



Original Article

A Note on Invariant Basis Number and Types for Strongly Graded Rings

Nguyen Quang Loc*

Hanoi National University of Education, 136 Xuan Thuy, Cau Giay, Hanoi, Vietnam

Received 11 April 2020

Revised 03 August 2020; Accepted 15 September 2020

Abstract: Given any pair of positive integers (n, k) and any nontrivial finite group G , we show that there exists a ring R of type (n, k) such that R is strongly graded by G and the identity component R_e has Invariant Basis Number. Moreover, for another pair of positive integers (n', k') with $n \leq n'$ and $k \mid k'$, it is proved that there exists a ring R of type (n, k) such that R is strongly graded by G and R_e has type (n', k') . These results were mentioned in [G. Abrams, Invariant basis number and types for strongly graded rings, *J. Algebra* 237 (2001) 32-37] without proofs.

Keywords: Strongly graded ring, Invariant Basis Number, type.

1. Introduction

In the study of abstract algebra, most of the rings that one first encounters are rings with Invariant Basis Number property. This class of rings includes all nonzero commutative rings and (left) Noetherian rings. A unital ring R is said to have *Invariant Basis Number* (IBN) if any two bases of any free left R -module have the same cardinality. Since this condition always holds for free modules with infinite bases (see, e.g., [1, Corollary 1.2]), the definition is equivalent to saying that for any pair of positive integers m and n , $R^m \cong R^n$ as left R -modules only if $m = n$. As a typical example of a ring without IBN, let V be an infinite dimensional vector space over a field K and let $R = \text{End}_K(V)$. Then

$$R = \text{Hom}_K(V, V) \cong \text{Hom}_K(V \oplus V, V) \cong \text{Hom}_K(V, V) \oplus \text{Hom}_K(V, V) \cong R \oplus R$$

*Corresponding author.

Email address: nqloc@hnue.edu.vn

<https://doi.org/10.25073/2588-1124/vnumap.4507>

as left R -modules. As a result, the free left R -modules R^m and R^n are isomorphic for any positive integers m, n .

An R -homomorphism $R^n \rightarrow R^m$ can be expressed by an $m \times n$ matrix A with entries in R , that is, $A \in M_{m \times n}(R)$. Thus the ring R fails to have IBN if and only if there exist positive integers $m \neq n$ and matrices $A \in M_{m \times n}(R)$, $B \in M_{n \times m}(R)$ such that $AB = I_m$ and $BA = I_n$. This statement does not involve left nor right modules, so the IBN property is indeed a left-right symmetric condition. It also shows that if there is a unit-preserving ring homomorphism $R' \rightarrow R$ and R has IBN, then R' has IBN.

If the ring R does not have IBN, then there is a smallest positive integer n for which $R^n \cong R^m$ as left R -modules with some integer $m > n$. We choose m minimal with this property for n and write $m = n + k$ with $k > 0$; then we say that R has type (n, k) . For example, the ring $R = \text{End}_K(V)$ considered above has type $(1, 1)$.

The type of a ring was studied by W. G. Leavitt in the seminal paper [2]. He showed that for any pair of positive integers (n, k) , there exists a unital ring of type (n, k) [2, Theorem 8]. Also, the following important results were proved.

Proposition 1.1. ([2, Theorem 2]) *Let R and R' be unital rings of type (n, k) and (n', k') , respectively. If there is a unit-preserving ring homomorphism $R' \rightarrow R$, then $n \leq n'$ and $k \mid k'$.*

Proposition 1.2. ([2, Theorem 3]) *Let R and R' be unital rings of type (n, k) and (n', k') , respectively. Then $R \oplus R'$ has type $(\max(n, n'), \text{lcm}(k, k'))$. On the other hand, if R and R' are IBN rings, then so is $R \oplus R'$.*

Let G be a multiplicative group with identity element e . Recall that a ring R is G -graded if $R = \bigoplus_{g \in G} R_g$, where each R_g is an additive subgroup of R and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The G -graded ring R is called *strongly graded* if $R_g R_h = R_{gh}$ for all $g, h \in G$. It is easy to see that the G -graded ring R is strongly graded if and only if $1_R \in R_g R_{g^{-1}}$ for any $g \in G$.

