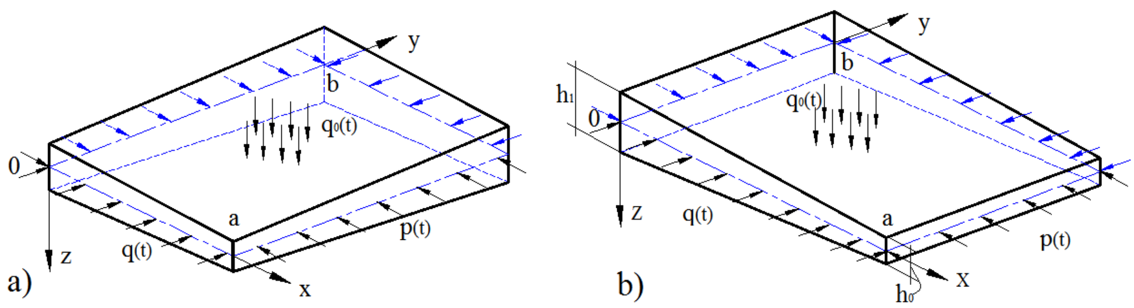

Fig. 1. Configuration of variable thickness FGM plate.

Consider a rectangular FGM plate with variable thickness subjected to mechanical load. The thickness of the plate can be expressed as: h=h(x,y) (Fig. 1a).

Assume that, plate made of FGM with the volume fraction of ceramic $V_c(z)$ changes according to the following rule:

$$V_c = \left(\frac{1}{2} + \frac{z}{h(x,y)}\right)^k \quad (1)$$

With the above rule, the Young's modulus E, Poisson ratio ν of FGM plate can be expressed as:

$$\begin{aligned} E(z) &= E_m.V_m + E_c.V_c = E_m + \left(E_c - E_m\right)\left(\frac{1}{2} + \frac{z}{h(x,y)}\right)^k \\ \rho(z) &= \rho_m.V_m + \rho_c.V_c = \rho_m + \left(\rho_c - \rho_m\right)\left(\frac{1}{2} + \frac{z}{h(x,y)}\right)^k \\ \upsilon(z) &= \upsilon_m.V_m + \upsilon_c.V_c = \upsilon_m + \left(\upsilon_c - \upsilon_m\right)\left(\frac{1}{2} + \frac{z}{h(x,y)}\right)^k \end{aligned} \quad (2)$$

According to [19], the strains at a distance z from the middle surface can be expressed as:

$$\varepsilon_{\text{ij}} = \varepsilon_{\text{ij}}^0 + zk_{\text{ij}} \text{ with } (i\,j = xx, yy, xy)$$

or

$$\varepsilon_{xx} = \varepsilon_{xx}^0 + zk_{xx}, \quad \varepsilon_y = \varepsilon_{yx}^0 + zk_{yy}, \quad \gamma_{xy} = \gamma_{xy}^0 + zk_{xy}, \quad (3)$$

Where: $\varepsilon_{xx}^0;\varepsilon_{yy}^0;\gamma_{xy}^0$, are the strains at the middle surface and $k_{xx};k_{yy}$ are curvatures and k_{xy} is the twist. They are related to the displacement components u, v, w in the x, y, z coordinate directions as:

$$\begin{aligned} &\varepsilon_{xx}^0 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial \text{w}}{\partial x}\right)^2; \varepsilon_{yy}^0 = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial \text{w}}{\partial y}\right)^2; \gamma_{xy}^0 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial \text{w}}{\partial x}\frac{\partial \text{w}}{\partial y} \\ &k_{xx} = -\frac{\partial^2 \text{w}}{\partial x^2}; \qquad k_{yy} = -\frac{\partial^2 \text{w}}{\partial y^2}; \qquad k_{xy} = -2\frac{\partial^2 \text{w}}{\partial x \partial y} \end{aligned} \quad (4)$$

Hooke's law applied to FGM plate under mechanical loads can be expressed as follows

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E_{(z)}}{1-\upsilon_{(z)}^2}\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix}.\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \text{ hay } \{\sigma\} = [\Re].\{\varepsilon\} \quad (5)$$

Integrating the stress-strain equations through the thickness of the plate we obtain the mechanical behavior equations of FGM plate with variable thickness:

$$\begin{Bmatrix} \{N_{ij}\} \\ \{M_{ij}\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} . \begin{Bmatrix} \{\varepsilon^0_{ij}\} \\ \{k_{ij}\} \end{Bmatrix} \text{ với } (i,j)=(xx, yy, xy) \quad (6)$$

In which:

$$\{[A];[B];[D]\} = \int_{-\frac{h(x,y)}{2}}^{\frac{h(x,y)}{2}} [\Re].(1, z, z^2)\, dz \quad (7)$$

The mechanical behavior equations of FGM plate with variable thickness can be rewritten as follows

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\ A_{21} & A_{22} & 0 & B_{21} & B_{22} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 \\ B_{21} & B_{22} & 0 & D_{21} & D_{22} & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D_{66} \end{bmatrix} . \begin{Bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \\ k_{xx} \\ k_{yy} \\ k_{xy} \end{Bmatrix} \quad (8)$$

in which:

$$A_{11} = A_{22} = \int_{-\frac{h(x,y)}{2}}^{\frac{h(x,y)}{2}} \frac{E(z)}{1-\upsilon^2}.dz = \frac{E_1.h(x,y)}{1-\upsilon^2}; A_{12} = A_{21} = \int_{-\frac{h(x,y)}{2}}^{\frac{h(x,y)}{2}} \frac{\upsilon.E(z)}{1-\upsilon^2}.dz = \frac{\upsilon.E_1.h(x,y)}{1-\upsilon^2};$$

$$B_{11} = B_{22} = \int_{-\frac{h(x,y)}{2}}^{\frac{h(x,y)}{2}} \frac{E(z).z}{1-\upsilon^2}.dz = \frac{E_2.h^2(x,y)}{1-\upsilon^2}; B_{12} = B_{21} = \int_{-\frac{h(x,y)}{2}}^{\frac{h(x,y)}{2}} \frac{\upsilon.E(z).z}{1-\upsilon^2}.dz = \frac{\upsilon.E_2.h^2(x,y)}{1-\upsilon^2};$$

$$D_{11} = D_{22} = \int_{-\frac{h(x,y)}{2}}^{\frac{h(x,y)}{2}} \frac{E(z).z^2}{1-\upsilon^2}.dz = \frac{E_3.h^3(x,y)}{1-\upsilon^2}; D_{12} = D_{21} = \int_{-\frac{h(x,y)}{2}}^{\frac{h(x,y)}{2}} \frac{\upsilon.E(z).z^2}{1-\upsilon^2}.dz = \frac{\upsilon.E_3.h^3(x,y)}{1-\upsilon^2};$$

$$A_{66} = \int_{-\frac{h(x,y)}{2}}^{\frac{h(x,y)}{2}} \frac{E(z)}{2(1+\upsilon)}.dz = \frac{E.h(x,y)}{2(1+\upsilon)}; B_{66} = \int_{-\frac{h(x,y)}{2}}^{\frac{h(x,y)}{2}} \frac{E(z).z}{2(1+\upsilon)}.dz = \frac{E_2.h^2(x,y)}{2(1+\upsilon)}$$

$$D_{66} = \int_{-\frac{h(x,y)}{2}}^{\frac{h(x,y)}{2}} \frac{E(z).z^2}{2(1+\upsilon)}.dz = \frac{E_3.h^3(x,y)}{2(1+\upsilon)}$$

where: $E_1 = E_m + \frac{E_c - E_m}{(k+1)}; E_2 = \frac{\left(E_c - E_m\right)k}{2(k+1)(k+2)};\ E_3 = \frac{E_m}{12} + \left(E_c - E_m\right)\left(\frac{1}{k+3} - \frac{1}{k+2} + \frac{1}{4(k+1)}\right)$

