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Original Article

Weak Laws of Large Numbers for Negatively Superadditive Dependent Random Vectors in Hilbert Spaces

Bui Khanh Hang*, Tran Manh Cuong, Ta Cong Son

VNU University of Science, 334 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam

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Abstract: Let $\{X_n, n \in \mathbb{N}\}$ be a sequence of negatively superadditive dependent random vectors taking values in a real separable Hilbert space. This paper presents some results on weak laws of large numbers for weighted sums (with or without random indices) of $\{X_n, n \in \mathbb{N}\}$.

Keywords: Large numbers, negatively superadditive dependent random vectors, Hilbert space.

1. Introduction

The weak laws of large numbers for weighted sums (with or without random indices) for random variables are studied by many authors (see, e.g., [1-5]). Recently, Hien and Thanh [6] obtained the weak law of large numbers for sums of negatively associated random vectors in Hilbert spaces. Dung et al. [7] established the weak laws of large numbers for weighted pairwise negative quadrant dependent random vectors in Hilbert spaces. In this paper, we investigate weak laws of large numbers for randomly weighted sums (with or without random indices) of sequences of negatively superadditive dependent random vectors in Hilbert spaces. We start with the definitions of negatively associated random variables and negatively superadditive dependent (NSD) random variables.

Let us consider a sequence $\{X_n, n \geq 1\}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. A finite family $\{X_1, \ldots, X_n\}$ is said to be negatively associated (NA) if for any disjoint subsets A, B of $\{1, \ldots, n\}$ and any real coordinate-wise nondecreasing functions f on $\mathbb{R}^{|A|}$, g on $\mathbb{R}^{|B|}$,

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*Corresponding author.
Email address: khanhhang.bui@gmail.com

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$$\text{Cov}(f(X_i, i \in A), g(X_j, j \in B)) \leq 0$$

whenever the covariance exists, where $|A|$ denotes the cardinality of A.

A function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is called superadditive if

$$\Phi(x \vee y) + \Phi(x \wedge y) \geq \Phi(x) + \Phi(y)$$

for all $x, y \in \mathbb{R}^n$, where $\vee$ is for componentwise maximum and $\wedge$ is for componentwise minimum.

The concept of negatively superadditive dependent random variables was introduced by Hu [8] based on the class of superadditive functions. A random vector $X = (X_1, X_2, ..., X_n)$ is said to be NSD random variables if

$$E\Phi(X_1, X_2, ..., X_n) \leq E\Phi(X_1^*, X_2^*, ..., X_n^*) \quad (1)$$

where $X_1^*, X_2^*, ..., X_n^*$ are independent with X_i^* and X_i having the same distribution for each i, and Φ is a superadditive function such that the expectations in (1) exist. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be NSD if for every $n \geq 1$, $(X_1, X_2, ..., X_n)$ is NSD.

Son et al. [9] gave the concept of NSD random vectors with values in Hilbert spaces. Now we recall the concept of NSD random vectors taking values in Hilbert spaces. Let H be a real separable Hilbert space with the norm $\|.\|$ generated by an inner product $\langle \cdot, \cdot \rangle$ and let $\{e_k, k \geq 1\}$ be an orthonormal basis in H.

Definition 1.1 A sequence $\{X_n, n \geq 1\}$ of H-valued random vectors is said to be NSD if for any $j \in B$, the sequence of random variables $\{\langle X_n, e_j \rangle, n \geq 1\}$ is NSD.

The following lemma plays an essential role in our main results.

Lemma 1.2. Let $\{X_n, n \geq 1\}$ be a sequence of H-valued NSD random vectors with mean 0 and finite second moments. Then there exists a positive constant C such that for each $n \geq 1$,

$$E\left(\max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} X_i \right\|^2\right) \leq C \sum_{i=1}^{n} E\| X_i \|^2.$$

2. The Main Results

Let $\{u_n, n \geq 1\}$ and $\{a_n, n \geq 1\}$ be sequences of positive real numbers. Let $\{a_{ni}, 1 \leq i \leq u_n\}$ be a bounded array of positive numbers.