For a G -graded ring R , the identity component R_e is a ring and there is a unit-preserving injection $R_e \rightarrow R$. Hence if R has IBN, then R_e also has IBN by above observation. On the other hand, if the type of R and R_e is (n', k') and (n, k) respectively, then $n' \leq n$ and $k' \mid k$ by Proposition 1.1.

In [3], the authors asked if R is a G -strongly graded ring with G a finite group, whether it is true that R_e has IBN if and only if R has IBN. This question was answered in the negative by G. Abrams in [4, Theorem A], where G is the group \square_2 . Moreover, [4, Theorem B] states that for any pairs of positive integers (n, k) and (n', k') with $n' \leq n$ and $k' \mid k$, there exists a ring R of type (n', k') such that R is strongly graded by \square_2 and R_e has type (n, k) . Abrams also mentioned that "it is not hard to show that results analogous to Theorems A and B hold with any finite group G taking the place of \square_2 " (see [4, page 36]).

The purpose of this paper is to prove Abrams's above remark (see Theorems 3.1 and 3.4). While the construction of the ring R is similar to that of [4], we apply a general method of [5, Theorem 2.10.1] to produce a strong G -grading on R . To deal with the IBN property and the type, we introduce the

notion of type of an object in an additive category, which is a generalisation of the corresponding concept for a ring. We show in Proposition 2.2 that how the IBN property or the type of an object V is related to the monogenic monoid generated by the isoclass $[V]$. This enables us to obtain a number of useful consequences, among others, Corollary 2.3, which generalises [2, Theorem 1].

We need some more notions on graded modules (see [5] for a complete account on graded rings and modules). Let R be a G -graded ring and M be a left R -module. The module M is called a *graded R -module* if $M = \bigoplus_{g \in G} M_g$, where each M_g is an additive subgroup of M and $R_h M_g \subseteq M_{hg}$ for all $h, g \in G$. A *graded homomorphism* between graded R -modules M, N is an R -homomorphism $f: M \rightarrow N$ such that $f(M_g) \subseteq N_g$ for all $g \in G$. For a graded R -module M , $END_R(M) = \bigoplus_{h \in G} END_R(M)_h$ is a G -graded ring, where

$$END_R(M)_h = \{f \in End_R(M) \mid f(M_g) \subseteq M_{gh} \text{ for all } g \in G\}.$$

The g -shifted graded R -module $M(g)$ is defined to be $M(g) = \bigoplus_{h \in G} M(g)_h$, where $M(g)_h = M_{gh}$. We denote by $R\text{-Mod}$ the category of left R -modules and by $R\text{-Gr}$ the category of graded left R -modules with graded homomorphisms.

Throughout this paper, we consider only rings with identity.

2. The IBN and Types of Objects

Let us start by extending [3, Definition 3.1] to non-IBN objects.

Definition 2.1. Let \mathcal{A} be an additive category. An object $V \in \mathcal{A}$ has IBN if for all positive integers m and n , $V^n \cong V^m$ implies $n = m$. If a nonzero object $V \in \mathcal{A}$ does not have IBN, then V has type (n, k) if n is the smallest positive integer such that $V^n \cong V^m$ for some $m > n$, and k is the smallest positive integer such that $V^n \cong V^{n+k}$.

Thus, when $\mathcal{A} = R\text{-Mod}$ and $V = R$, considered as a left R -module, we obtain the usual definitions of IBN property and type for a ring.

Next we point out the relationship between Definition 2.1 and the structure of monogenic semigroups (see [6, Section 1.2]). A *monogenic* semigroup (or a cyclic semigroup) is a semigroup generated by a single element. Let S be an additive semigroup and $a \in S$. By definition, the monogenic subsemigroup of S generated by a is

$$\langle a \rangle = \{a, 2a, 3a, \dots\},$$

where $ma = a + a + \dots + a$ (m summands). There are two possibilities for the structure of $\langle a \rangle$. If, whenever $ma = na$ implies $m = n$ for positive integers m and n , then clearly $\langle a \rangle$ is isomorphic to the additive semigroup \mathbb{N}^* of positive integers. On the other hand, if $ma = na$ for some $m \neq n$, then the *index* of $\langle a \rangle$ is the least positive integer n such that $na = ma$ for some $m > n$, and the *period* of $\langle a \rangle$ is the least positive integer k such that $na = (n+k)a$. In this case, it is proved that

$$\langle a \rangle = \{a, 2a, \dots, na, (n+1)a, \dots, (n+k-1)a\}$$

and $ma = m'a$ for $m \neq m'$ if and only if $m, m' \geq n$ and $m \equiv m' \pmod{k}$. The key point here is the elements $na, (n+1)a, \dots, (n+k-1)a$ form a group; in particular, these elements are distinct by cancellation law. A monogenic semigroup with index n and period k may also be expressed in terms of semigroup presentation $\langle a \mid na = (n+k)a \rangle$, which is a free semigroup generated by a subject to the relation $na = (n+k)a$ (see [6, Section 1.6]).

