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Original Article

Characteristics of Stability Boundary of Nonlinear Continuous Dynamical Systems

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Abstract: The theory of differential equations has been widely known and developed in recent years. One of the issues that many authors give their undivided attention to is the stability boundary of nonlinear dynamical systems. In this work, we first review several properties of equilibrium points on the stability boundary. We next extend the characteristics of the stability boundary for a fairly large class of nonlinear dynamical systems. These characteristics are the key to completely determine the stability boundary of nonlinear dynamical systems.

Keyword: Dynamical systems, equilibrium point, stability boundary, stability region.

1. Introduction

It is well-known that many nonlinear physical and engineering systems are designed to be operated at an equilibrium state [1-4]. In other words, the authors are constructed to be operated at an equilibrium point and are described by a nonlinear dynamical system. The most important requirement for successful operation of these systems is to maintain the stability of this equilibrium state. It is desired to have the robust stability of the equilibrium point with respect to small perturbation. More precisely, the system state returns the equilibrium point under small perturbations. Therefore, the theory of stability continues to play an important role in continuous dynamical systems. In this work, we will give several properties of equilibrium points on the stability boundary. Then we extend to a comprehensive theory of stability region for continuous dynamical systems. The characterizations presented here will be considered as a basis for approximating the stability boundary of continuous dynamical systems.

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The remaining of this paper is organized as follows: Section 2 presents some basic definitions about dynamical systems. Main results about the characterizations of the stability boundary are given in Section 3, while Section 4 discusses several examples to illustrate the theoretical result. The last section concludes main work in this paper.

2. Preliminaries

Throughout this paper, we always consider the following (autonomous) nonlinear system

$$\dot{x} = f(x), \tag{1}$$

where $x \in \mathbb{R}^n$ is a vector of state variables. It is natural to assume that the function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies some sufficient conditions for the existence and uniqueness of solution to (1). We now give some necessary definitions before exploring the main results in Section 3. The solution curve of (1) starting from x_0 at $t = 0$ is called the trajectory starting from x_0 and is denoted by $\phi(\cdot, x_0)$. The point $\hat{x} \in \mathbb{R}^n$ is said to be an equilibrium point of (1) if $f(\hat{x}) = 0,$ i.e. the equilibrium point is a particular type of solution that does not change in time. Therefore, equilibrium points are degenerated trajectories that do not move [5]. The set of all equilibrium points of (1) will be denoted by $E = \{x \in \mathbb{R}^n : f(x) = 0\}.$ Another important type of trajectory is a closed orbit. A trajectory γ is a closed orbit if γ is not an equilibrium point and for any $x \in \gamma,$ there exists $T > 0$ such that $\phi(T, x) = x.$ A set $M \subset \mathbb{R}^n$ is called a positively (negatively) invariant set of (1) if every trajectory of (1) starting in M remains in M for all $t.$ The union and intersection of invariant sets are also invariant. A set $M \subset \mathbb{R}^n$ is called a positive (negative) invariant set of (1) if every trajectory of (1) starting in M remain in M for all $t \geq 0$ ($t \leq 0$), respectively.

A point p is said to be in the w-limit set of x if corresponding to each $\varepsilon > 0$ and $T > 0,$ there is a $t > T$ such that $|\phi(t, x) - p| < \varepsilon.$ A point p is said to be in the α-limit set of x if corresponding to each $\varepsilon > 0$ and $T < 0,$ there is a $t < T$ such that $|\phi(t, x) - p| < \varepsilon.$ In other word, p is said to be in the w-limit set (α-limit set) of x if there exists a sequence $\{t_i\} \in \mathbb{R}$ such that $\phi(t, x) \to p$ as $t_i \to +\infty$ ($t_i \to -\infty$).

Definition 1. An equilibrium point $\hat{x}$ of the nonlinear dynamical system (1) is hyperbolic if the corresponding Jacobian matrix $Df(\hat{x})$ has no eigenvalues with zero real part. Otherwise, it is a non-hyperbolic equilibrium point. Furthermore, a hyperbolic equilibrium point is of type-k if k eigenvalues of the Jacobian matrix $Df(\hat{x})$ have positive real part and $n-k$ have negative real part. Specially, if $Df(\hat{x})$ has exactly one eigenvalue with a positive real part, we call it a type-one equilibrium point.

