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Original Article

Normal Impact of A Rigid Cone-shaped against A Viscoelastic Plate on Viscoelastic Foundation

Duong Tuan Manh*

VNU University of Engineering and Technology, 144 Xuan Thuy, Cau Giay, Hanoi, Vietnam

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Abstract: This paper considers the problem of a low-velocity normal impact of a rigid cone-shaped upon a viscoelastic plate. The contact force is defined by the modified Hertz's contact law. Approximate solutions of the system of nonlinear integro-differential equations for the contact force and local indentation have been obtained.

Keywords: Normal impact, viscoelastic plate, Kelvin-Voigt model.

1. Introduction

The problems with the analysis of the impact of two bodies have a widespread application in various fields of science and technology. Because these problems belong to dynamic contact interaction, their solutions are connected with severe mathematical and calculation difficulties. Some approaches and methods have been suggested in articles by Abrate [1], Rossikhin and Shitikova [2].

Rossikhin et al., [3] investigated the collision of two viscoelastic shells, viscoelastic features of which are described by the standard linear solid model. During the impact process, the local bearing of the materials of the colliding viscoelastic shells is taken into account, the solution in the contact domain is found via the modified Hertz contact theory involving the operator representation of viscoelastic analogs of Young's modulus and Poisson's ratio.

In the paper, we will consider a special but very important for engineering practice case when a rigid cone-shaped impacts a viscoelastic plate.

* Corresponding author.
E-mail address: iam.mr.manh@gmail.com

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2. Problem Formulation

Let us consider the problem on a transverse impact of a rigid cone-shaped upon a viscoelastic Kirchhoff-Love plate when the viscoelastic features of the surrounding medium are described by a Kelvin-Voigt model. In this case, the equations of motion of a rigid cone-shaped of mass ms and the viscoelastic rectangular plate with the dimensions a and b and of thickness h have, respectively, the form

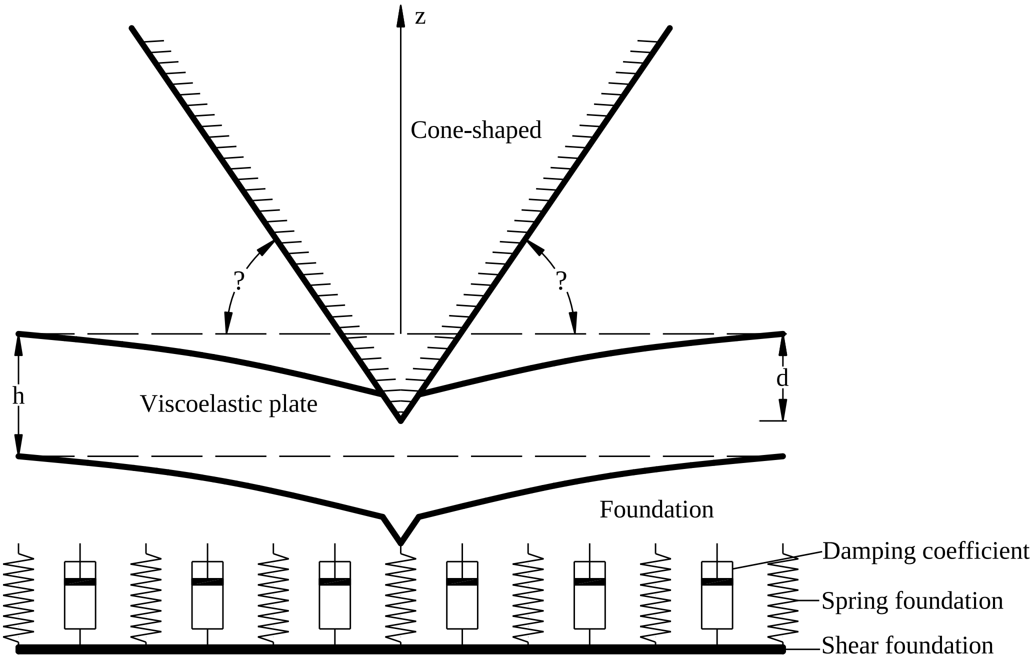

Figure 1. Scheme of the normal impact of a rigid cone-shaped against a plate.