The above definitions can be carried over monoids (i.e., semigroups with identity) as well, so we may talk about monogenic monoids (where $\langle a \rangle = \{0a, a, 2a, 3a, \dots\}$ with $0a$ equal the identity element).

The following proposition is now clear from Definition 2.1 and the previous discussion.

Proposition 2.2. *Let \mathcal{A} be an additive category and let \mathcal{P} be a set of objects in \mathcal{A} such that \mathcal{P} is closed under finite direct sums (including the empty sum, which gives the zero object). Let S be the abelian monoid of isomorphism classes of objects in \mathcal{P} , where the class of $A \in \mathcal{P}$ is denoted by $[A]$ and the operation on S is defined by*

$$[A] + [B] = [A \oplus B].$$

Then, for any object $V \in \mathcal{P}$, we have:

- (i) V has IBN if and only if the monogenic submonoid $\langle [V] \rangle$ of S is isomorphic to the additive monoid \mathbb{N} .*
- (ii) V has type (n, k) if and only if the monogenic submonoid $\langle [V] \rangle$ of S has index n and period k .*

As a direct consequence of Proposition 2.2, we obtain the corollary below. This result generalises [2, Theorem 1], which was proved by arguments on bases of free modules and such arguments cannot apply to our more general definition.

Corollary 2.3. *Let \mathcal{A} and \mathcal{P} be as in Proposition 2.2. For a pair of positive integers (n, k) , an equivalent condition for an object $V \in \mathcal{P}$ to have type (n, k) is:*

$$V^m \cong V^{m'} \text{ for } m \neq m' \text{ if and only if } m, m' \geq n \text{ and } m \equiv m' \pmod{k}.$$

Proof. Assume that we have the given condition. Then clearly $V^n \cong V^{n+k}$ and n is the smallest positive integer such that $V^n \cong V^m$ for some $m > n$. If $V^n \cong V^{n+h}$ for some $0 < h < k$, then we would have $n \equiv n+h \pmod{k}$, which is absurd. Hence the type of V is (n, k) . The converse follows from Proposition 2.2. \square

The next proposition is an extension of [3, Proposition 3.3] to the case of non-IBN objects. By using Corollary 2.3, the proof is similar to that of [3, Proposition 3.3], so we will omit it.

Proposition 2.4. *Let \mathcal{A} be an additive category and let \mathcal{P} be a set of objects in \mathcal{A} which is closed under finite direct sums. Then $V \in \mathcal{A}$ has IBN if and only if the ring $\text{End}_{\mathcal{A}}(V)$ has IBN. Similarly, $V \in \mathcal{P}$ has type (n, k) if and only if $\text{End}_{\mathcal{A}}(V)$ has type (n, k) .*

As an application of the above results, let R be a (unital) ring and take $\mathcal{A} = R\text{-Mod}$, \mathcal{P} to be the set of all finitely generated projective left R -modules. We denote by $\mathcal{V}(R)$ the monoid of isomorphism classes (denoted by $[P]$) of finitely generated projective left R -modules under operation given by direct sum. The monoid $\mathcal{V}(R)$ is *conical*, that is, if $x, y \in \mathcal{V}(R)$ and $x + y = 0$, then $x = y = 0$. Moreover, $[R]$ is an order-unit in $\mathcal{V}(R)$. Recall that for a monoid S , an *order-unit* in S is an element $u \in S$ such that

for each $x \in S$, there exist $y \in S$ and an integer $n \geq 1$ such that $x + y = nu$. By a result of Bergman [7, Theorem 6.2], any conical monoid with an order-unit appears as $V(R)$ for some ring R .

From Propositions 2.2 and 2.4 we obtain immediately:

Corollary 2.5. *Let R be a ring and V a finitely generated projective R -module. Let M denote the submonoid generated by $[V]$ in the monoid $V(R)$. Then we have:*

- (i) V has IBN if and only if $End_R(V)$ has IBN, if and only if M is isomorphic to \square .
- (ii) V has type (n, k) if and only if $End_R(V)$ has type (n, k) , if and only if M is a monogenic monoid with index n and period k .