Theorem 2.1 Let $\{X_n, n \geq 1\}$ be a sequence of NSD random vectors with mean 0 such that

$$\sum_{i=1}^{u_n} \sum_{j \in B} P(|X_i^j| > a_n) \to 0 \text{ as } n \to \infty, \quad (2)$$

$$\frac{1}{a_n^p} \sum_{i=1}^{u_n} a_{ni} \sum_{j \in B} E|X_i^j|^p \to 0 \text{ as } n \to \infty. \quad (3)$$

Then

$$\frac{1}{a_n}\sum_{i=1}^{u_n} a_{ni}(X_i - EY_{ni}) \xrightarrow{P} 0 \text{ as } n\to\infty,$$

where $1\le p\le 2$, $Y_{ni}=\sum_{j\in B} Y_{ni}^j e_j$ and $Y_{ni}^j = -a_n I_{\{X_i^j < -a_n\}} + a_n I_{\{X_i^j > a_n\}} + X_i^j I_{\{|X_i^j|\le a_n\}}$.

Proof. Let ϵ be an arbitrary positive number. We have

$$P\left(\left\|\frac{1}{a_n}\sum_{i=1}^{u_n} a_{ni}(X_i - EY_{ni})\right\| > 2\epsilon\right) \le P\left(\left\|\frac{1}{a_n}\sum_{i=1}^{u_n} a_{ni}(X_i - Y_{ni})\right\| > \epsilon\right) + P\left(\left\|\frac{1}{a_n}\sum_{i=1}^{u_n} a_{ni}(Y_{ni} - EY_{ni})\right\| > \epsilon\right).$$

Therefore, we have to prove that each term in the right-hand side tends to 0 as $n\to\infty$. Indeed,

$$\begin{aligned} P\left(\left\|\frac{1}{a_n}\sum_{i=1}^{u_n} a_{ni}(X_i - Y_{ni})\right\| > \epsilon\right) &\le \sum_{i=1}^{u_n} P(X_i \ne Y_{ni}) \\ &= \sum_{i=1}^{u_n}\sum_{j\in B} P(X_i^j \ne Y_{ni}^j) \\ &= \sum_{i=1}^{u_n}\sum_{j\in B} P(|X_i^j| > a_n) \to 0 \text{ as } n\to\infty. \end{aligned}$$

Since $\{a_{ni}(Y_{ni} - EY_{ni})\}$ is a sequence of NSD random vectors with mean 0, by Lemma 2, we get

$$\begin{aligned} &P\left(\left\|\frac{1}{a_n}\sum_{i=1}^{u_n} a_{ni}(Y_{ni} - EY_{ni})\right\| > \epsilon\right) \\ &\le \frac{1}{a_n^2\epsilon^2} E\left\|\sum_{i=1}^{u_n} a_{ni}(Y_{ni} - EY_{ni})\right\|^2 \\ &\le \frac{2}{a_n^2\epsilon^2}\sum_{i=1}^{u_n} a_{ni}^2 E\| Y_{ni} - EY_{ni}\|^2 \\ &\le \frac{2}{a_n^2\epsilon^2}\sum_{i=1}^{u_n} a_{ni}^2 E\| Y_{ni}\|^2 = \frac{2}{a_n^2\epsilon^2}\sum_{i=1}^{u_n} a_{ni}^2 \sum_{j\in B} E\| Y_{ni}^j\|^2 \\ &\le \frac{6}{a_n^2\epsilon^2}\sum_{i=1}^{u_n} a_{ni}^2 \sum_{j\in B}\left[a_n^2 P(|X_i^j| > a_n) + E|X_i^j|^2 I_{\{|X_i^j|\le a_n\}}\right] \\ &\le \frac{6}{a_n^2\epsilon^2}\sum_{i=1}^{u_n}\sum_{j\in B}[(a_{ni}a_n)^2 P(|a_{ni}X_i^j| > a_{ni}a_n) + E|a_{ni}X_i^j|^2 I_{\{|a_{ni}X_i^j|\le a_{ni}a_n\}}]. \end{aligned}$$

