Original Article

Wave Propagation and Localization in a Complex Random Potential with Power-law Correlations: A Numerical Study

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Abstract: In this paper, we investigate numerically wave propagation and localization in a complex random potential with power-law correlations. Using a discrete stationary Schrödinger equation with the simultaneous presence of the spatial correlation and the non-Hermiticity of the random potential in the diagonal on-site terms of the Hamiltonian, we calculate the disorder-averaged logarithmic transmittance and the localization length. From the numerical analysis, we find that the presence of power-law correlation in the imaginary part of the on-site disordered potential gives rise to the localization enhancement as compared with the case of absence of correlation. Depending on the disorder's strength, we show that there exist different behaviors of the dependence of the localization on the correlation strength.

Keywords: Non-Hermitian Hamiltonian, complex disordered potential, spatial correlated disorder, Anderson localization.

1. Introduction

Although Anderson localization of quantum particles and classical waves has been extensively studied over the last 60 years, it is still attracting the interest of many researchers in various areas of physics [1-6]. New aspects of localization phenomena have been discovered continually in diverse systems characterized by different kinds of wave equations or random potentials. Among them, the combined influence of the spatial correlation and the non-Hermiticity of the random potential is the main focus of this study.

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During the recent few decades, the influence of spatially correlated disorder on the wave propagation and localization properties has been attracted much attention from theoretical as well as experimental researchers [7-23]. It has been demonstrated that some special kinds of short-range correlated disorder can produce delocalized eigenstates even in one dimension [7, 8]. More interestingly, it was predicted that the introduction of long-range correlations into a disordered system would suppress localization and give rise to a continuum of delocalized states [9, 10]. At first, this prediction caused some controversy [9, 11]. However, it was experimentally verified later by a measurement of microwave transmission spectra through a single-mode waveguide with artificially introduced correlated disorder [12]. There is a common point of view that the strongest localization is obtained for strongly disordered uncorrelated potentials, which is based on the prediction that correlations suppress, rather than enhance, localization. In some specific situations [24-27], however, a strong decrease of the localization length and therefore a strong enhancement of localization has been observed, when the long-range correlated disorder is introduced into the Hermitian Hamiltonian system. In spite of a large number of publications devoted to this topic, the problem of understanding the effects of spatial correlations on Anderson localization has not been completely resolved yet.

In recent years, there has been considerable attention devoted to studying transport and localization properties in physical systems which are described by non-Hermitian Hamiltonians [28-39]. These kinds of Hamiltonians are usually regarded as effective Hamiltonians for which the non-Hermitian part can serve for different purposes. The possibility for the potential to possess an imaginary part makes these non-Hermitian Hamiltonians having many unique features such as non-Hermitian delocalization transition [28, 29], a transition from ballistic to diffusive transport [30], one-way scattering and transport [33, 34] and topological phase transitions [35, 36]. Basically, there are two classes of problems associated with non-Hermitian Hamiltonian: one in which the non-Hermitcity arises from the off-diagonal elements such as the Hatano-Nelson model, where an imaginary vector potential is applied to the Hamiltonian [28] and the other in which the non-Hermitcity originates from the complexity of on-site potentials [31, 37-39]. For the latter, a one-dimensional (1D) non-Hermitian lattice model with randomness only in the imaginary part of the on-site potential has been studied, with the result that the eigenstates of such a model are localized, but the properties of localization are attributed to be qualitatively different from those of the usual Anderson model [31]. The 2D non-Hermitian square lattice model was proposed in a recent work by Tzortzakakis and co-workers [38]. They demonstrated that the complex disordered potential acts in a dual way: its time-reversal symmetry breaking favors more delocalized states, while its disorder nature favors more localized ones. Very shortly after, Huang and Shklovskii extended the 2D square lattice model to 3D cubic one [39]. The authors showed that there exists a transition from delocalized to localized states as in the usual 3D Anderson model. However, this transition happens at a substantially lower critical value of the disorder strength than that of the Hermitian case ($W_c = 6.1$ vs $W_c = 16.5$).