Specialising Corollaries 2.5 and 2.3 to the case $V = R$, we recover the following known result.

Corollary 2.6. *Let M denote the submonoid generated by $[R]$ in the monoid $V(R)$. Then we have:*

- (i) R has IBN if and only if M is isomorphic to \square .
- (ii) R has type (n, k) if and only if M is a monogenic monoid with index n and period k , if and only if

$$R^m \cong R^{m'} \text{ for } m \neq m' \Leftrightarrow m, m' \geq n \text{ and } m \equiv m' \pmod{k}.$$

3. Strongly Graded Rings of Given Types

To construct a ring having IBN or having a particular type (n, k) , one possible way is using Corollary 2.5. Specifically, we will construct a module V of type (n, k) over some ring T via its monoid realisation. The desired ring will be $End_T(V)$. To show that this ring is strongly graded by a given finite group G , we use [5, Theorem 2.10.1]. Let us first recall several related notions in *loc. cit.*

Let \mathcal{A} be an abelian category and $M, N \in \mathcal{A}$. We say that N weakly divides M in \mathcal{A} if N is isomorphic to a direct summand of a finite direct sum M' of copies of M . We say that M, N are weakly isomorphic in \mathcal{A} , denoted by $M \sim N$, if and only if they weakly divide each other in \mathcal{A} .

In particular, we consider $\mathcal{A} = R\text{-Gr}$ for a G -graded ring R . A module $M \in R\text{-Gr}$ is said to be weakly G -invariant if $M \sim M(g)$ in $R\text{-Gr}$ for all $g \in G$, where $M(g)$ is the g -shifted of M . Then for a finite group G , [5, Theorem 2.10.1] states that the G -graded ring $End_R(M)$ is strongly graded if and only if M is weakly G -invariant. Note that the finiteness condition imposed on G implies that $End_R(M) = END_R(M)$, thus $End_R(M)$ is a graded ring (see [5, Corollary 2.4.6]).

We are now ready to state and prove the first main theorem of this section.

Theorem 3.1. *Let (n, k) be any pair of positive integers and G be any nontrivial finite group. Then there exists a ring R of type (n, k) such that R is strongly graded by G and R_e has IBN.*

Proof. We follow the idea of [4, Theorem A]. Assume that $|G| = m$. We consider the abelian monoid $(S, +)$ (with identity 0) presented by generators $\{x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_m\}$ and relations

$$n(x_1 + x_2 + \dots + x_m) = (n+k)(x_1 + x_2 + \dots + x_m), \quad (3.1)$$

$$x_1 + y_1 = x_2, x_2 + y_2 = x_3, \dots, x_m + y_m = x_1. \tag{3.2}$$

Since the given relations do not have 0 as their right-hand or left-hand sides, the monoid S is conical. Moreover, it is easy to verify that x_1 is an order-unit of S . Therefore, by [7, Theorem 6.2], there exists a ring T such that the monoid $V(T)$ is isomorphic to S . Let $X_1, X_2, \dots, X_m, Y_1, Y_2, \dots, Y_m$ denote the finitely generated projective left T -modules corresponding to the monoid elements $x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_m$, respectively.

We set $V = X_1 \oplus X_2 \oplus \dots \oplus X_m$ and $R = \text{End}_T(V)$. Then V is a finitely generated projective T -module, i.e., $[V] \in V(T)$. The relation (3.1) shows that the submonoid generated by $[V]$ in $V(T)$ is isomorphic to a monogenic monoid having presentation $\langle v \mid nv = (n+k)v \rangle$. By Corollary 2.5, both V and R have type (n, k) .

Assume that $G = \{e, g_2, \dots, g_m\}$. Let the ring T be graded trivially by G , i.e., $T_e = T$ and $T_g = 0$ for all $g \neq e$. The module V is graded as follows:

$$V = \bigoplus_{g \in G} V_g \quad \text{with} \quad V_e = X_1, V_{g_2} = X_2, \dots, V_{g_m} = X_m.$$

Clearly $V \in T\text{-Gr}$. The cyclic-type relations (3.2) imply that V_g and V_h are weakly isomorphic in $T\text{-Mod}$, for all $g, h \in G$ (precisely, each of them is a direct summand of the other). As a result, V is weakly G -invariant. Indeed, for a fixed $g \in G$, V_h is a direct summand of $V(g)_h = V_{gh}$ for all $h \in G$; thus V is a direct summand of $V(g)$ in $T\text{-Gr}$ and vice versa. By [5, Theorem 2.10.1], the ring $R = \text{End}_T(V)$ is strongly G -graded, as desired.