Internal force and moment resultants in Eq (8) can be expressed as:

$$\begin{cases} N_{xx} = A_{11}\left(\varepsilon_{xx}^0 + \upsilon.\varepsilon_{yy}^0\right) + B_{11}\left(k_{xx} + \upsilon.k_{yy}\right) \\ N_{yy} = A_{11}\left(\varepsilon_{yy}^0 + \upsilon.\varepsilon_{xx}^0\right) + B_{11}\left(k_{yy} + \upsilon.k_{xx}\right) \\ N_{xy} = A_{66}.\gamma_{xy}^0 + B_{66}.k_{xy} \end{cases} \tag{9}$$

$$\begin{cases} M_{xx} = B_{11}\left(\varepsilon_{xx}^0 + \upsilon.\varepsilon_{yy}^0\right) + D_{11}\left(k_{xx} + \upsilon.k_{yy}\right) \\ M_{yy} = B_{11}\left(\varepsilon_{yy}^0 + \upsilon.\varepsilon_{xx}^0\right) + D_{11}\left(k_{yy} + \upsilon.k_{xx}\right) \\ M_{xy} = B_{66}.\gamma_{xy}^0 + D_{66}.k_{xy} \end{cases} \tag{10}$$

Based on the classical plate theory, the motion equations of variable thickness FGM plate can be given as:

$$\begin{aligned} &\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \rho_1.h(x,y)\frac{\partial^2 u}{\partial t^2} \\ &\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = \rho_1.h(x,y)\frac{\partial^2 v}{\partial t^2} \\ &\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + N_{xx}.\frac{\partial^2 \mathrm{w}}{\partial x^2} + 2N_{xy}\frac{\partial^2 \mathrm{w}}{\partial x \partial y} + N_{yy}\frac{\partial^2 \mathrm{w}}{\partial y^2} - p.h(x,y)\frac{\partial^2 \mathrm{w}}{\partial x^2} \\ &\qquad - q.h(x,y)\frac{\partial^2 \mathrm{w}}{\partial y^2} + q_0 = \rho_1.h(x,y)\frac{\partial^2 \mathrm{w}}{\partial t^2} + 2\varepsilon\rho_1.h(x,y)\frac{\partial \mathrm{w}}{\partial t} \end{aligned} \tag{11}$$

here: $\rho_1 = \int_{-h(x,y)/2}^{h(x,y)/2} \rho_{(z)}.dz = \rho_m + \frac{\rho_c - \rho_m}{k+1}$

Substituting Eq. (9) and Eq. (10) in to Eq. (11) and consider Eq. (4) then, Eq. (11) can be rewritten as:

$$\begin{aligned} &L_{11}(U) + L_{12}(V) + L_{13}(\mathrm{W}) + P_1(\mathrm{W}) = \rho_1.h(x,y)\frac{\partial^2 u}{\partial t^2} \\ &L_{21}(U) + L_{22}(V) + L_{23}(\mathrm{W}) + P_2(\mathrm{W}) = \rho_1.h(x,y)\frac{\partial^2 v}{\partial t^2} \\ &L_{31}(U) + L_{32}(V) + L_{33}(\mathrm{W}) + P_3(\mathrm{W}) + P_4(U,\mathrm{W}) + P_5(V,\mathrm{W}) - p.h(x,y)\frac{\partial^2 \mathrm{w}}{\partial x^2} \\ &\qquad - q.h(x,y)\frac{\partial^2 \mathrm{w}}{\partial y^2} + q_0 = \rho_1.h(x,y)\frac{\partial^2 \mathrm{w}}{\partial t^2} + 2\varepsilon\rho_1.h(x,y)\frac{\partial \mathrm{w}}{\partial t} \end{aligned} \tag{12}$$

where:

$$L_{11}(U) = A_{11}\frac{\partial^2 u}{\partial x^2} + \frac{\partial A_{11}}{\partial x}\frac{\partial u}{\partial x} + A_{66}\frac{\partial^2 u}{\partial y^2} + \frac{\partial A_{66}}{\partial y}\frac{\partial u}{\partial y}$$

$$L_{12}(V)=A_{11}\upsilon\frac{\partial^2 v}{\partial x\partial y}+\upsilon.\frac{\partial A_{11}}{\partial x}\frac{\partial v}{\partial y}+A_{66}\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial A_{66}}{\partial y}\frac{\partial v}{\partial x}$$

$$L_{13}(\mathrm{W})=-B_{11}\frac{\partial^3 \mathrm{w}}{\partial x^3}-\left(B_{11}\upsilon+2B_{66}\right)\frac{\partial^3 \mathrm{w}}{\partial x\partial y^2}-\frac{\partial B_{11}}{\partial x}\left(\frac{\partial^2 \mathrm{w}}{\partial x^2}+\upsilon.\frac{\partial^2 \mathrm{w}}{\partial y^2}\right)-2\frac{\partial B_{66}}{\partial y}\frac{\partial^2 \mathrm{w}}{\partial x\partial y}$$

$$P_3(\mathrm{W}^2)=A_{11}\left(\frac{\partial \mathrm{w}}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial x^2}+\upsilon\frac{\partial \mathrm{w}}{\partial y}\frac{\partial^2 \mathrm{w}}{\partial x\partial y}\right)+\frac{1}{2}\frac{\partial A_{11}}{\partial x}\left(\left(\frac{\partial \mathrm{w}}{\partial x}\right)^2+\upsilon\left(\frac{\partial \mathrm{w}}{\partial y}\right)^2\right)+A_{66}\left(\frac{\partial^2 \mathrm{w}}{\partial x\partial y}\frac{\partial \mathrm{w}}{\partial y}+\frac{\partial \mathrm{w}}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial y^2}\right)+\frac{\partial A_{66}}{\partial y}\frac{\partial \mathrm{w}}{\partial x}\frac{\partial \mathrm{w}}{\partial y}$$

$$L_{21}(U)=A_{11}\upsilon\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial A_{11}}{\partial y}\upsilon\frac{\partial u}{\partial x}+A_{66}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial A_{66}}{\partial x}\frac{\partial u}{\partial y};L_{22}(V)=A_{66}\frac{\partial^2 v}{\partial x^2}+\frac{\partial A_{66}}{\partial x}\frac{\partial v}{\partial x}+A_{11}\frac{\partial^2 v}{\partial y^2}+\frac{\partial A_{11}}{\partial y}\frac{\partial v}{\partial y}$$

$$L_{23}(\mathrm{W})=-B_{11}\frac{\partial^3 \mathrm{w}}{\partial y^3}-B_{11}\upsilon.\frac{\partial^3 \mathrm{w}}{\partial x^2\partial y}-\frac{\partial B_{11}}{\partial y}\frac{\partial^2 \mathrm{w}}{\partial y^2}-\frac{\partial B_{11}}{\partial y}\upsilon.\frac{\partial^2 \mathrm{w}}{\partial x^2}-2B_{66}\frac{\partial^3 \mathrm{w}}{\partial x^2\partial y}-2\frac{\partial B_{66}}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial x\partial y}$$

$$P_2(\mathrm{W}^2)=A_{11}\frac{\partial \mathrm{w}}{\partial y}\frac{\partial^2 \mathrm{w}}{\partial y^2}+A_{11}\upsilon\frac{\partial \mathrm{w}}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial x\partial y}+\frac{1}{2}\frac{\partial A_{11}}{\partial y}\left(\frac{\partial \mathrm{w}}{\partial y}\right)^2+\frac{1}{2}\frac{\partial A_{11}}{\partial y}\upsilon\left(\frac{\partial \mathrm{w}}{\partial x}\right)^2+A_{66}\frac{\partial^2 \mathrm{w}}{\partial x^2}\frac{\partial \mathrm{w}}{\partial y}+A_{66}\frac{\partial \mathrm{w}}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial x\partial y}+\frac{\partial A_{66}}{\partial x}\frac{\partial \mathrm{w}}{\partial x}\frac{\partial \mathrm{w}}{\partial y}$$