$$m_s\ddot{z}(t) = -P(t), \tag{1}$$

$$\tilde{D}\Delta^2 w(x,y,t) + \rho h\ddot{w}(x,y,t) - L(w,F) = P(t)\delta\left(x-\frac{a}{2}\right)\delta\left(y-\frac{b}{2}\right)$$
$$-K_w w(x,y,t) - C_T\frac{\partial w(x,y,t)}{\partial t} + K_G\Delta w(x,y,t) \tag{2}$$

$$\frac{1}{\tilde{E}}\Delta^2 F = -\frac{1}{2}L(w,w), \tag{3}$$

where ρ is the density of plate, *w(x,y,t)* is the displacements of an arbitrary point in the plate in the z-direction, *P(t)* is the contact force, *F* is Airy function stress, Δ is the Laplace operator, $\delta\left(x-\frac{a}{2}\right)$, $\delta\left(x-\frac{b}{2}\right)$ is the Dirac delta function, *d* and *z(t)* are the penetration depths, θ is the angle between the plane and cone, K_w, K_G, C_T are the Winkler, Pasternak and Damper modulus parameters, respectively, and

$$\tilde{E} = E\left[1 + \frac{\eta}{E}\nabla_t\right]$$

$$\tilde{D} = \frac{\tilde{E}h^3}{12\left(1-\tilde{\nu}^2\right)}$$

Utilizing the KelvinVoigt model with the bulk modulus K remaining constant during the process of mechanical loading of this material [4], it is found [3] that

$$\frac{\tilde{E}}{1-\tilde{\nu}^2} = E\zeta\left[1 + \frac{\eta}{E}\nabla_t\right]\left[\tilde{L}(\alpha) - \tilde{L}(\gamma)\right],$$

where E,ν,η are the relaxed Young's modulus, Poisson's ratio and viscosity, respectively,

$\tau_\sigma = \lambda^{-1} = \dfrac{\eta}{E}$ is the creep time, and

$$\alpha = \frac{-2(\nu+1)\lambda}{(1-2\nu)};\gamma = \frac{-2(\nu-1)\lambda}{(1-2\nu)};\zeta = \frac{(1-2\nu)}{4\lambda};\tilde{L}(\delta) = \exp(-\delta t)\int_0^t \exp(\delta t')dt'$$

Equations (1), (2) and (3) are subjected to the following initial conditions:

$$\dot{z}(0) = V_0, z(0) = 0, P(0) = 0 \tag{4}$$

$$w(x,y,0) = 0, \dot{w}(x,y,0) = 0 \tag{5}$$

$$w(0,y,t) = 0, w(a,y,t) = w(x,0,t) = 0, w(x,b,t) = 0 \tag{6}$$

$$M_x(0,y,t) = M_x(a,y,t) = M_y(0,y,t) = M_y(b,y,t) = 0 \tag{7}$$

$$N_x(x,y,t) = p_{1;} N_y(x,y,t) = p_{2;} N_{xy}(x,y,t) = 0 - at: \begin{matrix} x = 0,a \\ y = 0,b \end{matrix}, \tag{8}$$

where all four edges of a rectangular plate are simply supported and immovable, V_0 is the initial velocity of the collision.

Integrating twice Eq. (1) yields

$$z(t) = -\frac{1}{m_s}\int_0^t P(t')(t-t')dt' + V_0 t \tag{9}$$

In the case of the collision of a rigid cone-shaped against a plate (Figure 1), the solution in the contact domain could be found using Hertz' theory. Thus, the contact force is defined as [5]

$$P(t) = \frac{2}{\pi}\frac{\tilde{E}}{\left(1-\tilde{\nu}^2\right)}\frac{z(t)^2}{\tan\theta}, \tag{10}$$

Replacing Eq. (1) into Eq. (10) yields

$$m_s\ddot{z}(t)+\frac{2}{\pi}\frac{\tilde{E}}{\left(1-\tilde{\nu}^2\right)}\frac{z(t)^2}{\tan\theta}=0 \tag{11}$$

The approximate solution of the equation (3) satisfying the boundary conditions (6), (7) and (8) can be written as

$$\mathrm{w}(x,y,t)=f\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{12}$$

$$F=\frac{E}{32}\left[1+\frac{\eta}{E}\nabla_t\right]f^2\left[\frac{a^2}{b^2}\cos\frac{2m\pi x}{a}+\frac{b^2}{a^2}\cos\frac{2n\pi y}{b}\right]+\frac{p_1y^2}{2}+\frac{p_2x^2}{2}\ , \tag{13}$$

where f is function of time t, m, n are odd numbers.

Replacing Eq. (13) into Eq. (2) yields

$$\begin{aligned}&\frac{\tilde{E}h^3}{12\left(1-\tilde{\nu}^2\right)}f\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left(2\frac{m^2n^2\pi^4}{a^2b^2}+\frac{m^4\pi^4}{a^4}+\frac{n^4\pi^4}{b^4}\right)+\rho h\ddot{f}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\&-\left\{\begin{aligned}&-f\frac{m^2\pi^2}{a^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left[-\frac{E}{32}\left[1+\frac{\eta}{E}\nabla_t\right]f^2 4n^2\pi^2a^{-2}\cos\frac{2n\pi y}{b}+p_1\right]+\\&-f\frac{n^2\pi^2}{b^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left[-\frac{E}{32}\left[1+\frac{\eta}{E}\nabla_t\right]f^2 4m^2\pi^2b^{-2}\cos\frac{2m\pi x}{a}+p_2\right]\end{aligned}\right\}\\&=P(t)\delta\left(x-\frac{a}{2}\right)\delta\left(y-\frac{b}{2}\right)-K_wf\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}-C_T\dot{f}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\&-K_Gf\pi^2\left(\frac{m^2}{a^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}+\frac{n^2}{b^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right)\end{aligned} \tag{14}$$