The identity component R_e of $R = \text{End}_T(V) = \bigoplus_{g, h \in G} \text{Hom}_T(V_g, V_h)$ consists of homomorphisms f such that $f(V_g) \subseteq V_g$ for all $g \in G$. Hence

$$R_e = \bigoplus_{g \in G} \text{End}_T(V_g) = \text{End}_T(X_1) \oplus \text{End}_T(X_2) \oplus \dots \oplus \text{End}_T(X_m).$$

It follows from the construction and Corollary 2.5 that X_1, X_2, \dots, X_m all have IBN. Consequently, the rings $\text{End}_T(X_1), \text{End}_T(X_2), \dots, \text{End}_T(X_m)$ have IBN. As a direct sum of IBN rings, the ring R_e also has IBN, by Proposition 1.2. \square

The following lemma is an obvious generalisation of an observation in [4].

Lemma 3.2. *Let (n', k') be any pair of positive integers and G be any finite group. Then there exists a ring R such that it is strongly graded by G and both R and R_e have type (n', k') .*

Proof. Let S be a unital ring of type (n', k') and let R be the group ring $S[G]$. Then clearly R is strongly graded by G and $R_e \cong S$. In particular, R_e has type (n', k') . There are unit-preserving ring homomorphisms from R_e to R (the inclusion map) and from R to R_e (which maps $\sum_{g \in G} s_g g$ to $(\sum_{g \in G} s_g)e$). It follows from Proposition 1.1 that R and R_e have the same type (n', k') . \square

Proposition 3.3. *Let (n, k) be any pair of positive integers and G be any nontrivial finite group. Then there exists a ring R of type $(1, 1)$ such that R is strongly graded by G and R_e has type (n, k) .*

Proof. By Lemma 3.2, the proposition holds true in the case $(n, k) = (1, 1)$. So we assume that $nk - 1 > 0$. Assume $G = \{e, g_2, \dots, g_m\}$. Let T be a unital ring of type (n, k) . We consider the free left T -module

$$V = \underbrace{T \oplus T^{nk-1} \oplus T^{nk} \oplus \dots \oplus T^{nk}}_{m \text{ summands}} \cong T^{(m-1)nk}.$$

By Corollary 2.6, $V \cong V^2$ as left T -modules. Thus V has type $(1, 1)$. Let $R = \text{End}_T(V)$, then R also has type $(1, 1)$ by Corollary 2.5.

Now let the ring T be graded trivially by G . The T -module V is graded as follows:

$$V = \bigoplus_{g \in G} V_g \quad \text{with} \quad V_e = T, V_{g_2} = T^{nk-1}, V_{g_3} = T^{nk}, \dots, V_{g_m} = T^{nk}.$$

Clearly $V \in T\text{-Gr}$; moreover, V is weakly G -invariant. By [5, Theorem 2.10.1], the ring $R = \text{End}_T(V)$ is strongly G -graded. The identity component R_e of R consists of homomorphisms f such that $f(V_g) \subseteq V_g$ for all $g \in G$. Thus, in the matrix form, we have

$$R_e = \begin{bmatrix} T & 0 & 0 & \dots & 0 \\ 0 & M_{nk-1}(T) & 0 & \dots & 0 \\ 0 & 0 & M_{nk}(T) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & M_{nk}(T) \end{bmatrix}.$$

There are unit-preserving ring homomorphisms from R_e to T (the projection from R_e into the upper left corner) and from T to R_e (the diagonal map). It follows from Proposition 1.1 that R_e and T have the same type (n, k) . \square

Using Lemma 3.2 and Proposition 3.3, the proof of the following theorem is similar to that of [4, Theorem B].