Type one equilibrium points play an important role in the characterization of stability boundary and quasi-stability boundary. From the above definition, we also see that if all of the eigenvalues of $Df(\hat{x})$ have negative real part, $\hat{x}$ is called a stable equilibrium point. Furthermore, we say that $\hat{x}$ is a source point if $\hat{x}$ is of type-$n.$ i.e. $k = n.$ Therefore, we assume that $\hat{x}$ is a hyperbolic equilibrium point and $U \subset \mathbb{R}^n$ is a neighborhood of $\hat{x}$ in most of the results of this paper. We now define the local stable and unstable manifolds as follows

$$W^s_{loc}(\hat{x}) := \{x \in U : \phi(t, x) \to \hat{x} \text{ as } t \to +\infty\},$$
$$W^u_{loc}(\hat{x}) := \{x \in U : \phi(t, x) \to \hat{x} \text{ as } t \to -\infty\}.$$

Note that $W^s_{loc}(\hat{x})$ and $W^u_{loc}(\hat{x})$ are positive and negative invariant sets, respectively. Figure 1 illustrates these local manifolds.

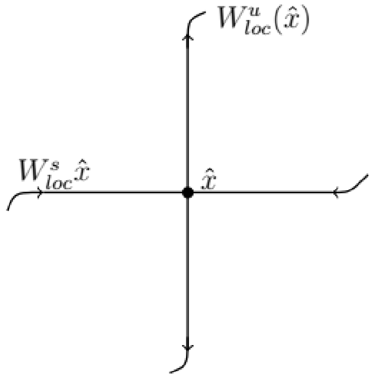

Figure 1. The local stable and unstable manifolds of an equilibrium point.

Remark 2 ([2]).

i) The equilibrium point $\hat{x}$ is the w-limit set of every point in $W^s(\hat{x})$ and it is the α-limit set of every point in $W^u(\hat{x})$. For a hyperbolic equilibrium point, the dimension of $W^s(\hat{x})$ equals to the number of eigenvalues of $Df(\hat{x})$ with negative real part. The sum of the dimension of $W^s(\hat{x})$ and that of $W^u(\hat{x})$ equals to the dimension of the state space;

ii) The existence and uniqueness of the solutions ensures that $W^s(\hat{x})$ and $W^u(\hat{x})$ can intersect with each other but not intersect themselves;

iii) Stable and unstable manifolds are invariant sets. Every trajectory in the stable manifold $W^s(\hat{x})$ converges to $\hat{x}$ as time goes to positive infinity, while every trajectory in the unstable manifold $W^u(\hat{x})$ converges to $\hat{x}$ as time goes to negative infinity.

In the rest of this section, we will deal with transversality and the stability region of a nonlinear dynamical systems. The idea of transversality is fundamental in the study of dynamical systems, and it was introduced by Palis, 1969; Palis and de Melo, 1981 and Smale, 1967, [2]. Before starting, we recall some relevant concepts. Let M be a smooth manifold with or without boundary. An immersed submanifold of M is a subset $A \subseteq M$ endowed with a topology (not necessarily the subspace topology) with respect to which A is a topological manifold (without boundary), and a smooth structure with respect to which the inclusion map $A \to M$ is a smooth immersion, [6]. For any two injectively immersed manifolds A and B in M, we say that they satisfy the transversality condition if one of the following two assertions hold true.

i) At every point of intersection $x \in A \cap B$, the tangent spaces of A and B span the tangent space of M at x, that is;

$$T_x(A) + T_x(B) = T_x(M), \quad x \in A \cap B.$$

ii) They do not intersect at all.

One of the most important characterizations of a hyperbolic point $\hat{x}$ is that its stable and unstable manifolds intersect transversely at $\hat{x}$. This transverse intersection is important because it persists under perturbation of the vector field.

For an asymptotically stable equilibrium point $\hat{x}$, there exists $\delta > 0$ such that for any x_0 satisfies $\| x_0 - \hat{x} \| < \delta$ then $\phi(t, x_0) \to \hat{x}$ as $t \to +\infty$. If δ is arbitrary large then $\hat{x}$ is called a global stable equilibrium point. However, many stable equilibrium points $\hat{x}$ are not globally stable. From now on, we denote the stable equilibrium point by x_s. The definition of the stability region of a stable equilibrium point x_s is considered as follows.

