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Original Article

$\mathcal{H}_\infty$ Finite-time Boundedness for Discrete-time Delay Neural Networks via Reciprocally Convex Approach

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Abstract: This paper addresses the problem of $\mathcal{H}_\infty$ finite-time boundedness for discrete-time neural networks with interval-like time-varying delays. First, a delay-dependent finite-time boundedness criterion under the finite-time $\mathcal{H}_\infty$ performance index for the system is given based on constructing a set of adjusted Lyapunov–Krasovskii functionals and using reciprocally convex approach. Next, a sufficient condition is drawn directly which ensures the finite-time stability of the corresponding nominal system. Finally, numerical examples are provided to illustrate the validity and applicability of the presented conditions.

Keywords: Discrete-time neural networks, $\mathcal{H}_\infty$ performance, finite-time stability, time-varying delay, linear matrix inequality.

1. Introduction

In recent years neural networks (NNs) have received remarkable attention because of many successful applications have been realised, e.g., in prediction, optimization, image processing, pattern recognization, association memory, data mining, etc. Time delay is one of important parameters of NNs and it can be considered as an inherent feature of both biological NNs and artificial NNs. Thus, analysis and synthesis of NNs with delay are important topics [1-3].

It is worth noting that Lyapunov's classical stability deals with asymptotic behaviour of a system over an infinite time interval, and does not usually specify bounds on state trajectories. In certain situations, finite-time stability, initiated from the first half of the 1950s, is useful to study behaviour of a system within a finite time interval (maybe short). More precisely, those are situations that state

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variables are not allowed to exceed some bounds during a given finite-time interval, for example, large values of the state are not acceptable in the presence of saturation [4-5]. By using the Lyapunov function approach and linear matrix inequality (LMI) techniques, a variety of results on finite-time stability, finite-time boundedness, finite-time stabilization and finite-time $\mathcal{H}_\infty$ control were obtained for continuous- or discrete-time systems in recent years [5-14]. In particular, within the framework of discrete-time NNs, there are two interesting articles [9, 10], which deal with finite-time stability and finite-time boundedness in that order.

To the best of our knowledge, $\mathcal{H}_\infty$ finite-time boundedness problem for discrete-time NNs with interval time-varying delay has not received adequate attention in the literature. This motivates our current study. For that purpose, in this paper, we first suggest conditions which guarantee finite-time boundedness of discrete-time delayed NNs and reduce the effect of disturbance input on the output to a prescribed level. Soon afterward, according to this scheme, finite-time stability of the nominal system is also obtained. Two numerical examples are presented to show the effectiveness of the achieved results.

Notation: $\mathbb{Z}_+$ denotes the set of all non-negative integers; $\mathbb{R}^n$ denotes the n-dimensional space with the scalar product $x^{\mathrm{T}}y$; $\mathbb{R}^{n\times r}$ denotes the space of $(n\times r)$ −dimension matrices; A^{T} denotes the transpose of matrix A; A is positive definite $(A>0)$ if $x^{\mathrm{T}}Ax>0$ for all $x\neq 0$; $A>B$ means $A-B>0$. The notation diag$\{\dots\}$ stands for a block-diagonal matrix. The symmetric term in a matrix is denoted by $*$.

2. Preliminaries

Consider the following discrete-time neural networks with time-varying delays and disturbances

$$\begin{cases} x(k+1)=Ax(k)+Wf(x(k))+W_1g(x(k-h(k)))+C\omega(k),\ k\in\mathbb{Z}_+,\\ z(k)=A_1x(k)+Dx(k-h(k))+C_1\omega(k),\\ x(k)=\varphi(k),\ k\in\{-h_2,-h_2+1,\dots,0\}, \end{cases} \tag{1}$$

where $x(k)\in\mathbb{R}^n$ is the state vector; $z(k)\in\mathbb{R}^p$ is the observation output; n is the number of neurals;

$$f(x(k))=[f_1(x_1(k)),f_2(x_2(k)),\dots,f_n(x_n(k))]^{\mathrm{T}},$$
$$g(x(k-h(k)))=[g_1(x_1(k-h(k))),g_2(x_2(k-h(k))),\dots,g_n(x_n(k-h(k)))]^{\mathrm{T}}$$

are activation functions, where $f_i,g_i,i=\overline{1,n}$, satisfy the following conditions

$$\begin{aligned} &\exists a_i>0:\quad |f_i(\xi)|\quad\le a_i|\xi|,\quad \forall i=\overline{1,n},\ \forall\xi\in\mathbb{R},\\ &\exists b_i>0:\quad |g_i(\xi)|\quad\le b_i|\xi|,\quad \forall i=\overline{1,n},\ \forall\xi\in\mathbb{R}. \end{aligned} \tag{2}$$

The diagonal matrix $A=\operatorname{diag}\{\overline{a}_1,\overline{a}_2,\dots,\overline{a}_n\}$ represents the self-feedback terms; the matrices $W,W_1\in\mathbb{R}^{n\times n}$ are connection weight matrices; $C\in\mathbb{R}^{n\times q},C_1\in\mathbb{R}^{p\times q}$ are known matrices; $A_1,D\in\mathbb{R}^{p\times n}$ are the observation matrices; the time-varying delay function $h(k)$ satisfies the condition

$$0<h_1\le h(k)\le h_2\quad \forall k\in\mathbb{Z}_+, \tag{3}$$

where h_1,h_2 are given positive integers; $\varphi(k)$ is the initial function; external disturbance $\omega(k)\in\mathbb{R}^q$ satisfies the condition

$$\textstyle\sum_{k=0}^{N}\omega^{\mathrm{T}}(k)\omega(k)<d, \tag{4}$$

where $d>0$ is a given number.

Definition 2.1. (Finite-time stability) Given positive constants c_1,c_2,N with $c_1<c_2,N\in\mathbb{Z}_+$ and a symmetric positive-definite matrix R, the discrete-time delay neural networks

$$\begin{aligned} x(k+1) &= Ax(k) + Wf(x(k)) + W_1 g(x(k-h(k))), \quad k \in \mathbb{Z}_+, \\ x(k) &= \varphi(k), \ k \in \{-h_2, -h_2+1, \dots, 0\}, \end{aligned} \tag{5}$$

is said to be finite-time stable w.r.t. (c_1, c_2, R, N) if

$$\max_{k \in \{-h_2, -h_2+1, \dots, 0\}} \varphi^{\mathrm{T}}(k) R \varphi(k) \le c_1 \implies x^{\mathrm{T}}(k) R x(k) < c_2 \quad \forall k \in \{1, 2, \dots, N\}.$$

Definition 2.2. (Finite-time boundedness) Given positive constants c_1, c_2, N with $c_1 < c_2, N \in \mathbb{Z}_+$ and a symmetric positive-definite matrix R, the discrete-time delay neural networks with disturbance

$$\begin{aligned} x(k+1) &= Ax(k) + Wf(x(k)) + W_1 g(x(k-h(k))) + C\omega(k), \quad k \in \mathbb{Z}_+, \\ x(k) &= \varphi(k), \ k \in \{-h_2, -h_2+1, \dots, 0\}, \end{aligned} \tag{6}$$

is said to be finite-time bounded w.r.t. (c_1, c_2, R, N) if

$$\max_{k \in \{-h_2, -h_2+1, \dots, 0\}} \varphi^{\mathrm{T}}(k) R \varphi(k) \le c_1 \implies x^{\mathrm{T}}(k) R x(k) < c_2 \quad \forall k \in \{1, 2, \dots, N\},$$

for all disturbances $\omega(k)$ satisfying (4).