$$L_{31}(U)=B_{11}\frac{\partial^3 u}{\partial x^3}+\left(2B_{66}+B_{11}\upsilon\right)\frac{\partial^3 u}{\partial x\partial y^2}+2\frac{\partial^2 B_{66}}{\partial x\partial y}\frac{\partial u}{\partial y}+\left(\frac{\partial^2 B_{11}}{\partial x^2}+\frac{\partial^2 B_{11}}{\partial y^2}\upsilon\right)\frac{\partial u}{\partial x}$$
$$+2\left(\frac{\partial B_{66}}{\partial x}\frac{\partial^2 u}{\partial y^2}+\frac{\partial B_{66}}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial B_{11}}{\partial x}\frac{\partial^2 u}{\partial x^2}+\upsilon\frac{\partial B_{11}}{\partial y}\frac{\partial^2 u}{\partial x\partial y}\right)$$

$$L_{32}(V)=$$
$$B_{11}\frac{\partial^3 v}{\partial y^3}+\left(B_{11}\upsilon+2B_{66}\right)\frac{\partial^3 v}{\partial x^2\partial y}+2\left(\frac{\partial B_{11}}{\partial x}\upsilon+\frac{\partial B_{66}}{\partial x}\right)\frac{\partial^2 v}{\partial x\partial y}+2\frac{\partial B_{11}}{\partial y}\frac{\partial^2 v}{\partial y^2}+2\frac{\partial B_{66}}{\partial y}\frac{\partial^2 v}{\partial x^2}+\left(\frac{\partial^2 B_{11}}{\partial y^2}+\upsilon.\frac{\partial^2 B_{11}}{\partial x^2}\right)\frac{\partial v}{\partial y}+2\frac{\partial^2 B_{66}}{\partial x\partial y}\frac{\partial v}{\partial x}$$

$$L_{33}(\mathrm{W})=-D_{11}\left(\frac{\partial^4 \mathrm{w}}{\partial x^4}+\frac{\partial^4 \mathrm{w}}{\partial y^4}\right)-2\left(D_{11}\upsilon+2D_{66}\right)\frac{\partial^4 \mathrm{w}}{\partial x^2\partial y^2}-2\frac{\partial D_{11}}{\partial x}\frac{\partial^3 \mathrm{w}}{\partial x^3}-2\frac{\partial D_{11}}{\partial y}\frac{\partial^3 \mathrm{w}}{\partial y^3}-2\left(\frac{\partial D_{11}}{\partial x}\upsilon+2\frac{\partial D_{66}}{\partial x}\right)\frac{\partial^3 \mathrm{w}}{\partial x\partial y^2}$$
$$-2\left(\upsilon\frac{\partial D_{11}}{\partial y}+2\frac{\partial D_{66}}{\partial y}\right)\frac{\partial^3 \mathrm{w}}{\partial x^2\partial y}-\frac{\partial^2 D_{11}}{\partial x^2}\left(\frac{\partial^2 \mathrm{w}}{\partial x^2}+\upsilon.\frac{\partial^2 \mathrm{w}}{\partial y^2}\right)-\frac{\partial^2 D_{11}}{\partial y^2}\left(\frac{\partial^2 \mathrm{w}}{\partial y^2}+\upsilon\frac{\partial^2 \mathrm{w}}{\partial x^2}\right)-4\frac{\partial^2 D_{66}}{\partial x\partial y}\frac{\partial^2 \mathrm{w}}{\partial x\partial y}$$

$$P_3(\mathrm{W}^2)=$$
$$\frac{1}{2}\frac{\partial^2 B_{11}}{\partial x^2}\left(\left(\frac{\partial \mathrm{w}}{\partial x}\right)^2+\upsilon\left(\frac{\partial \mathrm{w}}{\partial y}\right)^2\right)+2\left(B_{11}\upsilon-B_{66}\right)\left(\frac{\partial^2 \mathrm{w}}{\partial x\partial y}\right)^2+2\left(\frac{\partial B_{66}}{\partial x}+\frac{\partial B_{11}}{\partial x}\upsilon\right)\frac{\partial \mathrm{w}}{\partial y}\frac{\partial^2 \mathrm{w}}{\partial x\partial y}+2\frac{\partial B_{66}}{\partial x}\frac{\partial \mathrm{w}}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial y^2}+2\frac{\partial B_{11}}{\partial x}\frac{\partial \mathrm{w}}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial x^2}$$
$$+\left(B_{11}\upsilon+2B_{66}\right)\frac{\partial \mathrm{w}}{\partial y}\frac{\partial^3 \mathrm{w}}{\partial x^2\partial y}+2\left(B_{66}-B_{11}\upsilon\right)\frac{\partial^2 \mathrm{w}}{\partial x^2}\frac{\partial^2 \mathrm{w}}{\partial y^2}+\left(2B_{66}+B_{11}\upsilon\right)\frac{\partial \mathrm{w}}{\partial x}\frac{\partial^3 \mathrm{w}}{\partial x\partial y^2}+B_{11}\left(\frac{\partial \mathrm{w}}{\partial x}\frac{\partial^3 \mathrm{w}}{\partial x^3}+\frac{\partial \mathrm{w}}{\partial y}\frac{\partial^3 \mathrm{w}}{\partial y^3}\right)$$
$$+2\frac{\partial B_{66}}{\partial y}\frac{\partial \mathrm{w}}{\partial y}\frac{\partial^2 \mathrm{w}}{\partial x^2}+2\left(\upsilon\frac{\partial B_{11}}{\partial y}+\frac{\partial B_{66}}{\partial y}\right)\frac{\partial \mathrm{w}}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial x\partial y}+2\frac{\partial B_{11}}{\partial y}\frac{\partial \mathrm{w}}{\partial y}\frac{\partial^2 \mathrm{w}}{\partial y^2}+2\frac{\partial^2 B_{66}}{\partial x\partial y}\frac{\partial \mathrm{w}}{\partial x}\frac{\partial \mathrm{w}}{\partial y}+\frac{1}{2}\frac{\partial^2 B_{11}}{\partial y^2}\left(\left(\frac{\partial \mathrm{w}}{\partial y}\right)^2+\upsilon\left(\frac{\partial \mathrm{w}}{\partial x}\right)^2\right)$$

$$P_4(\mathrm{W}^3)=\frac{1}{2}A_{11}\left(\frac{\partial^2 \mathrm{w}}{\partial x^2}\left(\frac{\partial \mathrm{w}}{\partial x}\right)^2+\frac{\partial^2 \mathrm{w}}{\partial y^2}\left(\frac{\partial \mathrm{w}}{\partial y}\right)^2\right)+\frac{1}{2}A_{11}\upsilon\left(\frac{\partial^2 \mathrm{w}}{\partial x^2}\left(\frac{\partial \mathrm{w}}{\partial y}\right)^2+\frac{\partial^2 \mathrm{w}}{\partial y^2}\left(\frac{\partial \mathrm{w}}{\partial x}\right)^2\right)+2A_{66}\frac{\partial^2 \mathrm{w}}{\partial x\partial y}\frac{\partial \mathrm{w}}{\partial x}\frac{\partial \mathrm{w}}{\partial y}$$

$$P_5(U,\mathrm{W}) = A_{11}\frac{\partial u}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial x^2} + A_{11}\upsilon\frac{\partial u}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial y^2} + 2A_{66}\frac{\partial u}{\partial y}\frac{\partial^2 \mathrm{w}}{\partial x\partial y};$$

$$P_6(V,\mathrm{W}) = A_{11}\frac{\partial v}{\partial y}\frac{\partial^2 \mathrm{w}}{\partial y^2} + A_{11}\upsilon\frac{\partial v}{\partial y}\frac{\partial^2 \mathrm{w}}{\partial x^2} + 2A_{66}\frac{\partial v}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial x\partial y}$$

Eqs. (12) are the basic equations used to investigate the nonlinear dynamic response of variable thickness FGM plates subjected to mechanical load.