Using (3) and the following inequality

$$E|X|^2 I_{\{|X|\le b\}} + b^2 P(|X| > b) \le \frac{2}{p} b^{2-p} E|X|^p, \tag{4}$$

we obtain

$$P\left(\left\|\frac{1}{a_n}\sum_{i=1}^{u_n} a_{ni}(Y_{ni}-EY_{ni})\right\| > ò\right) \le \frac{12}{pa_n^2 ò^2}\sum_{i=1}^{u_n}\sum_{j\in B}(a_{ni}a_n)^{2-p}E\,|\,a_{ni}X_i^j\,|^p$$

$$= \frac{12}{pa_n^p ò^2}\sum_{i=1}^{u_n}\sum_{j\in B} a_{ni}^2 E\,|\,X_i^j\,|^p \to 0 \text{ as } n\to\infty.$$

Thus, the proof of Theorem (3) is completed.

The following result is a random index version of Theorem 3.

Theorem 2.2 If the conditions in Theorem 3 hold and $\{\tau_n, n\ge 1\}$ is a sequence of positive integer-valued random variables such that $\lim_{n\to\infty} P(\tau_n > u_n) = 0,$ then

$$\frac{1}{a_n}\sum_{i=1}^{\tau_n} a_{ni}(X_i - EY_{ni}) \xrightarrow{P} 0 \text{ as } n\to\infty. \tag{5}$$

Proof. For an arbitrary $ò > 0$,

$$P\left(\left\|\frac{1}{a_n}\sum_{i=1}^{\tau_n} a_{ni}(X_i - EY_{ni})\right\| > 2ò\right) \le P\left(\left\|\frac{1}{a_n}\sum_{i=1}^{\tau_n} a_{ni}(X_i - Y_{ni})\right\| > ò\right) + P\left(\left\|\frac{1}{a_n}\sum_{i=1}^{\tau_n} a_{ni}(Y_{ni} - EY_{ni})\right\| > ò\right)$$

$$:= A_n + B_n.$$

Therefore, we need to prove that A_n and B_n tend to 0 when $n\to\infty$.

For A_n, by (2) and (5),

$$\begin{aligned}
A_n &\le P\left(\bigcup_{i=1}^{\tau_n}(X_i \ne Y_{ni})\right)\\
&\le P\left(\bigcup_{i=1}^{\tau_n}(X_i \ne Y_{ni}, \tau_n \le u_n)\right) + P(\tau_n > u_n)\\
&\le \sum_{i=1}^{u_n} P(X_i \ne Y_{ni}) + P(\tau_n > u_n)\\
&\le \sum_{i=1}^{u_n}\sum_{j\in B} P(X_i^j \ne Y_{ni}^j) + P(\tau_n > u_n)\\
&\le \sum_{i=1}^{u_n}\sum_{j\in B} P(|\,X_i^j\,| > a_n) + P(\tau_n > u_n) \to 0 \text{ as } n\to\infty.
\end{aligned}$$

From Lemma 2 and (5), we have

$$B_n = P\left(\left\|\frac{1}{a_n}\sum_{i=1}^{\tau_n} a_{ni}(Y_{ni} - EY_{ni})\right\| > \varepsilon\right)$$

$$\leq P\left\{\left(\left\|\frac{1}{a_n}\sum_{i=1}^{\tau_n} a_{ni}(Y_{ni} - EY_{ni})\right\| > \varepsilon\right) \cap (\tau_n \leq u_n)\right\} + P(\tau_n > u_n)$$

$$= P\left\{\bigcup_{k=1}^{u_n}\left(\left\|\frac{1}{a_n}\sum_{i=1}^{k} a_{ni}(Y_{ni} - EY_{ni})\right\| > \varepsilon, \tau_n = k\right)\right\} + P(\tau_n > u_n)$$

$$\leq P\left(\max_{k\leq u_n}\left\|\frac{1}{a_n}\sum_{i=1}^{k} a_{ni}(Y_{ni} - EY_{ni})\right\| > \varepsilon\right) + P(\tau_n > u_n)$$

$$\leq \frac{1}{\varepsilon^2 a_n^2} E\left(\max_{k\leq u_n}\left\|\sum_{i=1}^{k} a_{ni}(Y_{ni} - EY_{ni})\right\|^2\right) + P(\tau_n > u_n)$$

$$\leq \frac{2}{\varepsilon^2 a_n^2}\sum_{i=1}^{u_n} a_{ni}^2 E\| Y_{ni} - EY_{ni}\|^2 + P(\tau_n > u_n) \to 0 \text{ as } n \to \infty$$

.