Definition 2.3. ($\mathcal{H}_\infty$ finite-time boundedness) Given positive constants c_1, c_2, γ, N with $c_1 < c_2, N \in \mathbb{Z}_+$ and a symmetric positive-definite matrix R, system (1) is $\mathcal{H}_\infty$ finite-time bounded w.r.t. (c_1, c_2, R, N) if the following two conditions hold:

(i) System (6) is finite-time bounded w.r.t. (c_1, c_2, R, N).

(ii) Under zero initial condition (i.e., $\varphi(k) = 0 \ \forall k \in \{-h_2, -h_2+1, \dots, 0\}$), the output $z(k)$ satisfies

$$\textstyle\sum_{k=0}^{N} z^{\mathrm{T}}(k) z(k) \le \gamma \sum_{k=0}^{N} \omega^{\mathrm{T}}(k) \omega(k) \tag{7}$$

for all disturbances $\omega(k)$ satisfying (4).

Next, we introduce some technical propositions that will be used to prove main results.

Proposition 2.1 (Discrete Jensen Inequality, [15]). *For any matrix $M \in \mathbb{R}^{n\times n}, M = M^T > 0$, positive integers r_1, r_2 satisfying $r_1 \le r_2$, a vector function $\omega: \{r_1, r_1+1, \dots, r_2\} \to \mathbb{R}^n$, then*

$$\left(\sum_{i=r_1}^{r_2} \omega(i)\right)^{\mathrm{T}} \mathrm{M} \left(\sum_{i=r_1}^{r_2} \omega(i)\right) \le (r_2 - r_1 + 1) \sum_{i=r_1}^{r_2} \omega^{\mathrm{T}}(i) M \omega(i).$$

Proposition 2.2 (Reciprocally Convex Combination Lemma, [16, 17]). *Let $R \in \mathbb{R}^{n\times n}$ be a symmetric positive-definite matrix. Then for all vectors $\zeta_1, \zeta_2 \in \mathbb{R}^n$, scalars $\alpha_1 > 0, \alpha_2 > 0$ with $\alpha_1 + \alpha_2 = 1$ and a matrix $S \in \mathbb{R}^{n\times n}$ such that*

$$\begin{bmatrix} R & S \\ S^{\mathrm{T}} & R \end{bmatrix} \ge 0,$$

the following inequality holds

$$\frac{1}{\alpha_1} \zeta_1^{\mathrm{T}} R \zeta_1 + \frac{1}{\alpha_2} \zeta_2^{\mathrm{T}} R \zeta_2 \ \ge \ \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R & S \\ S^{\mathrm{T}} & R \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}.$$

Proposition 2.3 (Schur Complement Lemma, [18]). *Given constant matrices X, Y, Z with appropriate dimensions satisfying $X = X^T, \ Y = Y^T > 0$. Then*

$$X + Z^{\mathrm{T}} Y^{-1} Z < 0 \iff \begin{bmatrix} X & Z^{\mathrm{T}} \\ Z & -Y \end{bmatrix} < 0.$$

3. Main results

In this section, we investigate the $\mathcal{H}_\infty$ finite-time boundedness of discrete-time neural networks in the form of (1) with interval time-varying delay. It will be seen from the following theorem that reciprocally convex approach is employed in our derivation. Let's define $h_{12} = h_2 - h_1,\ y(k) = x(k+1) - x(k)$ and assume there exists a real constant $\tau > 0$ such that

$$\max_{k\in\{-h_2,-h_2+1,\dots,-1\}} y^{\mathrm{T}}(k)y(k) < \tau.$$

Before present main results, we define the following matrices

$$F = \mathrm{diag}\{a_1,\dots,a_n\},\ G = \mathrm{diag}\{b_1,\dots,b_n\},$$
$$\Omega_{11} = -\delta(P+S_1) + (h_{12}+1)Q + R_1,\ \Omega_{12} = \delta S_1,\ \Omega_{18} = AP,$$
$$\Omega_{19} = h_1^2(A-I)S_1,\ \Omega_{1,10} = h_{12}^2(A-I)S_2,\ \Omega_{1,11} = A_1^{\mathrm{T}},\ \Omega_{1,12} = F,$$
$$\Omega_{22} = \delta^{h_1}(-R_1 + R_2 - \delta S_2) - \delta S_1,\ \Omega_{23} = \Omega_{34} = \delta^{h_1+1}(S_2 - S),\ \Omega_{24} = \delta^{h_1+1}S,$$
$$\Omega_{33} = -\delta^{h_1}Q - \delta^{h_1+1}(2S_2 - S - S^{\mathrm{T}}),\ \Omega_{3,11} = D^{\mathrm{T}},\ \Omega_{3,13} = G,$$
$$\Omega_{44} = -\delta^{h_2}R_2 - \delta^{h_1+1}S_2,\ \Omega_{55} = \Omega_{66} = \Omega_{11,11} = \Omega_{12,12} = \Omega_{13,13} = -I,$$
$$\Omega_{58} = W^{\mathrm{T}}P,\ \Omega_{59} = h_1^2W^{\mathrm{T}}S_1,\ \Omega_{5,10} = h_{12}^2W^{\mathrm{T}}S_2,$$
$$\Omega_{68} = W_1^{\mathrm{T}}P,\ \Omega_{69} = h_1^2W_1^{\mathrm{T}}S_1,\ \Omega_{6,10} = h_{12}^2W_1^{\mathrm{T}}S_2,$$
$$\Omega_{77} = -\frac{\gamma}{\delta^N}I,\ \Omega_{78} = C^{\mathrm{T}}P,\ \Omega_{79} = h_1^2C^{\mathrm{T}}S_1,\ \Omega_{7,10} = h_{12}^2C^{\mathrm{T}}S_2,\ \Omega_{7,11} = C_1^{\mathrm{T}},$$
$$\Omega_{88} = -P,\ \Omega_{99} = -h_1^2S_1,\ \Omega_{10,10} = -h_{12}^2S_2,$$
$$\Omega_{ij} = 0 \text{ for any other } i,j:\ j>i,\ \Omega_{ij} = \Omega_{ji}^{\mathrm{T}}, i>j,$$
$$\rho_1 = \tfrac{1}{2}c_1(h_1+h_2)(h_{12}+1)\delta^{N+h_2},\ \rho_2 = \tfrac{1}{2}\tau h_{12}^2(h_1+h_2+1)\delta^{N+h_2},$$
$$\Lambda_{11} = \gamma d - c_2\delta\lambda_1,\ \Lambda_{12} = c_1\delta^{N+1}\lambda_2,\ \Lambda_{13} = \rho_1\lambda_3,\ \Lambda_{14} = c_1h_1\delta^{N+h_1}\lambda_4,$$
$$\Lambda_{15} = c_1h_{12}\delta^{N+h_2}\lambda_5,\ \Lambda_{16} = \tfrac{1}{2}\tau h_1^2(h_1+1)\delta^{N+h_1}\lambda_6,\ \Lambda_{17} = \rho_2\lambda_7,$$
$$\Lambda_{22} = -c_1\delta^{N+1}\lambda_2,\ \Lambda_{33} = -\rho_1\lambda_3,\ \Lambda_{44} = -c_1h_1\delta^{N+h_1}\lambda_4,$$
$$\Lambda_{55} = -c_1h_{12}\delta^{N+h_2}\lambda_5,\ \Lambda_{66} = -\tfrac{1}{2}\tau h_1^2(h_1+1)\delta^{N+h_1}\lambda_6,\ \Lambda_{77} = -\rho_2\lambda_7,$$
$$\Lambda_{ij} = 0 \text{ for any other } i,j:\ j>i,\ \Lambda_{ij} = \Lambda_{ji}^{\mathrm{T}}, i>j.$$