For simplicity, we only consider the simply supported rectangular FGM plate with variable thickness which linearly changes in the x-axis (Fig. 1b). Assume that, the thickness of the plate can be determined as follows:

$$h(x) = \left(\frac{h_1 - h_0}{a}\right)x + h_0 \tag{13}$$

Where: a is the length of the plate's edge, h_1 and h_0 are the thickness of FGM plate at $x=0$ and $x=a$, respectively.

Then, Eqs. (12) will be rewritten as:

$$\begin{aligned}
&I_{11}(U) + I_{12}(V) + I_{13}(\mathrm{W}) + Q_1(\mathrm{W}) = \rho_1.h(x)\frac{\partial^2 u}{\partial t^2}\\
&I_{21}(U) + I_{22}(V) + I_{23}(\mathrm{W}) + Q_2(\mathrm{W}) = \rho_1.h(x)\frac{\partial^2 v}{\partial t^2}\\
&I_{31}(U) + I_{32}(V) + I_{33}(\mathrm{W}) + Q_3(\mathrm{W}) + Q_4(\mathrm{W}^2) + Q_5(U,\mathrm{W}) + Q_6(V,\mathrm{W})\\
&\qquad - p.h(x)\frac{\partial^2 \mathrm{w}}{\partial x^2} - q.h(x)\frac{\partial^2 \mathrm{w}}{\partial y^2} + q_0 = \rho_1.h(x)\frac{\partial^2 \mathrm{w}}{\partial t^2} + 2\varepsilon\rho_1.h(x)\frac{\partial \mathrm{w}}{\partial t}
\end{aligned} \tag{14}$$

in which:

$$I_{11}(U) = \frac{E_1.h(x)}{1-\upsilon^2}\frac{\partial^2 u}{\partial x^2} + \frac{E_1.h(x)}{2(1+\upsilon)}\frac{\partial^2 u}{\partial y^2} + \frac{E_1}{1-\upsilon^2}\left(\frac{h_1 - h_0}{a}\right)\frac{\partial u}{\partial x}$$

$$I_{12}(V) = \frac{(1+\upsilon)E_1.h(x)}{2(1-\upsilon^2)}\frac{\partial^2 v}{\partial x\partial y} + \frac{\upsilon E_1}{1-\upsilon^2}\left(\frac{h_1 - h_0}{a}\right).\frac{\partial v}{\partial y}$$

$$I_{13}(\mathrm{W}) = -\frac{E_2.h^2(x)}{1-\upsilon^2}\left(\frac{\partial^3 \mathrm{w}}{\partial x^3} + \frac{\partial^3 \mathrm{w}}{\partial x\partial y^2}\right) - \frac{2E_2 h(x)}{1-\upsilon^2}\left(\frac{h_1 - h_0}{a}\right)\left(\frac{\partial^2 \mathrm{w}}{\partial x^2} + \upsilon\frac{\partial^2 \mathrm{w}}{\partial y^2}\right)$$

$$\begin{aligned}
Q_1(\mathrm{W}) = {}&\frac{E_1.h(x)}{1-\upsilon^2}\left(\frac{\partial \mathrm{w}}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial x^2} + \upsilon\frac{\partial \mathrm{w}}{\partial y}\frac{\partial^2 \mathrm{w}}{\partial x\partial y}\right) + \frac{E_1.h(x)}{2(1+\upsilon)}\left(\frac{\partial \mathrm{w}}{\partial y}\frac{\partial^2 \mathrm{w}}{\partial x\partial y} + \frac{\partial \mathrm{w}}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial y^2}\right) +\\
&+ \frac{E_1(h_1 - h_0)}{2(1-\upsilon^2)a}\left(\left(\frac{\partial \mathrm{w}}{\partial x}\right)^2 + \upsilon\left(\frac{\partial \mathrm{w}}{\partial y}\right)^2\right)
\end{aligned}$$

$$I_{21}(U) = \frac{(1+\upsilon)E_1.h(x)}{2(1-\upsilon^2)}\frac{\partial^2 u}{\partial x\partial y} + \frac{E_1.}{2(1+\upsilon)}\left(\frac{h_1 - h_0}{a}\right).\frac{\partial u}{\partial y}$$

$$I_{22}(V)=\frac{E_1.h(x)}{2(1+\upsilon)}\frac{\partial^2 v}{\partial x^2}+\frac{E_1.h(x)}{1-\upsilon^2}\frac{\partial^2 v}{\partial y^2}+\frac{E_1.}{2(1+\upsilon)}\left(\frac{h_1-h_0}{a}\right).\frac{\partial v}{\partial x}$$

$$I_{23}(\mathrm{W})=-\frac{E_2.h^2(x)}{1-\upsilon^2}\frac{\partial^3 \mathrm{w}}{\partial y^3}-\frac{E_2.h^2(x)}{1-\upsilon^2}\frac{\partial^3 \mathrm{w}}{\partial x^2\partial y}-2\frac{E_2h(x)}{(1+\upsilon)}\left(\frac{h_1-h_0}{a}\right)\frac{\partial^2 \mathrm{w}}{\partial x\partial y}$$

$$Q_2(\mathrm{W}^2)=\frac{E_1.h(x)}{1-\upsilon^2}\left(\frac{\partial \mathrm{w}}{\partial y}\frac{\partial^2 \mathrm{w}}{\partial y^2}+\upsilon\frac{\partial \mathrm{w}}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial x\partial y}\right)+\frac{E_1.h(x)}{2(1+\upsilon)}\left(\frac{\partial \mathrm{w}}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial x\partial y}+\frac{\partial \mathrm{w}}{\partial y}\frac{\partial^2 \mathrm{w}}{\partial x^2}\right)+\frac{E_1.}{2(1+\upsilon)}\left(\frac{h_1-h_0}{a}\right)\frac{\partial \mathrm{w}}{\partial x}\frac{\partial \mathrm{w}}{\partial y}$$

$$I_{31}(U)=\frac{E_2.h^2(x)}{1-\upsilon^2}\left(\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 u}{\partial x\partial y^2}\right)+\frac{2E_2}{1-\upsilon^2}\left(\frac{h_1-h_0}{a}\right)^2\frac{\partial u}{\partial x}+2\frac{E_2h(x)}{1-\upsilon^2}\left(\frac{h_1-h_0}{a}\right)\left(2\frac{\partial^2 u}{\partial x^2}+(1-\upsilon)\frac{\partial^2 u}{\partial y^2}\right)$$

$$I_{32}(V)=\frac{E_2.h^2(x)}{1-\upsilon^2}\left(\frac{\partial^3 v}{\partial x^2\partial y}+\frac{\partial^3 v}{\partial y^3}\right)+2\frac{E_2h(x)}{(1-\upsilon^2)}\left(\frac{h_1-h_0}{a}\right)\left((1+\upsilon)\frac{\partial^2 v}{\partial x\partial y}+(1-\upsilon)\frac{\partial^2 v}{\partial x^2}\right)+\frac{2\upsilon E_2}{1-\upsilon^2}\left(\frac{h_1-h_0}{a}\right)^2\frac{\partial v}{\partial y}$$

$$I_{33}(\mathrm{W})=-\frac{E_3.h^3(x)}{1-\upsilon^2}\left(\frac{\partial^4 \mathrm{w}}{\partial x^4}+\frac{\partial^4 \mathrm{w}}{\partial y^4}\right)-2\frac{E_3.h^3(x)}{1-\upsilon^2}\frac{\partial^4 \mathrm{w}}{\partial x^2\partial y^2}-\frac{6E_3h^2(x)}{1-\upsilon^2}\left(\frac{h_1-h_0}{a}\right)\left(\frac{\partial^3 \mathrm{w}}{\partial x^3}+\frac{\partial^3 \mathrm{w}}{\partial x\partial y^2}\right)$$
$$-\frac{6E_3h(x)}{1-\upsilon^2}\left(\frac{h_1-h_0}{a}\right)^2\left(\frac{\partial^2 \mathrm{w}}{\partial x^2}+\upsilon\frac{\partial^2 \mathrm{w}}{\partial y^2}\right)$$