Definition 2. The stability region of a stable equilibrium point x_s for the nonlinear autonomous dynamical system (1), denoted $A(x_s)$, is defined as follows

$$A(x_s) := \{x \in \mathbb{R}^n : \lim_{t\to+\infty} \phi(t,x) \to x_s\}. \quad (2)$$

Clearly, the stability region can be expressed as $A(x_s) = \{x \in \mathbb{R}^n : w(x) = x_s\}$, where $w(x)$ is the w-limit set of x. The stability boundary of x_s is the boundary of the stability region $A(x_s)$ and is denoted by $\partial A(x_s)$. In fact, the stability region of a stable equilibrium point is the stable manifold. According to the topological properties of the stable manifold of x_s in [4] and [7], the stability region $A(x_s)$ is an open, invariant set that is diffeomorphic to $\mathbb{R}^n$. In other words, every trajectory starting in stability region lies entirely in it. Since the boundary of an invariant set is also invariant and the boundary of an open set is a closed set, we can conclude that the stability boundary $\partial A(x_s)$ is a closed invariant set of dimension less than n. If $A(x_s)$ is not dense in $\mathbb{R}^n$ then $\partial A(x_s)$ is of dimension $n-1$. Their characterizations will be discussed in the next section.

3. Characterizations of Stability Boundary

We now derive several dynamical and topological properties of a stability boundary. We first begin from a local characterization then extend to a global characterization of stability boundary. We know that a critical element of the vector field f is either a closed orbit (limit cycle) or an equilibrium point, and the characterizations of a critical element on the stability boundary closely resemble the characterizations of a equilibrium point on the stability boundary. However, we will focus on the results about equilibrium points on the stability boundary, which are further sharpened by imposing additional conditions on the nonlinear dynamical systems. Hence, throughout this section we always assume that $\hat{x}$ be a hyperbolic equilibrium point of (1) and U is a neighborhood of $\hat{x}$ in $W^s(\hat{x})$, where boundary ∂U is transversal to the vector field f. We call ∂U a fundamental domain of $W^s(\hat{x})$. Moreover, in order not to be trivial, we always consider x_s as a local stable equilibrium point.

A complete characterization for an equilibrium point lying on the stability boundary of the general nonlinear dynamical system (1) is given in next theorem.

Theorem 4 ([8]). Consider a general nonlinear continuous dynamical system (1). Let $A(x_s)$ be the stability region of an asymptotically stable equilibrium point x_s. Let $\hat{x} \neq x_s$ be a hyperbolic equilibrium point. Then

i) If $\{W^u(\hat{x}) - \hat{x}\} \cap \overline{A(x_s)} \neq \emptyset$ then $\hat{x} \in \partial A(x_s)$. Conversely, if $x \in \partial A(x_s)$ then $\{W^u(\hat{x}) - \hat{x}\} \cap \overline{A(x_s)} \neq \emptyset$.

ii) Suppose that $\hat{x}$ is not a source (i.e. $\{W^s(\hat{x}) - \hat{x}\} \neq \emptyset$) then $\hat{x} \in \partial A(x_s)$ if and only if $\{W^s(\hat{x}) - \hat{x}\} \cap \partial A(x_s) \neq \emptyset$.

iii) Suppose that $\hat{x} \in \partial A$, then $\hat{x}$ is a source if and only if $\{W^s(\hat{x}) - \hat{x}\} \cap \partial A(x_s) = \emptyset$.

Proof.

i) If $y \in W^u(\hat{x}) \cap \overline{A(x_s)}$ then $\lim_{t\to\infty} \phi(-t,y) = \hat{x}$. But since $\overline{A(x_s)}$ is invariant, we have

$$\phi(-t,y) \in \overline{A(x_s)} \text{ for all } t \in \mathbb{R}.$$

This follows that

$$\hat{x} \in \overline{A(x_s)}.$$

However, $\hat{x}$ can not be in the stability region, we imply that $\hat{x} \in \partial A(x_s)$.