One important application of Anderson localization in optical systems is to develop random lasers [40, 41], where an optical cavity is formed not by mirrors but by multiple scattering in a random gain medium. To achieve high-efficiency output from a random laser, it is crucial to produce strongly localized states. To do this, in our very recent work [42], we studied the effects of spatial correlations on the *transverse* localization of waves in 1D lattices with randomly distributed gain and loss. In the short-ranged case and when the correlation length is sufficiently small, we find that there exists a critical value of the disorder strength, below which enhancement and above which suppression of localization occurs as the correlation length increases. In the region where the correlation length is larger, localization is suppressed in all cases. Similar behavior is obtained for long-range correlations as the disorder strength and the correlation exponent are varied. In the region where localization is

enhanced in the presence of long-range correlations, we find that the enhancement occurs in the whole energy band, but is strongest near the band center.

In this paper, we study the effects of power-law correlations on the transmission and the *longitudinal* localization of plane waves in a 1D disordered lattice chain characterized by a random non-Hermitian Hamiltonian, where the imaginary part of the on-site potential is a correlated random function of the position. Using a calculation scheme for solving a disordered version of the stationary discrete Schrödinger equation, we calculate the disorder-averaged logarithmic transmittance and the localization length, when the intensity of the incident wave is held fixed. From the numerical results, we find that the presence of power-law correlation in the disorder distribution gives rise to the localization enhancement as compared with the case of absence of correlation. Specifically, in the region where the correlation is relatively weak, it is shown that the localization is enhanced rapidly with increasing the correlation strength. In the region where the disorder strength is weak, a non-monotonic dependence of the localization length on the correlation strength is found. This behavior is held in the entire energy band but is more pronounced at the band center than near the band edges. In the region where the disorder strength is large enough, the localization length is a monotonic decreasing function of the correlation strength.

The rest of this paper is organized as follows. In section 2, we describe the theoretical model and the method within the nearest-neighbor tight-binding approximation. The physical quantities of interest are also described briefly in this section. In section 3, we present our numerical results. Finally, in section 4, we conclude the paper.

2. Theoretical Model and Method

2.1. Model

We are interested in the stationary propagation of extended excitations through a 1D lattice chain of length L described by a discrete stationary Schrödinger equation

$$\varepsilon_n \psi_n + J(\psi_{n-1} + \psi_{n+1}) = E\psi_n \tag{1}$$

This is an open quantum system which is coupled to the external world through two identical semi-infinite perfect leads on either side. Here ε_n is the wave function amplitude at the nth lattice site, E is the energy of the incident wave, and J is the constant nearest-neighbor hopping term. It is well-known that when the on-site potential ε_n are assumed to be real, random, uncorrelated, and uniformly distributed in the interval $[-W/2, W/2]$ with W being the disorder strength, we have the usual Anderson model [1].

In the present work, the on-site potential ε_n is considered to be complex. Specifically, it takes the form,

$$\varepsilon_n = \varepsilon_n^R + i\varepsilon_n^I \tag{2}$$

where the real part ε_n^R is assumed to be the same for all lattice sites (below we put $\varepsilon_n^R = 0$ for simplicity), whereas the imaginary part ε_n^I is considered to be a sequence of random variables with power-law correlations. One of the most popular ways to numerically generate such a sequence is to consider the trace of the fractional Brownian motion defined by [9, 19]

$$\gamma_n = \sum_{k=1}^{L/2} \left[k^{-\alpha} \left(\frac{2\pi}{L} \right)^{(1-\alpha)} \right]^{1/2} \cos\left(\frac{2\pi nk}{L} + \phi_k \right) \tag{3}$$

where where the number of sites L is an even number and ϕ_k's are $L/2$ random phases uniformly distributed in the interval $[0, 2\pi]$. The Fourier transform of the two-point correlation function $\langle \gamma_i \gamma_j \rangle$, $S(k)$, is proportional to a power-law spectrum $k^{-\alpha}$. The exponent α determines the roughness of potential landscapes and characterizes the power-law correlation strength. When α is zero, we recover an uncorrelated random sequence. In order to study the dependence on the strength of disorder properly, we need to normalize the generated sequence γ_n by multiplying a suitable normalization constant [14]