$$Q_3(\mathrm{W}^2)=\frac{E_2}{1-\upsilon^2}\left(\frac{h_1-h_0}{a}\right)^2\left(\left(\frac{\partial \mathrm{w}}{\partial x}\right)^2+\upsilon\left(\frac{\partial \mathrm{w}}{\partial y}\right)^2\right)+(3\upsilon-1)\frac{E_2.h^2(x)}{(1-\upsilon^2)}\left(\frac{\partial^2 \mathrm{w}}{\partial x\partial y}\right)^2+2\frac{E_2.h(x)}{(1-\upsilon^2)}\left(\frac{h_1-h_0}{a}\right)\frac{\partial \mathrm{w}}{\partial y}\frac{\partial^2 \mathrm{w}}{\partial x\partial y}$$
$$+\frac{E_2.h^2(x)}{1-\upsilon^2}\frac{\partial \mathrm{w}}{\partial y}\frac{\partial^3 \mathrm{w}}{\partial y\partial x^2}+\frac{E_2.h^2(x)}{(1+\upsilon)}\frac{\partial \mathrm{w}}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial y^2}+2\frac{E_2.h(x)}{(1+\upsilon)}\left(\frac{h_1-h_0}{a}\right)\frac{\partial \mathrm{w}}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial y^2}+\frac{E_2.h^2(x)}{1-\upsilon^2}\upsilon\frac{\partial \mathrm{w}}{\partial x}\frac{\partial^3 \mathrm{w}}{\partial x\partial y^2}$$
$$+4\frac{E_2h(x)}{1-\upsilon^2}\left(\frac{h_1-h_0}{a}\right)\frac{\partial \mathrm{w}}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial x^2}+\frac{E_2.h^2(x)}{1-\upsilon^2}\frac{\partial \mathrm{w}}{\partial x}\frac{\partial^3 \mathrm{w}}{\partial x^3}+\frac{E_2.h^2(x)}{1-\upsilon^2}\frac{\partial \mathrm{w}}{\partial y}\frac{\partial^3 \mathrm{w}}{\partial y^3}+\left(2\frac{E_2.h^2(x)}{2(1+\upsilon)}-2\frac{E_2.h^2(x)}{1-\upsilon^2}\upsilon\right)\frac{\partial^2 \mathrm{w}}{\partial x^2}\frac{\partial^2 \mathrm{w}}{\partial y^2}$$

$$Q_4(\mathrm{W}^3)=\frac{E_1.h(x)}{2(1-\upsilon^2)}\frac{\partial^2 \mathrm{w}}{\partial x^2}\left(\left(\frac{\partial \mathrm{w}}{\partial x}\right)^2+\upsilon\left(\frac{\partial \mathrm{w}}{\partial y}\right)^2\right)+\frac{E_1.h(x)}{2(1-\upsilon^2)}\frac{\partial^2 \mathrm{w}}{\partial y^2}\left(\left(\frac{\partial \mathrm{w}}{\partial y}\right)^2+\upsilon\left(\frac{\partial \mathrm{w}}{\partial x}\right)^2\right)+\frac{E_1.h(x)}{(1+\upsilon)}\frac{\partial^2 \mathrm{w}}{\partial x\partial y}\frac{\partial \mathrm{w}}{\partial x}\frac{\partial \mathrm{w}}{\partial y}$$

$$Q_5(U,\mathrm{W})=\frac{E_1.h(x)}{1-\upsilon^2}\left(\frac{\partial u}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial x^2}+\upsilon\frac{\partial u}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial y^2}+(1-\upsilon)\frac{\partial u}{\partial y}\frac{\partial^2 \mathrm{w}}{\partial x\partial y}\right)$$
$$Q_6(V,\mathrm{W})=\frac{E_1.h(x)}{1-\upsilon^2}\left(\frac{\partial v}{\partial y}\frac{\partial^2 \mathrm{w}}{\partial y^2}+\upsilon\frac{\partial v}{\partial y}\frac{\partial^2 \mathrm{w}}{\partial x^2}+(1-\upsilon)\frac{\partial v}{\partial x}\frac{\partial^2 \mathrm{w}}{\partial x\partial y}\right)$$

Eqs. (14) are basic equations used to investigate nonlinear dynamic responses of FGM plates with thickness linearly changes in the x-axis subjected to mechanical load.

3. Solution method

Consider a variable thickness FGM rectangular plate subjected to uniformly distributed pressures p(t) and q(t) in x and y direction. The exciting force $q_0(t)$ acting on the plate's surface.

The plate is simply supported on 4 edges, then the boundary conditions are:

$$w=0, M_{xx}=0; N_{xx}=-ph(x) \text{ at x =0 and x=a.}$$

$$w=0, M_{y}=0; N_{yy}=-qh(x) \text{ at y =0 and y=b.}$$

Satisfying boundary conditions, the deflection of the plate can be chosen as:

$$u=U_{mn}(t)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}; v=V_{mn}(t)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}; w=W_{mn}(t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad (15)$$

Where: *m, n* are the numbers of half-wave along the x and y direction, respectively.

Substituting Eq. (15) into Eq. (14) then applying Galerkin procedure, at the same time, ignoring inertial components along x and y axes (because of u<<w, v<<w) [20], we obtain:

$$\begin{aligned} &l_{11}U + l_{12}V + l_{13}W + R_1 W^2 = 0 \\ &l_{21}U + l_{22}V + l_{23}W + R_2 W^2 = 0 \\ &l_{31}U + l_{32}V + l_{33}W + R_3 W^2 + R_4 W^3 + R_5 U.W + R_6 V.W + \frac{4abq_0}{mn\pi^2} = \rho_1^* \frac{d^2 w}{dt^2} + 2\varepsilon\rho_1^* \frac{dw}{dt} \end{aligned} \quad (16)$$

In which: $\rho_1^* = \frac{ab(h_1+h_0)}{8}\rho_1$

$$l_{11} = -\frac{E_1\pi^2(h_1+h_0)}{16(1-\upsilon^2)ab}\left((1-\upsilon)m^2b^2+2n^2a^2\right); l_{12}=l_{21}=-\frac{E_1.mn\pi^2(h_1+h_0)}{16(1-\upsilon)}$$

$$l_{13} = \left(\frac{E_2\pi(2m^2\pi^2+3)(h_1-h_0)^2}{24(1-\upsilon^2)ma^2b} + \frac{E_2m\pi^3h_1h_0}{4(1-\upsilon^2)ba^2}\right)\left(m^2b^2+n^2a^2\right) - \frac{E_2\pi(h_1-h_0)^2}{4(1-\upsilon^2)ma^2b}\left(m^2b^2+\upsilon n^2a^2\right)$$

$$R_1 = \frac{E_1(h_1+h_0)\pi}{9na^2b(1-\upsilon^2)}\left((3\upsilon-1)n^2a^2-4m^2b^2\right); \; l_{22} = -\frac{E_1\pi^2\left(m^2b^2(1-\upsilon)+2n^2a^2\right)(h_1+h_0)}{16(1-\upsilon^2)ab};$$

$$l_{23} = \left(\frac{E_2n\pi(2m^2\pi^2-3)(h_1-h_0)^2}{24(1-\upsilon^2)m^2b^2a} + \frac{E_2n\pi^3h_1h_0}{4(1-\upsilon^2)b^2a}\right)\left(n^2a^2+m^2b^2\right) + \frac{E_2(h_1-h_0)^2n\pi}{4(1+\upsilon)a}$$