Theorem 3.1. *Given positive constants c_1, c_2, γ, N with $c_1 < c_2, N \in \mathbb{Z}_+$ and a symmetric positive-definite matrix R. System* (1) *is $\mathcal{H}_\infty$ finite-time bounded w.r.t. (c_1, c_2, R, N) if there exist symmetric positive definite matrices $P, Q, R_1, R_2, S_1, S_2 \in \mathbb{R}^{n\times n}$, a matrix $S \in \mathbb{R}^{n\times n}$ and positive scalars $\lambda_i,\ i = \overline{1,7}$, $\delta \geq 1$, such that the following matrix inequalities hold:*

$$\lambda_1 R < P < \lambda_2 R,\ Q < \lambda_3 R,\ R_1 < \lambda_4 R,\ R_2 < \lambda_5 R,\ S_1 < \lambda_6 I,\ S_2 < \lambda_7 I, \tag{8}$$

$$\Xi = \begin{bmatrix} S_2 & S \\ S^{\mathrm{T}} & S_2 \end{bmatrix} > 0, \tag{9}$$

$$\Omega = [\Omega_{ij}]_{13\times 13} < 0, \tag{10}$$

$$\Lambda = [\Lambda_{ij}]_{7\times 7} < 0. \tag{11}$$

Proof. Consider the following Lyapunov–Krasovskii functional:

$$V(k)=\sum_{i=1}^{4} V_i(k),$$

where

$V_1(k)=x^{\mathrm{T}}(k)Px(k),$

$V_2(k)=\sum_{s=-h_2+1}^{-h_1+1}\sum_{t=k-1+s}^{k-1}\delta^{k-1-t}x^{\mathrm{T}}(t)Qx(t),$

$V_3(k)=\sum_{s=k-h_1}^{k-1}\delta^{k-1-s}x^{\mathrm{T}}(s)R_1x(s)+\sum_{s=k-h_2}^{k-h_1-1}\delta^{k-1-s}x^{\mathrm{T}}(s)R_2x(s),$

$V_4(k)=\sum_{s=-h_1+1}^{0}\sum_{t=k-1+s}^{k-1}h_1\delta^{k-1-t}y^{\mathrm{T}}(t)S_1y(t)+\sum_{s=-h_2+1}^{-h_1}\sum_{t=k-1+s}^{k-1}h_{12}\delta^{k-1-t}y^{\mathrm{T}}(t)S_2y(t).$

Denoting

$$\eta(k):=[x^{\mathrm{T}}(k)\quad f^{\mathrm{T}}(x(k))\quad g^{\mathrm{T}}(x(k-h(k)))\quad \omega^{\mathrm{T}}(k)]^{\mathrm{T}},\qquad \Gamma:=[A\quad W\quad W_1\quad C]$$

and taking the difference variation of $V_i(k), i=1,\ldots,4$, we have

$$\begin{aligned}
V_1(k+1)-\delta V_1(k)&=x^{\mathrm{T}}(k+1)Px(k+1)-\delta x^{\mathrm{T}}(k)Px(k)\\
&=\begin{bmatrix}x(k)\\ f(x(k))\\ g(x(k-h(k)))\\ \omega(k)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}A^{\mathrm{T}}\\ W^{\mathrm{T}}\\ W_1^{\mathrm{T}}\\ C^{\mathrm{T}}\end{bmatrix}P[A\quad W\quad W_1\quad C]\begin{bmatrix}x(k)\\ f(x(k))\\ g(x(k-h(k)))\\ \omega(k)\end{bmatrix}\\
&\quad -\delta x^{\mathrm{T}}(k)Px(k)\\
&=\eta^{\mathrm{T}}(k)\Gamma^{\mathrm{T}}P\Gamma\eta(k)-\delta x^{\mathrm{T}}(k)Px(k), \qquad (12)
\end{aligned}$$

$$\begin{aligned}
V_2(k+1)-\delta V_2(k)&=\sum_{s=-h_2+1}^{-h_1+1}\sum_{t=k+s}^{k}\delta^{k-t}x^{\mathrm{T}}(t)Qx(t)-\sum_{s=-h_2+1}^{-h_1+1}\sum_{t=k-1+s}^{k-1}\delta^{k-t}x^{\mathrm{T}}(t)Qx(t)\\
&=\sum_{s=-h_2+1}^{-h_1+1}\Bigg[x^{\mathrm{T}}(k)Qx(k)+\sum_{t=k+s}^{k-1}\delta^{k-t}x^{\mathrm{T}}(t)Qx(t)-\sum_{t=k+s}^{k-1}\delta^{k-t}x^{\mathrm{T}}(t)Qx(t)\\
&\quad-\delta^{k-(k-1+s)}x^{\mathrm{T}}(k-1+s)Qx(k-1+s)\Bigg]\\
&=\sum_{s=-h_2+1}^{-h_1+1}[x^{\mathrm{T}}(k)Qx(k)-\delta^{1-s}x^{\mathrm{T}}(k-1+s)Qx(k-1+s)]\\
&=(h_2-h_1+1)x^{\mathrm{T}}(k)Qx(k)-\sum_{s=-h_2+1}^{-h_1+1}\delta^{1-s}x^{\mathrm{T}}(k-1+s)Qx(k-1+s)
\end{aligned}$$