$$\varepsilon_n^I = \frac{W}{\sqrt{12}} \frac{\gamma_n}{C} \tag{4}$$

where W measures the disorder strength and C is the normalization constant which is obtained by the condition $\sqrt{\langle \gamma_n^2 \rangle - \langle \gamma_n \rangle^2} = 1$. In Figure 1, we show an example of typical random configurations of ε_n^I corresponding to $W = 2.0$ and $\alpha = 0, 1.5, 2.5$. It is clearly that the normalization effectively modifies the local disorder strength while keeping the global disorder strength the same for any α.
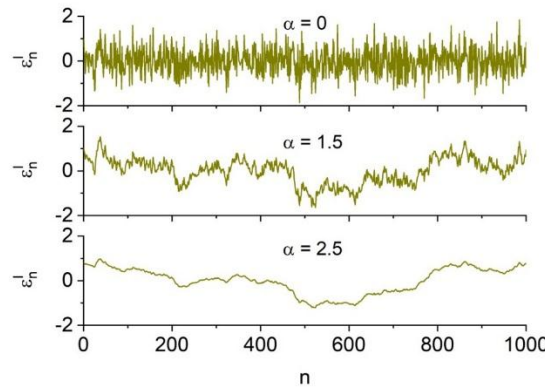

Figure 1. Typical random configurations of ε_n^I, when $W = 2.0$ and the power-law correlation exponent $\alpha = 0, 1.5, 2.5$. It is clearly that the normalization effectively modifies the local disorder strength while keeping the global disorder strength the same for different values of α.

2.2. Method

We assume that a plan wave is incident from the right onto the disordered sample of L lattice sites and define the transmission problem by using

$$\begin{cases} \psi_{n+1} = (E - \varepsilon_n)\psi_n - \psi_{n-1}, & 0 < n < L \\ \psi_n = r_0 e^{iq(L-n)} + r_1 e^{iq(n-L)}, & n \geq L \\ \psi_n = t e^{-iqn}, & n \leq 0 \end{cases} \tag{5}$$

where the wave number q is related to the energy of the incident wave E by $E = 2\cos q$; r_0, r_1 and t are the complex amplitudes of the incident, reflected and transmitted waves, respectively. Since our model is non-Hermitian, the conservation of probability current is not satisfied, therefore $|r_1|^2 + |t|^2 \neq |r_0|^2$. The wave number q is assumed to be the same inside and outside the medium. This restriction corresponds to the case where the creation of other harmonics is neglected.

For the transmission problem, in principle, there are two cases for which we need to be carefully distinguished. The case where the value of $|t|^2$ is fixed is called the fixed output case, while that with a fixed value of $|r_0|^2$ is called the fixed input case. In actual experiments, however, one usually fixes the incident wave intensity while varying other parameters. Therefore, we developed a reliable theoretical method for solving Eqs. (4) and (5) numerically in the situation where the incident wave intensity, $|r_0|^2$, is fixed [43-45]. In particular, we first assume that t is a certain positive real number. Using the relationships $\psi_{-1} = te^{iq}$, $\psi_0 = t$ and $\psi_1 = E\psi_0 - \psi_{-1}$ and solving the first of Eq. (5) iteratively, we obtain ψ_L and ψ_{L+1} for a given L. Next, using the relationships $\psi_L = r_0 + r_1$ and $\psi_{L+1} = r_0 e^{-iq} + r_1 e^{iq}$, we express r_0 and r_1 as

$$r_0 = \frac{\psi_L e^{iq} - \psi_{L+1}}{e^{iq} - e^{-iq}} \quad \text{and} \quad r_1 = \frac{\psi_{L+1} - \psi_L e^{-iq}}{e^{iq} - e^{-iq}} \tag{6}$$

And then the expressions for the transmittance T and the reflectance R are given by

$$T = \left|\frac{t}{r_0}\right|^2 = |t|^2 \frac{4\sin^2 q}{\left|\psi_L e^{iq} - \psi_{L+1}\right|^2} \quad \text{and} \quad R = \left|\frac{r_1}{r_0}\right|^2 = \frac{\left|\psi_L e^{-iq} - \psi_{L+1}\right|^2}{\left|\psi_L e^{iq} - \psi_{L+1}\right|^2} \tag{7}$$

In order to calculate T and R in the fixed input problem, we first choose the values of E (or q), $|r_0|^2$ and a random configuration of ε_n. Then we solve Eq. (1) repeatedly for many different initial values of t ($t = 0$, δ, 2δ, 3δ,...) until we obtain the final value of $|r_0|^2$ sufficiently close to the chosen value. The step size δ needs to be chosen appropriately to achieve desired accuracy.
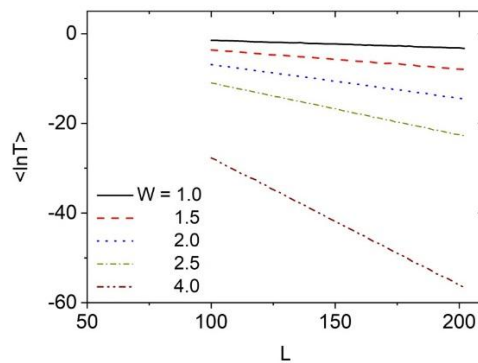

Figure 2. In the absence of power-law correlation $(\alpha = 0)$: Disorder-averaged logarithmic transmittance $\langle \ln T \rangle$ versus L at $E = 0$ for various values of the disorder strength W.