$$R_2 = -\frac{E_1\pi(h_1+h_0)}{(1-\upsilon^2)9mb^2a}\left(4n^2a^2-(3\upsilon-1)m^2b^2\right)$$

$$l_{31} = \frac{E_2\pi(h_1-h_0)^2(2m^2\pi^2-3)(m^2b^2+n^2a^2)}{24(1-\upsilon^2)mba^2} + \frac{E_2h_1h_0m\pi^3(m^2b^2+n^2a^2)}{4(1-\upsilon^2)a^2b} + \frac{E_2n^2\pi(h_1-h_0)^2}{(1+\upsilon)4mb}$$

$$l_{32}=\left(\frac{E_2 n\pi\left(h_1-h_0\right)^2\left(2m^2\pi^2-3\right)}{24\left(1-\upsilon^2\right)m^2ab^2}+\frac{E_2\pi^3 nh_1h_0}{4\left(1-\upsilon^2\right)b^2a}\right)\left(m^2b^2+\mathrm{n}^2a^2\right)+(1-\upsilon)\frac{E_2\mathrm{n}\pi\left(h_1-h_0\right)^2}{4\left(1-\upsilon^2\right)a}$$

$$l_{33}=-\frac{E_3\pi^2\left(\mathrm{m}^2b^2+\mathrm{n}^2a^2\right)^2}{\left(1-\upsilon^2\right)4a^3b^3}\left[\left(h_1-h_0\right)^3\left(\frac{m^2\pi^2-3}{4m^2}\right)+\left(h_1-h_0\right)^2h_0\left(\frac{2m^2\pi^2-3}{2m^2}\right)+\frac{h_0^2\pi^2}{2}\left(3h_1-h_0\right)\right]$$
$$-\frac{3E_3\pi^2\left(h_1+h_0\right)\left(h_1-h_0\right)^2}{\left(1-\upsilon^2\right)4a^2b}\left(\mathrm{m}^2b^2+\mathrm{n}^2a^2\right)+\frac{3E_3\pi^2\left(h_1+h_0\right)\left(h_1-h_0\right)^2}{4\left(1-\upsilon^2\right)a^3b}\left(\upsilon\mathrm{n}^2a^2+\mathrm{m}^2b^2\right)+$$
$$+\frac{\left(h_1+h_0\right)\pi^2}{8ab}\left(p\mathrm{m}^2b^2+q\mathrm{n}^2a^2\right)$$

$$R_3=\frac{8E_2\left(h_1-h_0\right)^2}{\left(1-\upsilon^2\right)9a^3bmn}\left(\mathrm{m}^2b^2+\upsilon n^2a^2\right)+\frac{2E_2\pi^2m}{3\left(1-\upsilon^2\right)na^3b}\left[\frac{9m^2\pi^2-28}{27m^2\pi^2}\left(h_1-h_0\right)^2+6h_1h_0\right]\left(\left(\upsilon-1\right)n^2a^2-2m^2b^2\right)$$
$$+\frac{4E_2n\pi^2}{9\left(1-\upsilon^2\right)mb^3a}\left(\frac{9m^2\pi^2-45}{9m^2\pi^2}\left(h_1-h_0\right)^2+2h_1h_0\right)\left(\left(1-6\upsilon\right)m^2b^2-n^2a^2\right)$$
$$+\frac{32E_2\left(h_1-h_0\right)}{27mnb\left(1-\upsilon^2\right)}\left[\left(1-\upsilon\right)n^2a^2h_1+2m^2b^2\left(h_1-h_0\right)\right]$$

$$R_4=-\frac{2E_1\pi^2\left(h_1+h_0\right)\left(\mathrm{n}^4a^4+\mathrm{m}^4b^4\right)}{9\left(1-\upsilon^2\right)mna^3b^3}+\left(1-3\upsilon\right)\frac{E_1\left(h_1+h_0\right)\mathrm{nm}\pi^2}{9\left(1-\upsilon^2\right)ab};$$

$$R_5=\frac{2E_1\pi\left(h_1+h_0\right)}{9\left(1-\upsilon^2\right)nba^2}\left(4\mathrm{b}^2\mathrm{m}^2+\left(1+3\upsilon\right)\mathrm{n}^2a^2\right)\quad R_6=\frac{2E_1\pi\left(h_1+h_0\right)}{9\left(1-\upsilon^2\right)mab^2}\left[\left(1+3\upsilon\right)\mathrm{m}^2b^2-4\mathrm{n}^2a^2\right]$$

The first two equations of Eq. (16) are two linear algebraic equations for the amplitudes U_{mn} and V_{mn}. Solving U_{mn} and V_{mn} in terms of W_{mn} then substituting into the third equation of Eq. (16), we obtain:

$$\rho_1^*\frac{d^2\mathrm{w}}{dt^2}+2\varepsilon\rho_1^*\frac{\mathrm{dw}}{dt}+a_1\mathrm{W}+a_2\mathrm{W}^2+a_3\mathrm{W}^3=\frac{4abq_0}{mn\pi^2}\tag{17}$$

In which:

$$a_1=-l_{33}-\frac{l_{31}\left(l_{12}l_{23}-l_{13}l_{22}\right)+l_{32}\left(l_{13}l_{21}-l_{11}l_{23}\right)}{l_{11}l_{22}-l_{12}l_{21}}$$
$$a_2=-R_3-\frac{R_1\left(l_{32}l_{21}-l_{31}l_{22}\right)+R_2\left(l_{31}l_{12}-l_{32}l_{11}\right)+R_5\left(l_{12}l_{23}-l_{13}l_{22}\right)+R_6\left(l_{13}l_{21}-l_{11}l_{23}\right)}{l_{11}l_{22}-l_{12}l_{21}}$$
$$a_3=-R_4-\frac{R_5\left(R_2l_{12}-R_1l_{22}\right)+R_6\left(R_1l_{21}-R_2l_{11}\right)}{l_{11}l_{22}-l_{12}l_{21}}$$

Vibration analysis

Suppose that the plate is subjected to uniform compression loads q(t) and p(t) on each edge and the exciting force in form $q_0=Q\sin\Omega t$, Eq.(17) can be rewritten as follows

$$\rho_1^* \frac{d^2\mathrm{w}}{dt^2} + 2\varepsilon\rho_1^* \frac{d\mathrm{w}}{dt} + a_1\mathrm{W} + a_2\mathrm{W}^2 + a_3\mathrm{W}^3 = \frac{4abQ\sin\Omega t}{mn\pi^2} \quad (18)$$

** Natural-vibration frequency of plate: the natural frequency of the variable thickness FGM plate can be defined as.*

$$\omega_0 = \sqrt{a_1 / \rho_1^*} \quad (19)$$

** Nonlinear response of variable thickness FGM plate:*

Nonlinear responses of variable thickness FGM plates are received from Eq. (17) by using Runger-Kutta method.

Dynamic stability analysis

For dynamic stability analysis, this paper studies a rectangular plate with variable thickness subjected to linear compression in terms of time p(t)= $-c_1t$ and q(t)= $-c_2t$. In which, c_1 and c_2 are loading speed. Dynamic responses of plate can be determined by solving equation (17). The dynamic critical time t_{cr} can be obtained by using Budiansky–Roth criterion [21]. The dynamic critical load can be expressed as $p_{cr}= c_1t_{cr}$ and $q_{cr}= c_2t_{cr}$.

4. Numerical and discussion

Validation

According to the authors' knowledge, there has been no publication on the nonlinear dynamic response of the FGM plate with variable thickness. Thus, the results in this paper are compared with the constant thickness plates ($h(x) = h_o = h_1 = const$). The natural frequencies of constant thickness plate are compared with the ones of Uymaz and Aydogdu [19] (Tab. 1). Natural frequency parameters ω^* determined as follows:

$$\omega^* = \omega_0 . \sqrt{\frac{12(1-\upsilon^2)\rho_c a^2 b^2}{\pi^4 E_c h^2}}$$

In which: ω_0 is nature frequency of plate and calculated from Eq. (19).