In order to prove the rest of this part, we assume that $\hat{x} \in \partial A(x_s)$. Let $G \subset \{W^u(\hat{x}) - \hat{x}\}$ be a fundamental domain of $W^u(\hat{x})$, i.e., G is a compact set such that

$$\bigcup_{t\in\mathbb{R}} \phi(t,G) = \{W^u(\hat{x}) - \hat{x}\}.$$

Let G_ε be the ε-neighborhood of G in $\mathbb{R}^n$. The set G_ε is a fundamental neighborhood associated with $W^u(\hat{x})$. It follows that for $t<0$, $\cup\phi(t,G_\varepsilon)$ contains a set of the form $\{U - W^s(\hat{x})\}$, where U is a neighborhood of $\hat{x}$. Therefore, we have $U \cap A(x_s) \neq \varnothing$. from $\hat{x} \in \partial A(x_s)$. Indeed, from $W^s(\hat{x}) \cap A(x_s) = \varnothing$, we have

$$\{U - W^s(\hat{x})\} \cap A(x_s) \neq \varnothing,$$

or

$$\bigcup_{t<0} \phi(t,G_\varepsilon) \cap A(x_s) \neq \varnothing,$$

since $\{U - W^s(\hat{x})\} \subset \cup\phi(t,G_\varepsilon), t<0$. It also means that $\phi(t,G_\varepsilon) \cap A(x_s) \neq \varnothing$ for some $t<0$. On the other hand, stability region is invariant under the flow so we obtain

$$G_\varepsilon \cap A(x_s) \neq \varnothing.$$

Since G is a compact set and ε is arbitrary, it follows that G contain at least one element of $\overline{A(x_s)}$. Hence, $\{W^u(\hat{x}) - \hat{x}\} \cap \overline{A(x_s)} \neq \varnothing$ and part (a) is proved.

ii) The proof of this part is similar to part a). From the assumption that $\hat{x}$ is not a source, we have $\{W^s(\hat{x}) - \hat{x}\} \neq \varnothing$. Firstly, we suppose that $\{W^s(\hat{x}) - \hat{x}\} \cap \partial A(x_s) \neq \varnothing$. Then there exists $y \in \{W^s(\hat{x}) - \hat{x}\} \cap \partial A(x_s) \neq \varnothing$ such that

$$\lim_{t\to+\infty} \phi(t,y) = \hat{x}.$$

Since $\partial A(x_s)$ is also an invariant set, we get

$$\phi(t,y) \in \partial A(x_s) \text{ for all } t \in \mathbb{R}.$$

Hence, $\hat{x}$ is on the stability region.

Conversely, we assume that $\hat{x} \in \partial A(x_s)$. Let $H \subset \{W^s(\hat{x}) - \hat{x}\}$ be a fundamental domain of $W^s(\hat{x})$. It is preserved since $\{W^s(\hat{x}) - \hat{x}\} \neq \varnothing$. It means that H is a compact set such that

$$\bigcup_{t\in\mathbb{R}} \phi(t,H) = \{W^s(\hat{x}) - \hat{x}\}.$$

Let H_ε be the ε-neighborhood of H in $\mathbb{R}^n$. The set H_ε is a fundamental neighborhood associated with $W^s(\hat{x})$. It follows that for $t>0$, $\cup\phi(t,H_\varepsilon)$ contains a set of the form $\{U - W^u(\hat{x})\}$, where U is a neighborhood of $\hat{x}$. Therefore, we have $U \cap \partial A(x_s) \neq \varnothing$ from $\hat{x} \in \partial A(x_s)$. Indeed, from $W^u(\hat{x}) \cap \partial A(x_s) = \varnothing$, we have

$$\bigcup_{t>0} \phi(t,H_\varepsilon) \cap \partial A(x_s) \neq \varnothing,$$

Since $\{U - W^u(\hat{x})\} \subset \cup\phi(t,H_\varepsilon)$, $t>0$. It also means that $\phi(t,H_\varepsilon) \cap \partial A(x_s) \neq \varnothing$ for some $t>0$. On the other hand, stability boundary is also invariant under the flow so we imply

$$H_\varepsilon \cap \partial A(x_s) \neq \varnothing.$$

Since H is a compact set and ε is arbitrary, this follows that H contain at least one point of $\partial A(x_s)$. Hence, $\{W^s(\hat{x})-\hat{x}\}\cap\partial A(x_s)\neq\varnothing$ and part (b) is proved.