$$= (h_{12}+1)x^{\mathrm{T}}(k)Qx(k) - \sum_{s=k-h_2}^{k-h_1} \delta^{k-s}x^{\mathrm{T}}(s)Qx(s)$$
$$\leq (h_{12}+1)x^{\mathrm{T}}(k)Qx(k) - \delta^{k-(k-h(k))}x^{\mathrm{T}}(k-h(k))Qx(k-h(k))$$
$$\leq (h_{12}+1)x^{\mathrm{T}}(k)Qx(k) - \delta^{h_1}x^{\mathrm{T}}(k-h(k))Qx(k-h(k)), \quad (13)$$

$$V_3(k+1) - \delta V_3(k) = \sum_{s=k+1-h_1}^{k} \delta^{k-s}x^{\mathrm{T}}(s)R_1x(s) - \sum_{s=k-h_1}^{k-1} \delta^{k-s}x^{\mathrm{T}}(s)R_1x(s)$$
$$+ \sum_{s=k+1-h_2}^{k-h_1} \delta^{k-s}x^{\mathrm{T}}(s)R_2x(s) - \sum_{s=k-h_2}^{k-h_1-1} \delta^{k-s}x^{\mathrm{T}}(s)R_2x(s)$$
$$= x^{\mathrm{T}}(k)R_1x(k) + x^{\mathrm{T}}(k-h_1)[\delta^{h_1}(-R_1+R_2)]x(k-h_1)$$
$$-\delta^{h_2}x^{\mathrm{T}}(k-h_2)R_2x(k-h_2), \quad (14)$$

$$V_4(k+1) - \delta V_4(k) = \sum_{s=-h_1+1}^{0} \sum_{t=k+s}^{k} h_1\delta^{k-t}y^{\mathrm{T}}(t)S_1y(t) - \sum_{s=-h_1+1}^{0} \sum_{t=k-1+s}^{k-1} h_1\delta^{k-t}y^{\mathrm{T}}(t)S_1y(t)$$
$$+ \sum_{s=-h_2+1}^{-h_1} \sum_{t=k+s}^{k} h_{12}\delta^{k-t}y^{\mathrm{T}}(t)S_2y(t) - \sum_{s=-h_2+1}^{-h_1} \sum_{t=k-1+s}^{k-1} h_{12}\delta^{k-t}y^{\mathrm{T}}(t)S_2y(t)$$
$$= \sum_{s=-h_1+1}^{0} h_1[y^{\mathrm{T}}(k)S_1y(k) - \delta^{1-s}y^{\mathrm{T}}(k-1+s)S_1y(k-1+s)]$$
$$+ \sum_{s=-h_2+1}^{-h_1} h_{12}[y^{\mathrm{T}}(k)S_2y(k) - \delta^{1-s}y^{\mathrm{T}}(k-1+s)S_2y(k-1+s)]$$
$$= h_1^2y^{\mathrm{T}}(k)S_1y(k) - h_1 \sum_{s=-h_1+1}^{0} \delta^{1-s}y^{\mathrm{T}}(k-1+s)S_1y(k-1+s)$$
$$+ h_{12}^2y^{\mathrm{T}}(k)S_2y(k) - h_{12} \sum_{s=-h_2+1}^{-h_1} \delta^{1-s}y^{\mathrm{T}}(k-1+s)S_2y(k-1+s)$$
$$= y^{\mathrm{T}}(k)[h_1^2S_1 + h_{12}^2S_2]y(k) - h_1 \sum_{s=k-h_1}^{k-1} \delta^{k-s}y^{\mathrm{T}}(s)S_1y(s)$$
$$- h_{12} \sum_{s=k-h_2}^{k-1-h_1} \delta^{k-s}y^{\mathrm{T}}(s)S_2y(s)$$
$$\leq y^{\mathrm{T}}(k)[h_1^2S_1 + h_{12}^2S_2]y(k) - h_1\delta \sum_{s=k-h_1}^{k-1} y^{\mathrm{T}}(s)S_1y(s)$$

$$- h_{12}\delta^{h_1+1}\textstyle\sum_{s=k-h_2}^{k-1-h_1} y^{\mathrm{T}}(s)S_2y(s) \tag{15}$$

By Proposition 2.1,

$$\begin{aligned}-h_1\delta\sum_{s=k-h_1}^{k-1} y^{\mathrm{T}}(s)S_1y(s) &\le -\frac{h_1\delta}{(k-1)-(k-h_1)+1}\left[\sum_{s=k-h_1}^{k-1} y(s)\right]^{\mathrm{T}} S_1\left[\sum_{s=k-h_1}^{k-1} y(s)\right]\\ &= -\delta[x(k)-x(k-h_1)]^{\mathrm{T}}S_1[x(k)-x(k-h_1)],\end{aligned} \tag{16}$$

$$\begin{aligned}&- h_{12}\delta^{h_1+1}\sum_{s=k-h_2}^{k-1-h_1} y^{\mathrm{T}}(s)S_2y(s)\\ &\quad= - h_{12}\delta^{h_1+1}\left[\sum_{s=k-h(k)}^{k-h_1-1} y^{\mathrm{T}}(s)S_2y(s) + \sum_{s=k-h_2}^{k-h(k)-1} y^{\mathrm{T}}(s)S_2y(s)\right]\\ &\quad\le \delta^{h_1+1}\Bigg(-\frac{h_{12}}{(k-h_1-1)-(k-h(k))+1}\left[\sum_{s=k-h(k)}^{k-h_1-1} y(s)\right]^{\mathrm{T}} S_2\left[\sum_{s=k-h(k)}^{k-h_1-1} y(s)\right]\\ &\qquad-\frac{h_{12}}{(k-h(k)-1)-(k-h_2)+1}\left[\sum_{s=k-h_2}^{k-h(k)-1} y(s)\right]^{\mathrm{T}} S_2\left[\sum_{s=k-h_2}^{k-h(k)-1} y(s)\right]\Bigg)\\ &\quad= \delta^{h_1+1}\left(-\frac{1}{(h(k)-h_1)/h_{12}}\zeta_1^{\mathrm{T}}S_2\zeta_1-\frac{1}{(h_2-h(k))/h_{12}}\zeta_2^{\mathrm{T}}S_2\zeta_2\right)\end{aligned}$$

where $\zeta_1 = x(k-h_1)-x(k-h(k))$ and $\zeta_2 = x(k-h(k))-x(k-h_2)$. From note that