Hence, the proof is completed.

Theorem 2.3 Let $\{k_n, n \geq 1\}$ be a sequence of positive integer numbers and $\{a_n, n \geq 1\}$ be a sequence of positive real numbers such that $k_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{k_n}{a_n^2} = 0. \tag{6}$$

Suppose that $g : [0, +\infty) \to \mathbb{R}^+$ is a nondecreasing function such that $\frac{g^2(k_n)}{a_n^2}$ is bounded and

$$\lim_{a\to 0} g(a) = 0, \sum_{j=1}^{\infty} g^2\left(\frac{1}{j}\right) < \infty, \tag{7}$$

$$\sup_{n\geq 1} \frac{k_n}{a_n^2} \sum_{j=1}^{k_n - 1} \frac{g^2(j+1) - g^2(j)}{j} < \infty. \tag{8}$$

If

$$\sup_{a>0}\sup_{n\geq 1} \frac{1}{k_n}\sum_{i=1}^{u_n}\sum_{j\in B} aP(|X_i^j| > g(a)) < \infty, \tag{9}$$

and

$$\lim_{a\to\infty}\sup_{n\geq 1} \frac{1}{k_n}\sum_{i=1}^{u_n}\sum_{j\in B} aP(|X_i^j| > g(a)) = 0, \tag{10}$$

then

$$\frac{1}{a_n}\left\|\sum_{i=1}^{u_n}(X_i - EY_{ni})\right\| \xrightarrow{P} 0 \text{ as } n \to \infty, \tag{11}$$

Where $Y_{ni} = \sum_{j\in B} Y_{ni}^j e_j$, $Y_{ni}^j = -g(k_n)\mathrm{I}_{\{X_i^j < -g(k_n)\}} + g(k_n)\mathrm{I}_{\{X_i^j > g(k_n)\}} + X_i^j \mathrm{I}_{\{|X_i^j| \le g(k_n)\}}$.

Proof. For an arbitrary $\varepsilon > 0$,

$$P\left(\frac{1}{a_n}\left\|\sum_{i=1}^{u_n}(X_i - EY_{ni})\right\| > 2\varepsilon\right) \le P\left(\frac{1}{a_n}\left\|\sum_{i=1}^{u_n}(X_i - Y_{ni})\right\| > \varepsilon\right) + P\left(\frac{1}{a_n}\left\|\sum_{i=1}^{u_n}(Y_{ni} - EY_{ni})\right\| > \varepsilon\right)$$

$$:= A_n + B_n.$$

For A_n, by (10) with $a = k_n$, we get

$$A_n \le \sum_{i=1}^{u_n} P(X_i \ne Y_{ni}) \le \sum_{i=1}^{u_n}\sum_{j\in B} P(X_i^j \ne Y_{ni}^j)$$

$$= \sum_{i=1}^{u_n}\sum_{j\in B} P(|X_i^j| > g(k_n)) \to 0 \text{ as } n \to \infty.$$

Since $\{Y_{ni} - EY_{ni}, i \ge 1\}$ is a sequence of NSD random vectors with mean 0, by Lemma 2,

$$B_n \le \frac{1}{a_n^2\varepsilon^2} E\left\|\sum_{i=1}^{u_n}(Y_{ni} - EY_{ni})\right\|^2$$

$$\le \frac{1}{a_n^2\varepsilon^2}\sum_{i=1}^{u_n} E\| Y_{ni} - EY_{ni}\|^2$$

$$\le \frac{1}{a_n^2\varepsilon^2}\sum_{i=1}^{u_n} E\| Y_{ni}\|^2 = \frac{1}{a_n^2\varepsilon^2}\sum_{i=1}^{u_n}\sum_{j\in B} E\| Y_{ni}^j\|^2 .$$