The plate made of Aluminium and Zirconia with material properties are: $\nu_c = \nu_m = \nu = 0.3$, $E_m = 70.10^9$ N/m^2, $\rho_m = 2702$ kg/m^3 and $E_c = 151.10^9$ N/m^2, $\rho_c = 3000$ kg/m^3.

Table 1. Comparison of natural frequencies ω^* of constant thickness FGM plates

Source	a/b=1, (m, n)=(1, 1), a/h=100					
	k=0	k=0.5	k=1	k=5	k=10	k=∞
Ref [19]	1.9974	1.7972	1.7117	1.6062	1.5652	1.4317
Present	2.0	1.7987	1.7153	1.6105	1.5677	1.4349
Difference (%)	0.13	0.08	0.21	0.27	0.16	0.22

Results in Table 1 show that, the comparison obtain a good agreement with above publication. There for, the results of this article are reliable.

Vibration results

Consider a rectangular variable thickness FGM plate simply supported on four edges. Geometric parameters of plate are: a=1,5m, b=0,8m, h_1=0.008m, h_0=0.005m, Plate made of Aluminium and Alumina with properties of the material are: E_m = 70.10^9 N/m^2, ρ_m = 2702 kg/m^3 and E_c = 380.10^9 N/m^2, ρ_c = 3800 kg/m^3, respectively. Assume that, Poisson's ratio $\nu_m=\nu_c$= 0.3.

Natural-vibration frequency of variable plate:

Table 2. Natural frequencies (1/s) of variable thickness plate

k	a=1,5m, b=0,8m, h_1=0.008m, h_0=0.005m				
	(m, n)=(1, 1)	(m, n)= (1, 3)	(m, n)=(1, 5)	(m, n)=(1, 7)	(m, n)=(1, 9)
0	382.70	2835,5	7739,3	15095	24902
0.5	323.94	2402,0	6556,4	12788	21097
1	291.70	2165,4	5910,8	11529	19020
3	256.64	1907,9	5208,2	10159	16759
5	251.39	1867,7	5098,4	99443	16406

Table 2 shows natural frequencies of variable thickness plate with various modes shapes (m, n). As can be seen, the lowest nature frequency corresponding to vibration mode of considered plate is (m, n) = (1, 1).

Nonlinear dynamic response of variable thickness plate subjected to exciting force $q_0=Q\sin\Omega t$.

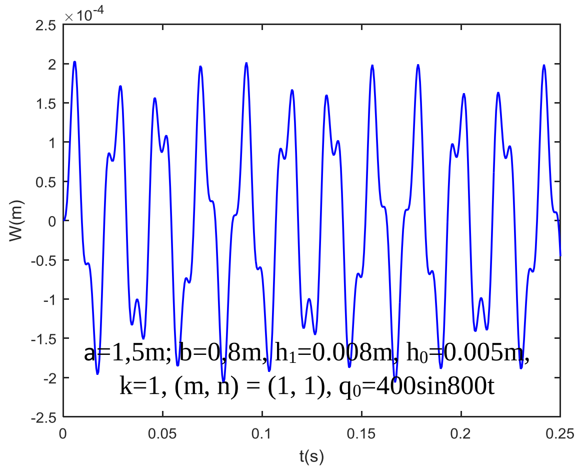

Fig. 2. Dynamic responses of variable thickness plates.

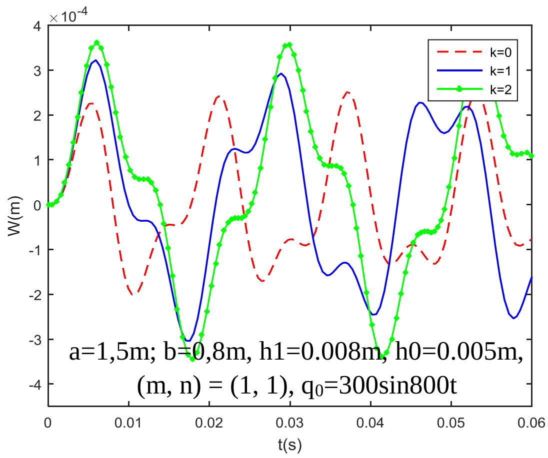

Fig. 3. Dynamic responses of variable thickness plate with various k.

Figure 2 shows dynamic response of variable thickness plate subjected to mechanical load. As can be seen that, the bound of dynamic response amplitude changes according to sine-shape law.

Figure 3 predicts effects of volume fraction index k on nonlinear vibration of variable thickness plates. The graph shows that, amplitude of dynamic responses increase with the increasing of k. this is reasonable because when k increase, the metal constituent in the plate increase, therefore, stiffness of the plate decrease.

Effect of geometric factors on nonlinear dynamic responses of variable thickness are illustrated in figure 4 and figure 5.

Figure 4 shows the effect of ratio a/b on nonlinear vibration of FGM variable thickness plate. From the graph, we can see that, dynamic responses amplitude of the plate increases when increasing the ratio a/b, that means the stiffness of the plate decreases.

Figure 5 shows the effect of ratio h_0/h_1 on dynamic responses of plate. As can be seen that, dynamic response amplitude decrease when ratio h_0/h_1 increase. That means, stiffness of plate increase when h_0 increase and the stiffness of plate reaches the maximum value when $h_0=h_1$ (constant thickness plate).

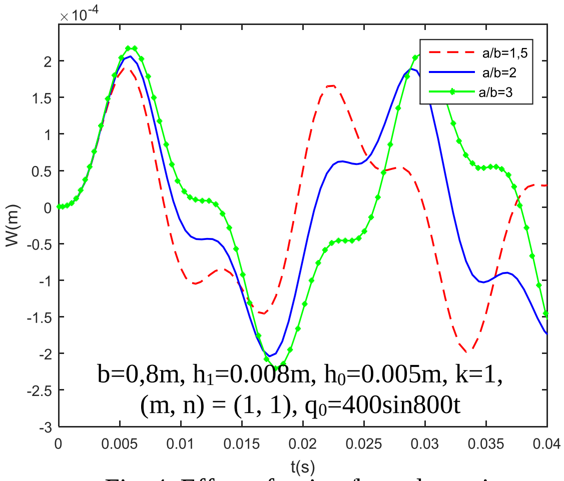

Fig. 4. Effect of ratio a/b on dynamic response of variable thickness plate.

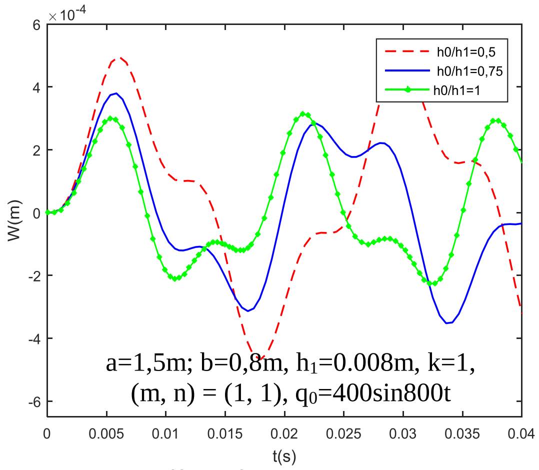

Fig. 5. Effect of ratio h_0/h_1 on dynamic response of variable thickness plate.

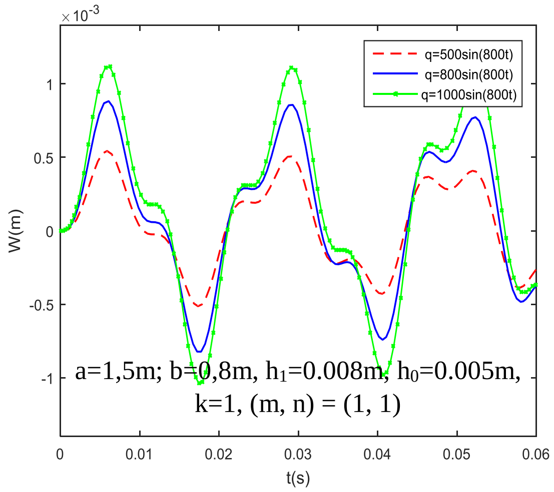

Fig. 6. Influnce of exciting load on dynamic response of plate.