$$\frac{h(k)-h_1}{h_{12}} \ge 0,\quad \frac{h_2-h(k)}{h_{12}} \ge 0,\quad \frac{h(k)-h_1}{h_{12}}+\frac{h_2-h(k)}{h_{12}} = 1,$$

$$\zeta_1 = 0 \text{ if } (h(k)-h_1)/h_{12} = 0 \quad\text{and}\quad \zeta_2 = 0 \text{ if } (h_2-h(k))/h_{12} = 0,$$

and the hypothesis (9), Proposition 2.2 gives us

$$\begin{aligned}-h_{12}\delta^{h_1+1}\sum_{s=k-h_2}^{k-1-h_1} y^{\mathrm{T}}(s)S_2y(s) &\le -\delta^{h_1+1}\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}S_2 & S\\ S^{\mathrm{T}} & S_2\end{bmatrix}\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix}\\ &= -\delta^{h_1+1}\left[\zeta_1^{\mathrm{T}}S_2\zeta_1+\zeta_1^{\mathrm{T}}S\zeta_2+\zeta_2^{\mathrm{T}}S^{\mathrm{T}}\zeta_1+\zeta_2^{\mathrm{T}}S_2\zeta_2\right].\end{aligned} \tag{17}$$

Substitute (16), (17) into (15) and combine with (12)-(14), we get

$$\begin{aligned}V(k+1)-\delta V(k) \le{}& \eta^{\mathrm{T}}(k)\Gamma^{\mathrm{T}}P\Gamma\eta(k)+x^{\mathrm{T}}(k)[-\delta P+(h_{12}+1)Q+R_1-\delta S_1]x(k)\\ &+x^{\mathrm{T}}(k)[2\delta S_1]x(k-h_1)\\ &+x^{\mathrm{T}}(k-h(k))[-\delta^{h_1}Q-\delta^{h_1+1}(2S_2-S-S^{\mathrm{T}})]x(k-h(k))\\ &+x^{\mathrm{T}}(k-h(k))[2\delta^{h_1+1}(S_2-S^{\mathrm{T}})]x(k-h_1)\\ &+x^{\mathrm{T}}(k-h(k))[2\delta^{h_1+1}(S_2-S)]x(k-h_2)\\ &+x^{\mathrm{T}}(k-h_1)[\delta^{h_1}(-R_1+R_2)-\delta S_1-\delta^{h_1+1}S_2]x(k-h_1)\end{aligned}$$

$$
\begin{aligned}
&+ x^{\mathrm{T}}(k-h_1)[2\delta^{h_1+1}S]x(k-h_2)\\
&+ x^{\mathrm{T}}(k-h_2)[-\delta^{h_2}R_2-\delta^{h_1+1}S_2]x(k-h_2)+\ y^{\mathrm{T}}(k)[h_1^2S_1+h_{12}^2S_2]y(k)\\
&+ z^{\mathrm{T}}(k)z(k)-\frac{\gamma}{\delta^N}\omega^{\mathrm{T}}(k)\omega(k)+\frac{\gamma}{\delta^N}\omega^{\mathrm{T}}(k)\omega(k)-z^{\mathrm{T}}(k)z(k).\\
=\ &\eta^{\mathrm{T}}(k)\Gamma^{\mathrm{T}}P\Gamma\eta(k)\\
&+ x^{\mathrm{T}}(k)[-\delta P+(h_{12}+1)Q+R_1-\delta S_1+A_1^{\mathrm{T}}A_1]x(k)\\
&+ x^{\mathrm{T}}(k)[2\delta S_1]x(k-h_1)+x^{\mathrm{T}}(k)[2A_1^{\mathrm{T}}D]x(k-h(k))\\
&+ x^{\mathrm{T}}(k)[2A_1^{\mathrm{T}}C_1]\omega(k)\\
&+ x^{\mathrm{T}}(k-h_1)[\delta^{h_1}(-R_1+R_2)-\delta S_1-\delta^{h_1+1}S_2]x(k-h_1)\\
&+ x^{\mathrm{T}}(k-h_1)[2\delta^{h_1+1}(S_2-S)]x(k-h(k))\\
&+ x^{\mathrm{T}}(k-h_1)[2\delta^{h_1+1}S]x(k-h_2)\\
&+ x^{\mathrm{T}}(k-h(k))[-\delta^{h_1}Q-\delta^{h_1+1}(2S_2-S-S^{\mathrm{T}})+D^{\mathrm{T}}D]x(k-h(k))\\
&+ x^{\mathrm{T}}\big(k-h(k)\big)[2\delta^{h_1+1}(S_2-S)]x(k-h_2)+\ x^{\mathrm{T}}(k-h(k))[2D^{\mathrm{T}}C_1]\omega(k)\\
&+ x^{\mathrm{T}}(k-h_2)[-\delta^{h_2}R_2-\delta^{h_1+1}S_2]x(k-h_2)\\
&+ \omega^{\mathrm{T}}(k)\left[-\frac{\gamma}{\delta^N}I+C_1^{\mathrm{T}}C_1\right]\omega(k)+y^{\mathrm{T}}(k)[h_1^2S_1+h_{12}^2S_2]y(k)\\
&+ \tfrac{\gamma}{\delta^N}\omega^{\mathrm{T}}(k)\omega(k)-z^{\mathrm{T}}(k)z(k).
\end{aligned}
\tag{18}
$$

Besides, from (2), it can be verified that

$$
\begin{aligned}
0\ &\le -f^{\mathrm{T}}(x(k))f(x(k))+x^{\mathrm{T}}(k)F^2x(k),\\
0\ &\le -g^{\mathrm{T}}(x(k-h(k)))g(x(k-h(k)))+x^{\mathrm{T}}(k-h(k))G^2x(k-h(k)).
\end{aligned}
\tag{19}
$$

Moreover, by setting

$$\xi(k) \coloneqq [x^{\mathrm{T}}(k)\quad x^{\mathrm{T}}(k-h_1)\quad x^{\mathrm{T}}(k-h(k))\quad x^{\mathrm{T}}(k-h_2)\quad f^{\mathrm{T}}(x(k))\quad g^{\mathrm{T}}(x(k-h(k)))\quad \omega^{\mathrm{T}}(k)]^{\mathrm{T}}$$

$$\Upsilon := \begin{bmatrix} PA & 0 & 0 & 0 & PW & PW_1 & PC\\ h_1^2S_1(A-I) & 0 & 0 & 0 & h_1^2S_1W & h_1^2S_1W_1 & h_1^2S_1C\\ h_{12}^2S_2(A-I) & 0 & 0 & 0 & h_{12}^2S_2W & h_{12}^2S_2W_1 & h_{12}^2S_2C \end{bmatrix},$$

we can rewrite

$\eta^{\mathrm{T}}(k)\Gamma^{\mathrm{T}}P\Gamma\eta(k)+y^{\mathrm{T}}(k)[h_1^2S_1+h_{12}^2S_2]y(k)$