Moreover, we have

$$E|Y_{ni}^j|^2 \le 3g^2(k_n)P(|X_i^j| > g(k_n)) + 3E|X_i^j|^2\, \mathrm{I}_{\{|X_i^j| \le g(k_n)\}}.$$

It follows that

$$B_n \le \frac{3}{a_n^2\varepsilon^2}\sum_{i=1}^{u_n}\sum_{j\in B} g^2(k_n)P(|X_i^j| > g(k_n) + \frac{3}{a_n^2\varepsilon^2}\sum_{i=1}^{u_n}\sum_{j\in B} E|X_i^j|^2\, \mathrm{I}_{\{|X_i^j| \le g(k_n)\}}$$

$$:= C_n + D_n.$$

By the boundedness of $\dfrac{g^2(k_n)}{a_n^2}$ and (10) with $a = k_n$,

$$C_n = \frac{g^2(k_n)}{a_n^2}\frac{1}{k_n}\sum_{i=1}^{u_n}\sum_{j\in B} k_n P(|X_i^j| > g(k_n)) \to 0 \text{ as } n \to \infty.$$

To prove the rest of Theorem 5, we need to show that $D_n \to 0$ as $n \to \infty$. Observe that

$$D_n = \frac{3}{a_n^2\varepsilon^2}\sum_{i=1}^{u_n}\sum_{j\in B} E|X_i^j|^2\, \mathrm{I}_{\{|X_i^j| \le g(1)\}} + \frac{3}{a_n^2\varepsilon^2}\sum_{i=1}^{u_n}\sum_{j\in B}\sum_{l=2}^{k_n} E|X_i^j|^2\, \mathrm{I}_{\{g(l-1) \le |X_i^j| \le g(l)\}}$$

$$:= \frac{3}{\varepsilon^2}(M_n + N_n).$$

For M_n, we have

$$M_n = \frac{1}{a_n^2}\sum_{i=1}^{u_n}\sum_{j\in B}\sum_{l=1}^{\infty} E|X_i^j|^2 \mathrm{I}_{\{g(\frac{1}{l+1})\le |X_i^j|\le g(\frac{1}{l})\}}$$
$$\le \frac{1}{a_n^2}\sum_{i=1}^{u_n}\sum_{j\in B}\sum_{l=1}^{\infty} g^2(\frac{1}{l})P\left[g(\frac{1}{l+1})\le |X_i^j|\le g(\frac{1}{l})\right]$$
$$\le \frac{1}{a_n^2}\sum_{i=1}^{u_n}\sum_{j\in B}\sum_{l=2}^{\infty}\left(g^2(\frac{1}{l-1})-g^2(\frac{1}{l})\right)P\left(|X_i^j|> g(\frac{1}{l})\right)$$
$$\le \frac{k_n}{a_n^2}\sum_{l=2}^{\infty} l\left[g^2(\frac{1}{l-1})-g^2(\frac{1}{l})\right].\sup_{n\ge 1}\left[\frac{1}{k_n}\sum_{i=1}^{u_n}\sum_{j\in B}\frac{1}{l}P\left(|X_i^j|> g(\frac{1}{l})\right)\right]$$

$$\le \frac{k_n}{a_n^2}\left(g^2(1)+\sum_{l=1}^{\infty} g^2(\frac{1}{l})\right).\sup_{a>0}\sup_{n\ge 1}\left[\frac{1}{k_n}\sum_{i=1}^{u_n}\sum_{j\in B} aP\left(|X_i^j|> g(a)\right)\right]$$

.

We have $\frac{k_n}{a_n^2}\to 0$ as $n\to\infty$ by (6), $g^2(1)+\sum_{l=1}^{\infty} g^2(\frac{1}{l})<\infty$ by (7) and

$$\sup_{a>0}\sup_{n\ge 1}\left[\frac{1}{k_n}\sum_{i=1}^{u_n}\sum_{j\in B} aP\left(|X_i^j|> g(a)\right)\right]<\infty, \quad \text{by (9).}$$

Hence, $M_n \to 0$ as $n\to\infty$.