Theorem 3.4. *Let (n', k') be any pair of positive integers and G be any nontrivial finite group. For any pair (n, k) of positive integers with $n' \leq n$ and $k' \mid k$, there exists a ring R of type (n', k') such that R is strongly graded by G and R_e has type (n, k) .*

Proof. By Proposition 3.3, there exists a ring T which is strongly graded by G in such a way that T has type $(1, 1)$ and T_e has type (n, k) . By Lemma 3.2, for the given pair (n', k') there exists a ring T' strongly graded by G such that both T' and T'_e have type (n', k') . Let $R = T \oplus T'$. Then the ring R , with the component-wise grading, is strongly graded by G . In particular, we have $R_e \cong T_e \oplus T'_e$. Applying Proposition 1.2 yields that the type of R is $(\max(1, n'), \text{lcm}(1, k')) = (n', k')$, while the type of R_e is $(\max(n, n'), \text{lcm}(k, k')) = (n, k)$. \square

We conclude this paper with some remarks concerning strongly G -graded rings, where G is an arbitrary group (see also [3]). In general, we do not know whether Theorems 3.1 and 3.4 are still true if

G is any infinite group. However, we are able to prove Theorem 3.1 for $G = \square$ and for any pair $(1, k)$ with $k \geq 1$. Indeed, let R_{k+1} be the directed graph consisting of one vertex and $k+1$ loops. We consider the Leavitt path algebra $L_K(R_{k+1})$ of R_{k+1} with coefficients in a field K (see [8, 9]), which has type $(1, k)$ (in general, it follows from the separative cancellation proved in [9] that the Leavitt path algebra of a finite directed graph either has IBN, or has type $(1, m)$ for some $m \geq 1$; we thank the referee for pointing out this fact). It is well-known that Leavitt path algebras are naturally \square -graded. Moreover, since R_{k+1} has no sinks, the algebra $L_K(R_{k+1})$ is strongly \square -graded by [10, Theorem 3.15] (this could also be deduced directly from definition of the \square -grading on $L_K(R_{k+1})$). Finally, it follows from the proof of [9, Theorem 5.3] that the identity component $L_K(R_{k+1})_0$ is an ultramatricial K -algebra, i.e., it is a direct limit of finite direct products of matrix algebras over K . It is known (see, e.g., [1, Exercise 1.17]) that every ultramatricial K -algebra has IBN, and so $L_K(R_{k+1})_0$ has IBN. As a conclusion, the algebra $L_K(R_{k+1})$ is strongly \square -graded of type $(1, k)$, whereas its identity component has IBN.

4. Conclusion

In this paper, we have considered the IBN property and the type of the identity component R_e of a strongly G -graded ring R in relationship with the type of R , where G is any nontrivial finite group. More concretely, we have shown that for any pair of positive integers (n, k) , there exist strongly G -graded rings R such that R has type (n, k) and R_e has IBN, or R_e has a given type (n', k') with $n \leq n'$ and $k \mid k'$ (Theorems 3.1 and 3.4).

Acknowledgments

We would like to express our deep gratitude to the referee for her/his careful reading and valuable suggestions, which led to the final shape of the paper.

References

- [1] T.Y. Lam, Lectures on Modules and Rings, Graduate Texts in Mathematics 189, Springer-Verlag New York, 1999.
- [2] W.G. Leavitt, The module type of a ring, Trans. Amer. Math. Soc. 103 (1962) 113-130.
- [3] C. Năstăsescu, B. Torrecillas, F. Van Oystaeyen, IBN for graded rings, Comm. Algebra 28 (2000) 1351-1360.
- [4] G. Abrams, Invariant basis number and types for strongly graded rings, J. Algebra 237 (2001) 32-37.
- [5] C. Năstăsescu, F. Van Oystaeyen, Methods of Graded Rings, Lecture Notes in Mathematics 1836, Springer-Verlag Berlin Heidelberg, 2004.
- [6] J.M. Howie, Fundamentals of Semigroup Theory, London Math. Soc. Monographs New Series 12, Oxford University Press, 2003.
- [7] G.M. Bergman, Coproducts and some universal ring constructions, Trans. Amer. Math. Soc. 200 (1974) 33-88.
- [8] G. Abrams, G. Aranda Pino, The Leavitt path algebra of a graph, J. Algebra 293 (2005) 319-334.
- [9] P. Ara, M.A. Moreno, E. Pardo, Nonstable K-theory for graph algebras, Algebr. Represent. Theory 10 (2007) 157-178.
- [10] R. Hazrat, The graded structure of Leavitt path algebras, Israel J. Math. 195 (2013) 833-895.