$$= \ \xi^{\mathrm{T}}(k)\begin{bmatrix} A^{\mathrm{T}}\\ 0\\ 0\\ 0\\ W^{\mathrm{T}}\\ W_1^{\mathrm{T}}\\ C^{\mathrm{T}} \end{bmatrix} P[A\quad 0\quad 0\quad 0\quad W\quad W_1\quad C]\xi(k)$$

$$+\xi^{\mathrm{T}}(k)\begin{bmatrix}(A-I)^{\mathrm{T}}\\0\\0\\0\\W^{\mathrm{T}}\\W_1^{\mathrm{T}}\\C^{\mathrm{T}}\end{bmatrix}[h_1^2S_1+h_{12}^2S_2][(A-I)\quad 0\quad 0\quad 0\quad W\quad W_1\quad C]\xi(k)$$

$$=\ \xi^{\mathrm{T}}(k)\Upsilon^{\mathrm{T}}\begin{bmatrix}P & 0 & 0\\0 & h_1^2S_1 & 0\\0 & 0 & h_{12}^2S_2\end{bmatrix}^{-1}\Upsilon\xi(k). \tag{20}$$

Consequently, combining (18), (19) and (20) gives

$$V(k+1)-\delta V(k)\le\xi^{\mathrm{T}}(k)\left(\Phi+\Upsilon^{\mathrm{T}}\begin{bmatrix}P & 0 & 0\\0 & h_1^2S_1 & 0\\0 & 0 & h_{12}^2S_2\end{bmatrix}^{-1}\Upsilon\right)\xi(k)$$
$$+\frac{\gamma}{\delta^N}\omega^{\mathrm{T}}(k)\omega(k)-z^{\mathrm{T}}(k)z(k), \tag{21}$$

where

$$\Phi:=\begin{bmatrix}\Omega_{11}+A_1^{\mathrm{T}}A_1+F^2 & \Omega_{12} & A_1^{\mathrm{T}}D & 0 & 0 & 0 & A_1^{\mathrm{T}}C_1\\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} & 0 & 0 & 0\\ * & * & \Omega_{33}+D^{\mathrm{T}}D+G^2 & \Omega_{34} & 0 & 0 & D^{\mathrm{T}}C_1\\ * & * & * & \Omega_{44} & 0 & 0 & 0\\ * & * & * & * & -I & 0 & 0\\ * & * & * & * & * & -I & 0\\ * & * & * & * & * & * & -\frac{\gamma}{\delta^N}I+C_1^{\mathrm{T}}C_1\end{bmatrix}.$$

Next, by using Proposition 2.3, it can be deduced that

$$\Phi+\Upsilon^{\mathrm{T}}\begin{bmatrix}P & 0 & 0\\0 & h_1^2S_1 & 0\\0 & 0 & h_{12}^2S_2\end{bmatrix}^{-1}\Upsilon<0\quad\Leftrightarrow\quad\Omega<0.$$

This, together with (21), gives

$$V(k+1)-\delta V(k)\le\frac{\gamma}{\delta^N}\omega^{\mathrm{T}}(k)\omega(k)\quad\forall k\in\mathbb{Z}_+.$$

This estimation can be rewritten as

$$V(k)\le\delta V(k-1)+\frac{\gamma}{\delta^N}\omega^{\mathrm{T}}(k-1)\omega(k-1)\quad\forall k\in\mathbb{N}.$$

By iteration, and take assumption (4) into account, it follows that

$$V(k)\le\delta^kV(0)+\frac{\gamma}{\delta^N}\sum_{s=0}^{k-1}\delta^{k-1-s}\omega^{\mathrm{T}}(s)\omega(s)$$
$$\le\delta^NV(0)+\frac{\gamma}{\delta^N}\delta^{N-1}\sum_{s=0}^{N-1}\omega^{\mathrm{T}}(s)\omega(s)$$

$$< \delta^N V(0) + \tfrac{\gamma}{\delta} d \quad \forall k \in \mathbb{Z}_+. \tag{22}$$

From assumption (8) and $x(k) = \varphi(k)\ \ \forall k \in \{-h_2, -h_2+1, \dots, 0\}$, it is obvious that

$$\begin{aligned}
V(0) &= x^{\mathrm{T}}(0)Px(0) + \sum_{s=-h_2+1}^{-h_1+1} \sum_{t=-1+s}^{-1} \delta^{-1-t} x^{\mathrm{T}}(t)Qx(t) \\
&\quad + \sum_{s=-h_1}^{-1} \delta^{-1-s} x^{\mathrm{T}}(s)R_1x(s) + \sum_{s=-h_2}^{-h_1-1} \delta^{-1-s} x^{\mathrm{T}}(s)R_2x(s) \\
&\quad + \sum_{s=-h_1+1}^{0} \sum_{t=-1+s}^{-1} h_1\delta^{-1-t} y^{\mathrm{T}}(t)S_1y(t) + \sum_{s=-h_2+1}^{-h_1} \sum_{t=-1+s}^{-1} h_{12}\delta^{-1-t} y^{\mathrm{T}}(t)S_2y(t) \\
&< \lambda_2 x^{\mathrm{T}}(0)Rx(0) + \lambda_3\delta^{h_2-1} \sum_{s=-h_2+1}^{-h_1+1} \sum_{t=-1+s}^{-1} x^{\mathrm{T}}(t)Rx(t) \\
&\quad + \lambda_4\delta^{h_1-1} \sum_{s=-h_1}^{-1} x^{\mathrm{T}}(s)Rx(s) + \lambda_5\delta^{h_2-1} \sum_{s=-h_2}^{-h_1-1} x^{\mathrm{T}}(s)Rx(s) \\
&\quad + \lambda_6 h_1\delta^{h_1-1} \sum_{s=-h_1+1}^{0} \sum_{t=-1+s}^{-1} y^{\mathrm{T}}(t)y(t) + \lambda_7 h_{12}\delta^{h_2-1} \sum_{s=-h_2+1}^{-h_1} \sum_{t=-1+s}^{-1} y^{\mathrm{T}}(t)y(t) \\
&\le \left[\lambda_2 + \lambda_3\delta^{h_2-1}\frac{h_2(h_2+1) - h_1(h_1-1)}{2} + \lambda_4\delta^{h_1-1}h_1 + \lambda_5\delta^{h_2-1}(h_2-h_1)\right]c_1 \\
&\quad + \left[\lambda_6\delta^{h_1-1}h_1\tfrac{h_1(h_1+1)}{2} + \lambda_7\delta^{h_2-1}h_{12}\tfrac{h_2(h_2+1)-h_1(h_1+1)}{2}\right]\tau.
\end{aligned} \tag{23}$$