In the study of the Anderson localization phenomenon of waves in disordered media, the localization length ξ is the central quantity which is defined by [2]

$$\lim_{L \to \infty} \langle \ln T \rangle = -\frac{L}{\xi} \tag{8}$$

where $\langle \cdots \rangle$ denotes averaging over a large number of random configurations of ε_n. The localization length measures the exponential decay rate of a localized state in the thermodynamic limit ($L \to \infty$ or $T \to 0$).

3. Numerical Results

In our calculations, the disorder-averaged quantities are obtained by averaging over 10^4 distinct disorder configurations. Although the chosen system sizes are finite, they are large enough so that the asymptotic behaviors in the thermodynamic limit should be achieved. Indeed, as seen below that the exponential decay of $\langle T \rangle$ (or the linear decay of $\langle \ln T \rangle$) with L approaches to some very small values ($\langle T \rangle \to 0$) when the calculations were performed for the system size $L = 100$ up to $L = 200$. All of our results were obtained for the fixed input intensity of $|r_0|^2 = 1$. The step size for t was $\delta = 10^{-6}$. The error in the calculated value of $|r_0|^2 = 1$ was smaller than 10^{-5}.

First, we consider the problem in the case of absence of power-law correlation. In Figure 2, we plot $\langle \ln T \rangle$ versus L at the band center ($E = 0$) for various values of the disorder strength W when $\alpha = 0$. From the numerical results we find that for a given W, $\langle \ln T \rangle$ decays linearly as L increases to infinity. This is a signature of Anderson localization. We have also verified and found that this behavior is held for all values of E in the entire energy band. Similarly to the usual Anderson model where the real part of the on-site potential is random, in the model under consideration, the decay rate of $\langle \ln T \rangle$ increases, hence the localization is enhanced as the disorder strength W increases. These are in good agreement with the results of previous studies [31, 37-39], in which the transverse localization of waves in 1D, 2D and 3D optical waveguide arrays with non-Hermitian disorder has been investigated in details.
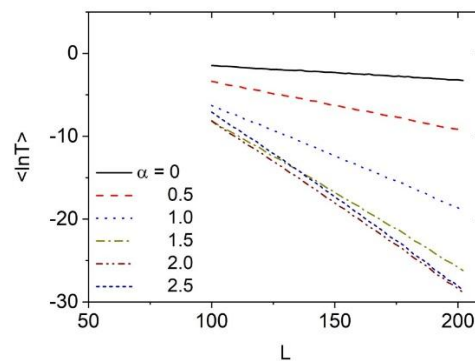
Figure 3. In the presence of power-law correlation $(\alpha \neq 0)$: Disorder-averaged logarithmic transmittance $\langle \ln T \rangle$ versus L at $E = 0$ for various different values of α when $W = 1.0$.

Next, we study the effects of spatially correlated disorder on the longitudinal localization of waves for which the amplitude of the wave decays exponentially along the propagation direction. This is the main goal of the present work. In Figure 3, we plot $\langle \ln T \rangle$ versus L at $E = 0$ when $W = 1.0$ for various values of the power-law correlation exponent α. We find that in the presence of power-law correlation, the linear decay of $\langle \ln T \rangle$ with L is maintained. The decay rate of $\langle \ln T \rangle$ is increased, hence the behavior of localization is enhanced as α increases. Remarkably, in the correlation region with $\alpha < 2$, $\langle \ln T \rangle$ decays more and more rapidly with L as α increases. However, if we keep increasing α further, the decay rate of $\langle \ln T \rangle$ slows down and stops at a certain critical value α_c.