We will show that $N_n \to \infty$ as $n\to\infty$ in the rest of this proof. We have

$$N_n \le \frac{1}{a_n^2}\sum_{i=1}^{u_n}\sum_{j\in B}\sum_{l=2}^{k_n} g^2(l)P\left[g(l-1)\le |X_i^j|\le g(l)\right]$$
$$= \frac{g^2(1)}{a_n^2}\sum_{i=1}^{u_n}\sum_{j\in B} P\left[|X_i^j|> g(1)\right]$$
$$+\frac{1}{a_n^2}\sum_{i=1}^{u_n}\sum_{j\in B}\sum_{l=2}^{k_n-1}\left[g^2(l+1)-g^2(l)\right]P[|X_i^j|> g(l)$$
$$= g^2(1)\frac{k_n}{a_n^2}\left[\frac{1}{k_n}\sum_{i=1}^{u_n}\sum_{j\in B} P\left(|X_i^j|> g(1)\right)\right]$$
$$+\frac{1}{a_n^2}\sum_{l=1}^{k_n-1}\frac{g^2(l+1)-g^2(l)}{l}\left[\sum_{i=1}^{u_n}\sum_{j\in B} lP(|X_i^j|> g(l))\right]$$
$$\le g^2(1)\frac{k_n}{a_n^2}\sup_{a>0}\sup_{n\ge 1}\frac{1}{k_n}\sum_{i=1}^{u_n}\sum_{j\in B} aP\left[|X_i^j|> g(a)\right]$$
$$+\frac{k_n}{a_n^2}\sum_{l=1}^{k_n-1}\frac{g^2(l+1)-g^2(l)}{l}\left[\frac{1}{k_n}\sum_{i=1}^{u_n}\sum_{j\in B} lP(|X_i^j|> g(l))\right].$$

Since $\frac{k_n}{a_n^2}\to 0$ as $n\to\infty$ by (6), $\sup_{a>0}\sup_{n\ge 1}\frac{1}{k_n}\sum_{i=1}^{u_n}\sum_{j\in B} aP\left[|X_i^j|> g(a)\right]<\infty$ by (9) and

$$\frac{k_n}{a_n^2}\sum_{l=1}^{k_n-1}\frac{g^2(l+1)-g^2(l)}{l}\left[\frac{1}{k_n}\sum_{i=1}^{u_n}\sum_{j\in B} lP(|X_i^j|>g(l))\right]\to 0,$$

by (8), (10) and by the Toeplitz lemma, $N_n \to 0$ as $n\to\infty$.
Thus, the result is proved.

Remark. It is difficult to check the condition (8). By the same argument as in Proposition 1 of D. H. Hong et al. in [4], we can prove the sufficient condition for (8) given as follows:

$$\frac{g(k_n)}{a_n}=O(1) \text{ and } \sum_{l=1}^{k_n}\frac{g^2(l)}{l^2}=O(\frac{a_n^2}{k_n}). \quad (12)$$

Take $g(t)=\frac{1}{t^{1/r}}$ and $a_n = k_n^{1/r}$. It is easy to check that the conditions (6) and (7) hold. By the inequality $\sum_{l=1}^{k_n} l^{2/r-2} \le Ck_n^{2/r-1}$, it follows that (12) holds. Therefore, the condition (8) holds and we have the following corollary.

Corollary 2.4 If

$$\sup_{a>0}\sup_{n\ge 1}\frac{1}{k_n}\sum_{i=1}^{u_n}\sum_{j\in B} aP(|X_i^j|^r > a)<\infty, \quad (13)$$

and

$$\lim_{a\to\infty}\sup_{n\ge 1}\frac{1}{k_n}\sum_{i=1}^{u_n}\sum_{j\in B} aP(|X_i^j|^r > a)=0, \quad (14)$$

then

$$\frac{1}{k_n^{1/r}}\left\|\sum_{i=1}^{u_n}(X_i - EY_{ni})\right\|\xrightarrow{P} 0 \text{ as } n\to\infty. \quad (15)$$

3. Conclusion

Thus, we have stated and proved the weak laws of large numbers for randomly weighted sums (with or without random indices) of sequences of negatively superadditive dependent random vectors in Hilbert spaces.

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