From (22) and (23), we obtain

$$V(k) < \delta^N \sigma + \tfrac{\gamma}{\delta} d \quad \forall k \in \mathbb{Z}_+. \tag{24}$$

where

$$\begin{aligned}
\sigma := &\left[\lambda_2 + \lambda_3\delta^{h_2-1}\frac{h_2(h_2+1) - h_1(h_1-1)}{2} + \lambda_4\delta^{h_1-1}h_1 + \lambda_5\delta^{h_2-1}(h_2-h_1)\right]c_1 \\
&+ \left[\lambda_6\delta^{h_1-1}h_1\frac{h_1(h_1+1)}{2} + \lambda_7\delta^{h_2-1}h_{12}\frac{h_2(h_2+1) - h_1(h_1+1)}{2}\right]\tau.
\end{aligned}$$

On the other hand, from (8) it follows that

$$V(k) \ge x^{\mathrm{T}}(k)Px(k) \ge \lambda_1 x^{\mathrm{T}}(k)Rx(k) \quad \forall k \in \mathbb{Z}_+. \tag{25}$$

Note that by Proposition 2.3, the inequality (11) is equivalent to

$$\begin{aligned}
\gamma d - c_2\delta\lambda_1 + c_1\delta^{N+1}\lambda_2 + \rho_1\lambda_3 + c_1h_1\delta^{N+h_1}\lambda_4 + c_1h_{12}\delta^{N+h_2}\lambda_5 \\
+ \frac{1}{2}\tau h_1^2(h_1+1)\delta^{N+h_1}\lambda_6 + \rho_2\lambda_7 < 0,
\end{aligned}$$

or

$$\gamma d - c_2\delta\lambda_1 + \delta^{N+1}\sigma < 0. \tag{26}$$

Consequently, we get from (24), (25) and (26) that:

$$x^{\mathrm{T}}(k)Rx(k) < \frac{1}{\delta\lambda_1}[\delta^{N+1}\sigma + \gamma d] < c_2 \quad \forall k = 1,2,\ldots,N.$$

This implies that system (6) is finite-time bounded with respect to (c_1, c_2, R, N). To complete the proof, it remains to show the finite-time γ-level condition (7). For this, bearing (21) in mind, we see that

$$V(k+1) \le \delta V(k) + \frac{\gamma}{\delta^N}\omega^{\mathrm{T}}(k)\omega(k) - z^{\mathrm{T}}(k)z(k) \quad \forall k \in \mathbb{Z}_+$$

and by iteration, the following estimate holds

$$0 \le V(k) \le \delta^k V(0) + \sum_{s=0}^{k-1} \delta^{k-1-s}\left[\frac{\gamma}{\delta^N}\omega^{\mathrm{T}}(s)\omega(s) - z^{\mathrm{T}}(s)z(s)\right]. \tag{27}$$

Under zero initial condition, it is clear that $V(0) = 0$, thus (27) implies

$$0 \le \sum_{s=0}^{k-1} \delta^{k-1-s}\left[\frac{\gamma}{\delta^N}\omega^{\mathrm{T}}(s)\omega(s) - z^{\mathrm{T}}(s)z(s)\right]$$

$$\Longrightarrow \sum_{s=0}^{k-1} \delta^{k-1-s} z^{\mathrm{T}}(s)z(s) \le \sum_{s=0}^{k-1} \delta^{k-1-s}\frac{\gamma}{\delta^N}\omega^{\mathrm{T}}(s)\omega(s).$$

Let $k = N+1$, we have

$$\sum_{s=0}^{N} \delta^{N-s} z^{\mathrm{T}}(s)z(s) \le \gamma \sum_{s=0}^{N} \frac{\delta^{N-s}}{\delta^N}\omega^{\mathrm{T}}(s)\omega(s). \tag{28}$$

Note that $1 \le \delta^{N-s} \le \delta^N \ \forall s \in \{0,1,\ldots,N\}$, (28) immediately yields

$$\sum_{s=0}^{N} z^{\mathrm{T}}(s)z(s) \le \gamma \sum_{s=0}^{N} \omega^{\mathrm{T}}(s)\omega(s).$$

This estimation holds for all non-zero exogenous disturbance $\omega(k)$ satisfying (4) and hence the condition (7) is derived. This completes the proof of the theorem. ■

Corollary 3.1. *Given positive constants c_1, c_2, γ, N with $c_1 < c_2, N \in \mathbb{Z}_+$ and a symmetric positive-definite matrix R. System* (5) *is finite-time stable w.r.t. (c_1, c_2, R, N) if there exist symmetric positive definite matrices $P, Q, R_1, R_2, S_1, S_2 \in \mathbb{R}^{n\times n}$, a matrix $S \in \mathbb{R}^{n\times n}$ and positive scalars $\lambda_i,\ i = \overline{1,7}$, $\delta \ge 1$, such that the LMIs* (8), (9) *and the following matrix inequalities hold*

$$\bar{\Omega} = \left[\bar{\Omega}_{ij}\right]_{11\times 11} < 0, \tag{29}$$

$$\bar{\Lambda} = \left[\bar{\Lambda}_{ij}\right]_{7\times 7} < 0, \tag{30}$$

where $\bar{\Omega}$ is derived from Ω by deleting the 7th and 11th rows and columns and $\bar{\Lambda}_{11} = -c_2\delta\lambda_1$, $\bar{\Lambda}_{ij} = \Lambda_{ij}$ for any other i, j.

Proof. The proof is similar to that of Theorem 3.1, thus is omitted. ■

Remark 3.1. As in papers [6, 13, 14], to prove Theorem 3.1 (and Corollary 3.1), we construct a set of adjusted Lyapunov–Krasovskii functionals involving variable ratios δ^{k-1-s} and δ^{k-1-t}. By doing so, we do not need to transform the original system into two interconnected subsystems as the authors did in [7] that the obtained conditions (8), (10), (11) of Theorem 3.1 and (29), (30) of Corollary 3.1 are still in the form of matrix inequalities as in [7]. The parameter δ has the role as an adjustable parameter and (10)-(11), (29)-(30) will become LMIs when we fix this parameter, so they can be easily programmed and calculated by using the LMI toolbox in MATLAB [19]. This is also a remarkable advantage of our two above results in comparison with: condition (5) in [6] and conditions (45), (56) in [13].

Remark 3.2. Employing unknowns and free-weighting matrices will complicate the system analysis and significantly increase the computational demand. Meanwhile, an outstanding advantage of reciprocally convex combination technique is that it can significantly reduce the number of decision variables compared to other methods [16, 17]. As a result, in this paper, based on reciprocally convex combination technique, we used minimum number of variables, e.g., (8)-(11) and (29), (30) have exactly one free-weighting matrix. Consequently, our criteria are more compact and effective in comparison with others. This advantage will be illustrated by means of the following examples.