The next question we ask is how this behavior changes as α is increased beyond the critical value of α_c. The answer is shown clearly in Figure 4, which plots the inverse localization length $1/\xi$ as a function of the correlation parameter α at $E = 0$ for several different values of the disorder strength W. In the parameter region under consideration, we find that the behavior of localization is always enhanced in the case of $\alpha \neq 0$ as compared with the case of $\alpha = 0$. When W is small, we find that the inverse localization length $1/\xi$ shows a non-monotonic dependence on α. Particularly, in the region where α is relatively small, $1/\xi$ increases rapidly with increasing α. However, increasing $1/\xi$ with α will slow down and $1/\xi$ attains a maximum value at a critical value α_c. If we continue increasing α further, $1/\xi$ returns to decrease slowly with α. This implies that the correlation-induced localization enhancement is the strongest at α_c. It is clearly seen that the value of α_c strongly depends on the disorder strength W, specifically, α_c increases with increasing W. As a result, when W is sufficiently large ($W > 4$: bandwidth), the localization length ξ would be a monotonic decreasing function of α. These results are completely consistent with those presented in our recent work [42], in which the effects of spatially correlated disorder on the transverse localization of waves in a 1D array of optical waveguides have been studied.
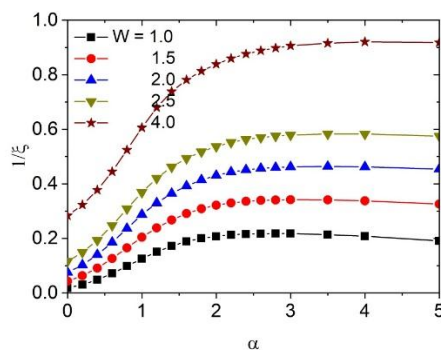

Figure 4. Inverse localization length $1/\xi$ versus power-law correlation strength α, when $E = 0$ for various values of the disorder strength W.

Finally, in Figure 5, we depict the inverse localization length $1/\xi$ as a function of the correlation parameter α at $W = 1.0$ for three typical values of the energy of the incident wave, $E = 0$, 1 and $\sqrt{3}$. We find that the behavior of non-monotonic dependence of the inverse localization length on the correlation strength is held in the whole energy band. It appears to be more pronounced at the band center than near the band edges.

It is important to mention that although it is not presented in this paper, we have checked numerically that all the results reported above are also valid in the situations where the average value of the imaginary part of the on-site potential ε_n^I takes any nonzero value. These include the cases where ε_n^I takes only positive or negative random values.
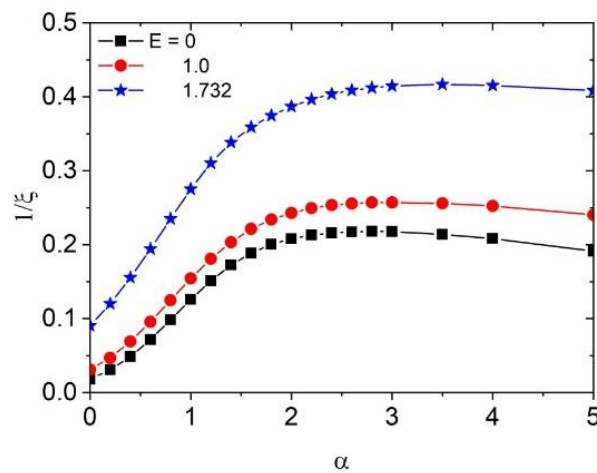

Figure 5. Inverse localization length $1/\xi$ versus power-law correlation strength α, when $W = 1.0$ for several typical values of the incident wave energy E.

4. Conclusion

In summary, we have studied the effects of power-law correlations on the transmission and the Anderson localization of plane waves in a 1D disordered lattice chain characterized by a random non-Hermitian Hamiltonian, where the imaginary part of the on-site potential is a correlated random function of the position. Using a calculation scheme for solving a disordered version of the stationary discrete Schrödinger equation, we have calculated the disorder-averaged logarithmic transmittance and the localization length, when the intensity of the incident wave is held fixed. From the numerical results, we have found that the presence of power-law correlation in the disorder distribution gives rise to the localization enhancement as compared with the case of absence of correlation. Specifically, in the region where the correlation is relatively weak, we have found that the localization is enhanced rapidly with increasing the correlation strength. In the region where the disorder strength is weak $(W < 4)$, a non-monotonic dependence of the localization length on the correlation strength has been found. This behavior is held in the entire energy band but is more pronounced at the band center than near the band edges. In the region where the disorder strength is large enough $(W > 4)$, the localization length is a monotonic decreasing function of the correlation strength. Devices such as random lasers utilize

highly disordered gain materials. Our results can provide a useful guideline for achieving correlation-enhanced localized states in such structures, thereby enhancing the device efficiency.

Acknowledgments

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