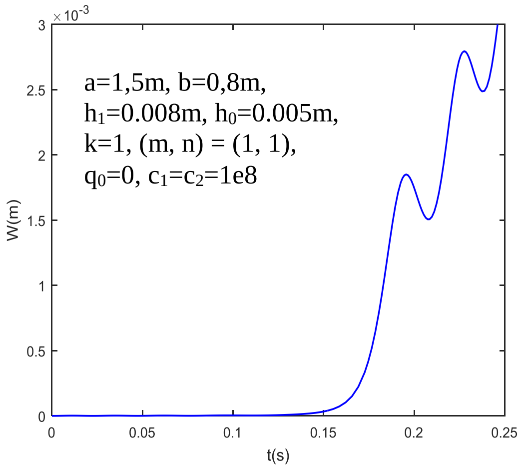

Fig 7. Dynamic response of variable thickness plate.

Figure 6 indicates the effect of excited force amplitude on nonlinear vibration of plate. When amplitude of excited force increase, the amplitudes nonlinear dynamic response of variable thickness FGM plate increase.

Dynamic buckling analysis results

In case plate subjected to linear compression in terms of time $q = c_1 t$ and $p = c_2 t$. In which, c_1 and c_2 are loading speed. The critical time t_{cr} can be obtained by using Budiansky–Roth criterion. The dynamic critical force $q_{cr} = c_1 t_{cr} = c_1.t_{cr}$ (or $p_{cr} = c_2 t_{cr}$).

Nonlinear dynamic responses of variable thickness FGM plate are indicated in Figure 7 to Figure 11.

Nonlinear dynamic response of variable thickness FGM plate is demonstrated in figure 7. The critical force obtained in this case is p_{cr} = 19,56 Mpa. Figure 8 illustrates influence of volume fraction index k on dynamic responses of variable thickness plate. From the graph we can see that the critical forces decrease with the increasing of volume fraction index k. For k=1, k=2 and k=3, critical forces are p_{cr} = 19,56 Mpa, p_{cr} = 16,32 Mpa and p_{cr} = 15,28 Mpa, respectively.

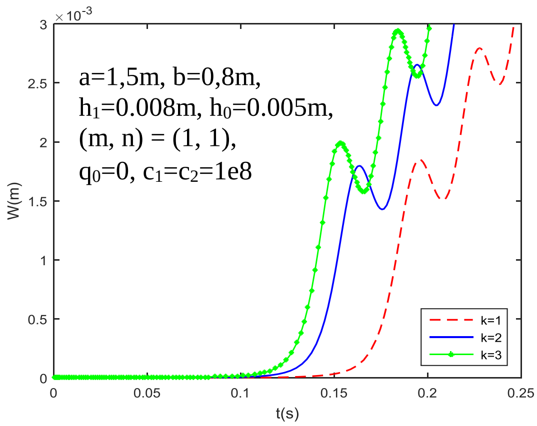

Fig 8. Effect of k on dynamic response of variable thickness plates.

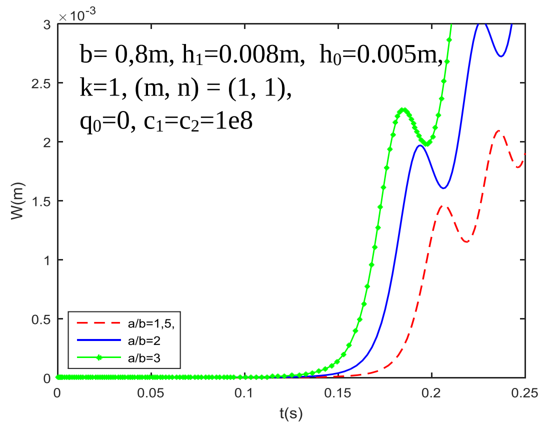

Fig 9. Effect of ratio a/b on dynamic response of variable thickness plate.

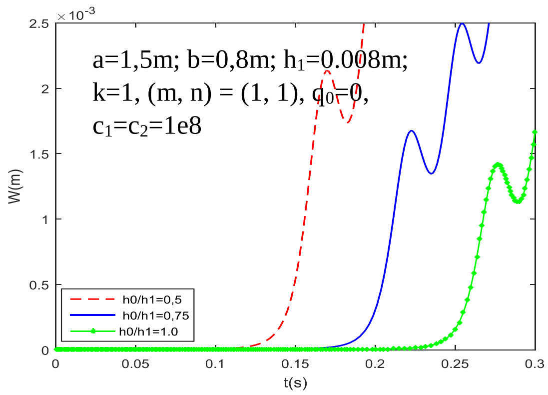

Fig 10. Effect of ratio h0/h1 on dynamic response of variable thickness plate.

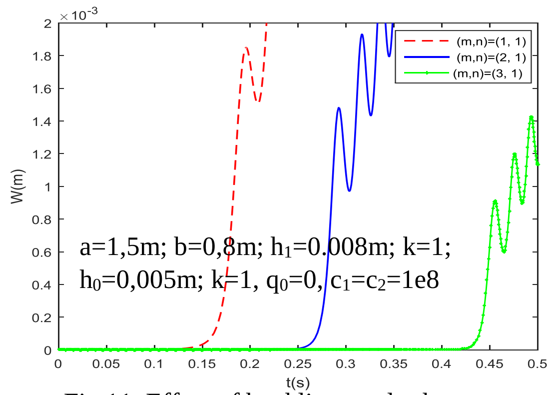

Fig 11. Effect of buckling mode shapes on dynamic responses of variable thickness plate.

Effect of ratio a/b on dynamic responses of variable thickness plate is shown in Figure 9. As can be seen that, if the ratio a/b increases, the critical load will decrease. For a/b=1,5; a/b=2 and a/b=3, critical forces are $p_{cr\ (a/b=1,5)}$ = 20,57 Mpa, $p_{cr\ (a/b=2)}$ = 19,32 Mpa and $p_{cr\ (a/b=3)}$ = 18,43Mpa, respectively. On other words, the load-bearing capacity of the plate will decrease when the plate's length increase.

Figure 10 illustrates the influence of ratio h_0/h_1 on dynamic responses of variable thickness plate. Results show that, ratio h_0/h_1 increases, the critical forces also increase. (p_{cr} = 16,92 Mpa in case h_0/h_1=0,5 and p_{cr} = 27,62 Mpa in case h_0/h_1=1). That means, when ratio h_0/h_1 increases, the plate will work more stability.

Figure 11 shows the effect of buckling mode shapes on dynamic responses of variable thickness FGM plate subjected to mechanical load. Clearly, the smallest critical dynamic buckling load corresponds to the buckling mode shape (m, n)=(1, 1).

5. Conclusions

This paper established the governing equations of variable thickness FGM plate according to the classical plate theory and the geometrical nonlinearity in von Karman-Donnell sense. The basics of vibration and dynamic stability problems of a variable thickness FGM plate have been investigated by using Galerkin method, Runger-Kutta method and Budiansky-Roth criterion.

Some conclusions can be drawn:

i) The lowest nature frequency corresponding to vibration mode of variable thickness FGM plate is (m, n) =(1,1).

ii) The vibration amplitude of variable thickness FGM plate increases and critical load of the plate decreases with the rise of ratio a/b. That mean, the length of the plate increases reducing the stability of the plate.

iii) The dynamic critical load of plate increases and vibration amplitude of plate decreases when ratio h_o/h_1 increasing. On other words, the stability of the plate increases with increasing plate thickness.

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