Example 3.1. Consider the system (1), where

$$A = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.7 \end{bmatrix}, \quad W = \begin{bmatrix} -0.025 & 0.025 \\ 0.02 & 0.035 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.05 & 0.025 \\ -0.05 & 0.025 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.05 \\ 0.15 \end{bmatrix}, \quad A_1 = [0.35 \quad -0.25], \quad D = [0.2 \quad -0.15], \quad C_1 = [0.1],$$

$$F = \begin{bmatrix} 0.45 & 0 \\ 0 & 0.35 \end{bmatrix}, \quad G = \begin{bmatrix} 0.35 & 0 \\ 0 & 0.25 \end{bmatrix}, \quad R = \begin{bmatrix} 1.25 & 0 \\ 0 & 1.35 \end{bmatrix},$$

$$h(k) = 2 + 12\sin^2\frac{k\pi}{2}, k \in \mathbb{Z}_+.$$

For given $h_1 = 2,\ h_2 = 14,\ N = 90,\ d = 1,\ \tau = 1,\ c_1 = 1,\ c_2 = 9$ and $\gamma = 1$, the LMIs (8)-(11) are feasible with $\delta = 1.0001$ and

$$P = \begin{bmatrix} 18.1478 & -5.5689 \\ -5.5689 & 18.8968 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.0533 & -0.0506 \\ -0.0506 & 0.0481 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0.4532 & -0.0187 \\ -0.0187 & 0.1914 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0.2154 & -0.0044 \\ -0.0044 & 0.0852 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0.0030 & 0.0002 \\ 0.0002 & 0.0023 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.0475 & 0.0019 \\ 0.0019 & 0.0171 \end{bmatrix},$$

$$S = \begin{bmatrix} -0.0474 & -0.0022 \\ -0.0017 & -0.0170 \end{bmatrix}, \quad \lambda_1 = 9.9628, \quad \lambda_2 = 18.5547, \quad \lambda_3 = 0.0783,$$

$$\lambda_4 = 0.3645, \quad \lambda_5 = 0.1726, \quad \lambda_6 = 0.0036, \quad \lambda_7 = 0.0476.$$

Hence, by Theorem 3.1, the system is $\mathcal{H}_\infty$ finite-time bounded w.r.t. $(1, 9, R, 90)$.

Example 3.2. Consider the nominal system (5) with matrices A, W, W_1, F, G, R are the same as in Example 3.1. Then, with parameters $h_1, N, d, \tau, c_1, c_2$ and γ having the exact same value as in Example 3.1 except $h_2 = 25$, the LMIs (8), (9), (29) and (30) are feasible with $\delta = 1.0001$ and

$$P = \begin{bmatrix} 27.0262 & 2.2062 \\ 2.2062 & 33.9481 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.0702 & 0.0037 \\ 0.0037 & 0.0528 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0.3156 & 0.1518 \\ 0.1518 & 1.6792 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0.1365 & 0.0074 \\ 0.0074 & 0.1819 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0.7089 & 0.0199 \\ 0.0199 & 0.6848 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.0165 & -0.0002 \\ -0.0002 & 0.0139 \end{bmatrix},$$

$$S = \begin{bmatrix} -0.0162 & 0.0010 \\ 0.0010 & -0.0054 \end{bmatrix}, \quad \lambda_1 = 20.8563, \quad \lambda_2 = 26.6829, \quad \lambda_3 = 0.0593,$$

$$\lambda_4 = 2.0304, \quad \lambda_5 = 0.2069, \quad \lambda_6 = 1.0048, \quad \lambda_7 = 0.0168.$$

For this reason, Corollary 3.1 enable us to assert that the system is finite-time stable w.r.t. $(1, 9, R, 90)$. Figure 1 shows the response solution with the initial condition

$$\varphi(k) = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix} \quad \forall k \in \{-25, -24, \dots, 0\}.$$

Remark 3.3. It is well-known that improved conditions can be derived by using tighter refined Jensen summation inequality, see [20] and references therein. However, unlike the linear systems mentioned in [20], system (1) is a nonlinear system so the system analysis is generally more complex. Therefore, for technical reasons, refined Jensen summation inequalities have not been utilized in this paper. We believe that using these tools may be a good idea for improving the results mentioned above, but this also leads to exciting new challenges that need to be overcome in our future studies.

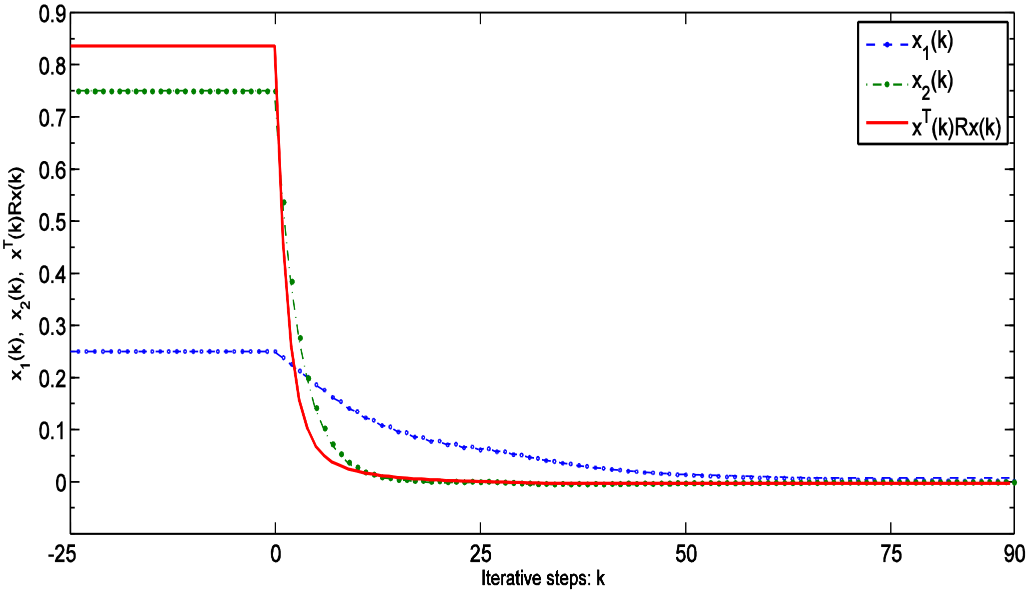

Figure 1. Response solution of the system in Example 3.2.

4. Conclusion

In this paper, we investigate the finite-time stability and $\mathcal{H}_\infty$ performance for a class of discrete-time neural networks subjected to interval-like time-varying delay and norm-bounded disturbances. By constructing a set of improved Lyapunov–Krasovskii functionals and using reciprocally convex approach, delay-dependent sufficient conditions are obtained which can be easily calculated by the LMI Toolbox in MATLAB.

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