With due account for orthogonality of sines on the segments $0\le x\le a;\ 0\le y\le b$, we obtain:

$$\int_0^a\int_0^b\left\{\begin{aligned}&\frac{\tilde{E}h^3}{12\left(1-\tilde{\nu}^2\right)}f\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left(2\frac{m^2n^2\pi^4}{a^2b^2}+\frac{m^4\pi^4}{a^4}+\frac{n^4\pi^4}{b^4}\right)\\&+\rho h\ddot{f}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\&-\left\{\begin{aligned}&f\frac{m^2\pi^2}{a^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left[\frac{E}{32}\left[1+\frac{\eta}{E}\nabla_t\right]f^2 4n^2\pi^2a^{-2}\cos\frac{2n\pi y}{b}-p_1\right]+\\&f\frac{n^2\pi^2}{b^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left[\frac{E}{32}\left[1+\frac{\eta}{E}\nabla_t\right]f^2 4m^2\pi^2b^{-2}\cos\frac{2m\pi x}{a}-p_2\right]\end{aligned}\right\}\\&-P(t)\delta\left(x-\frac{a}{2}\right)\delta\left(y-\frac{b}{2}\right)+K_wf\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}+C_T\dot{f}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\&+K_Gf\pi^2\left(\frac{m^2}{a^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}+\frac{n^2}{b^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right)\end{aligned}\right\}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dydx=0$$

The result of which could be written in the form of

$$P(t) = A_1\left[\tilde{L}(\alpha) - \tilde{L}(\gamma)\right] f + A_2\left[\tilde{L}(\alpha) - \tilde{L}(\gamma)\right]\dot{f} + A_3 f + A_4\ddot{f} + A_5 f^3 + A_6 f^2\dot{f}\,, \qquad (15)$$

where

$$A_1 = E\zeta\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 \frac{h^3 ab\pi^4}{48}$$

$$A_2 = \eta\zeta\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 \frac{h^3 ab\pi^4}{48} + \frac{ab}{4}C_T$$

$$A_3 = \left(\frac{m^2}{a^2}p_1 + \frac{n^2}{b^2}p_2\right)\frac{ab\pi^2}{4} + \frac{ab}{4}K_w + \frac{ab\pi^2}{4}K_G\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

$$A_4 = \rho h\frac{ab}{4}$$

$$A_5 = E\frac{m^2n^2\pi^4 b}{64a^3} + E\frac{m^2n^2\pi^4 a}{64b^3}$$

$$A_6 = \frac{m^2n^2\pi^4 b}{64a^3}\eta + \frac{m^2n^2\pi^4 a}{64b^3}\eta$$

$$A_7 = \frac{2}{\pi\tan\theta}; \qquad (16)$$

The system of nonlinear integro-differential equations describing the dynamic behavior of two bodies during their collision takes the form of

$$\begin{cases} P(t) = -m_s\ddot{z}(t) \\ m_s\ddot{z}(t) + A_7\dfrac{\tilde{E}}{\left(1-\tilde{\nu}^2\right)}z(t)^2 = 0 \\ m_s\ddot{z}(t) + A_1\left[\tilde{L}(\alpha) - \tilde{L}(\gamma)\right] f + A_2\left[\tilde{L}(\alpha) - \tilde{L}(\gamma)\right]\dot{f} + A_3 f + A_4\ddot{f} + A_5 f^3 + A_6 f^2\dot{f} = 0 \end{cases} \qquad (17)$$

3. Approximate Calculations

First we substitute the approximate solution of the equation (11) to form

$$\tilde{H}z(t) = \tilde{N}z(t)^2 + \tilde{M}z(t)\dot{z}(t)\,, \qquad (18)$$

where

$$\tilde{N} = -\frac{A_7 E\zeta}{m_s}\left[\tilde{L}(\alpha) - \tilde{L}(\gamma)\right]$$

$$\tilde{M} = -\frac{A_7\eta\zeta}{m_s}\left[\tilde{L}(\alpha) - \tilde{L}(\gamma)\right]$$

$$z(t)=\sum_{k=0}^{\infty}a_k(t) \tag{19}$$

Replacing Eq. (19) into Eqs. (18) yields

$$\begin{aligned}\sum_{k=0}^{\infty}a_k(t)&=V_0t-A_8\int_0^t\int_0^{t'}\Big(e^{-\alpha t''}\int_0^{t''}e^{\alpha t'''}\sum_{k=0}^{\infty}A_k(t''')-e^{-\gamma t''}\int_0^{t''}e^{\gamma t'''}\sum_{k=0}^{\infty}A_k(t''')\Big)dt'''dt''dt'\\&-A_9\int_0^t\int_0^{t'}\Big(e^{-\alpha t''}\int_0^{t''}e^{\alpha t'''}\sum_{k=0}^{\infty}B_k(t''')-e^{-\gamma t''}\int_0^{t''}e^{\gamma t'''}\sum_{k=0}^{\infty}B_k(t''')\Big)dt'''dt''dt'\end{aligned}, \tag{20}$$

where

$$A_8=\frac{A_7E\zeta}{m_s}$$

$$A_9=\frac{A_7\eta\zeta}{m_s}$$

Note that

$$e^{-\alpha t}\int_0^t e^{\alpha x}x^k dx=\sum_{i=0}^{k}\frac{k!}{(k-i)!}\frac{(-1)^i}{\alpha^{i+1}}t^{k-i}-k!\frac{(-1)^k}{\alpha^{k+1}}e^{-\alpha t} \tag{21}$$

The first two of them have the form

$$\begin{aligned}a_0(t)&=V_ot\\a_1(t)&=-\frac{V_0^2A_8}{12}\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)t^4+V_0^2\left(\frac{A_8}{3}\left(\frac{1}{\alpha^2}-\frac{1}{\gamma^2}\right)-\frac{A_9}{3}\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)\right)t^3\\&-V_0^2\left(A_8\left(\frac{1}{\alpha^3}-\frac{1}{\gamma^3}\right)+A_9\left(\frac{1}{\alpha^2}-\frac{1}{\gamma^2}\right)\right)t^2+6A_8V_0^2\left(\frac{1}{\alpha^4}-\frac{1}{\gamma^4}\right)t\\&-4A_9V_0^2\left(\frac{1}{\alpha^3}-\frac{1}{\gamma^3}\right)t+6A_8V_0^2\left(\frac{e^{-\alpha t}}{\alpha^5}-\frac{e^{-\gamma t}}{\gamma^5}\right)-4A_9V_0^2\left(\frac{e^{-\alpha t}}{\alpha^4}-\frac{e^{-\gamma t}}{\gamma^4}\right)\\&-6A_8V_0^2\left(\frac{1}{\alpha^5}-\frac{1}{\gamma^5}\right)+4A_9V_0^2\left(\frac{1}{\alpha^4}-\frac{1}{\gamma^4}\right)\end{aligned} \tag{22}$$

We obtain the solution

$$
\begin{aligned}
z(t) \approx\ & V_o t - \frac{V_0^2 A_8}{12}\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)t^4 + V_0^2\left(\frac{A_8}{3}\left(\frac{1}{\alpha^2}-\frac{1}{\gamma^2}\right)-\frac{A_9}{3}\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)\right)t^3 \\
& -V_0^2\left(A_8\left(\frac{1}{\alpha^3}-\frac{1}{\gamma^3}\right)+A_9\left(\frac{1}{\alpha^2}-\frac{1}{\gamma^2}\right)\right)t^2 + 6A_8V_0^2\left(\frac{1}{\alpha^4}-\frac{1}{\gamma^4}\right)t \\
& -4A_9V_0^2\left(\frac{1}{\alpha^3}-\frac{1}{\gamma^3}\right)t + 6A_8V_0^2\left(\frac{e^{-\alpha t}}{\alpha^5}-\frac{e^{-\gamma t}}{\gamma^5}\right) - 4A_9V_0^2\left(\frac{e^{-\alpha t}}{\alpha^4}-\frac{e^{-\gamma t}}{\gamma^4}\right) \\
& -6A_8V_0^2\left(\frac{1}{\alpha^5}-\frac{1}{\gamma^5}\right)+4A_9V_0^2\left(\frac{1}{\alpha^4}-\frac{1}{\gamma^4}\right)
\end{aligned} \tag{23}
$$

In the experiments of the collision problems, the collision time is about 10^{-3}s, so the series (19) converges very quickly. Here, we only need the first two terms to approximate the value of the solution. Another method such as balancing the coefficients of the series can be found in [3].

The obtained equation by replacing Eq. (23) into Eq. (15) can be used for nonlinear dynamical analysis of the collision.

4. Conclusion

In the present paper, the problem on the normal impact of a rigid cone-shaped upon a viscoelastic plate has been studied using the damping features of the impactor modeled by the linear Kelvin-Voigt model. An approximate analytical solution has also been found.

Acknowledgments

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