iii) We now suppose that $\hat{x}$ is a source. This follows that $\{W^s(\hat{x})-\hat{x}\}=\varnothing.$ Obviously,

$$\{W^s(\hat{x})-\hat{x}\}\cap\partial A(x_s)=\varnothing\cap\partial A(x_s)=\varnothing.$$

Conversely, if $\{W^s(\hat{x})-\hat{x}\}\cap\partial A(x_s)=\varnothing$ we suppose contradiction that $\hat{x}$ is not a source. Since $\hat{x}\in\partial A(x_s),$ according to part (b) we obtain $\{W^s(\hat{x})-\hat{x}\}\cap\partial A(x_s)\neq\varnothing,$ contradiction. Thus, $\hat{x}$ must be source.

□

Up to now, we have assumed only that all the equilibrium points are hyperbolic. This is a generic property for nonlinear dynamical systems. Besides, another property is also generic: the intersection of the stable and unstable manifolds of the equilibrium points satisfy the transversality condition, [2], [7] and [8]. In the below of this section, we will present the theorem which characterizes an equilibrium point which is on the stability boundary in term of both its stable and unstable manifolds. Furthermore, this theorem is used to check numerical computation whether or not an equilibrium point belongs a stability boundary. Before starting, we make the following assumptions concerning the vector field.

(A1) All the equilibrium points on the stability boundary are hyperbolic.

(A2) The stable and unstable manifolds of equilibrium points on the stability boundary satisfy the transverality condition.

(A3) Every trajectory on the stability boundary approaches one of the equilibrium points as $t\to\infty.$

The following lemmas are used in the proof of the next theorem.

Lemma 5 ([2])**.** Let x_i and x_j be hyperbolic critical elements of the nonlinear dynamical system (1). Assume that the intersection of the stable and unstable manifolds of x_i, x_j satisfies the transversality condition and $\{W^u(x_i)-x_i\}\cap\{W^s(x_j)-x_j\}\neq\varnothing.$ Then $\dim W^u(x_i)\geq\dim W^u(x_j),$ where the equality sign is true only when x_i is an equilibrium point and x_j is a closed orbit.

Lemma 6 ([2])**.** Let $\hat{\upsilon}$ be a hyperbolic critical element of the nonlinear dynamical system (1) with $\dim W^u(\hat{\upsilon})=m.$ If $\hat{\upsilon}$ is an equilibrium point, let D be an m-disk in $W^u(\hat{\upsilon}).$ If $\hat{\upsilon}$ is a closed orbit, let D be an $(m-1)$-disk in $W^u(\hat{\upsilon})\cap S,$ where S is a cross section at $p\in\hat{\upsilon}.$ Let N be an m-disk (if $\hat{\upsilon}$ is an equilibrium point) or $(m-1)$-disk (if $\hat{\upsilon}$ is a closed orbit) having a point of transversal intersection with $W^s(\hat{\upsilon}).$ Then D is contained in the closure of the set $\bigcap_{t\geq0}\phi(t,N).$

Theorem 7 ([8])**.** Let $A(x_s)$ be the stability region of an asymptotically stable equilibrium point x_s of the nonlinear dynamical system (1) and $\hat{x}$ be an equilibrium point. If assumptions (A1)-(A3) are satisfied, then

i) $\hat{x}\in\partial A(x_s)$ if and only if $W^u(\hat{x})\cap A(x_s)\neq\varnothing,$

ii) $\hat{x}\in\partial A(x_s)$ if and only if $W^s(\hat{x})\subseteq\partial A(x_s).$

Proof.

i) Let n_u denote the type of equilibrium point x i.e the dimension of its unstable manifold. From assumption (A1), it follows that $n_u(x)\geq1$ for all equilibrium points $x\in\partial A(x_s).$ Let $\hat{x}$ be an equilibrium point on $\partial A(x_s)$ and $n_u(\hat{x})=h.$ By Theorem 4, there exists a point $y\in\{W^u(\hat{x})-\hat{x}\}\cap\overline{A(x_s)}.$ If $y\in A(x_s),$ the proof is completed. Therefore, we only need to prove for case $y\in\partial A(x_s).$ From assumption (A3), there exists an equilibrium point $\hat{z} \in \partial A(x_s)$ such that $y \in \{W^s(\hat{z}) - \hat{z}\}$. Let $n_u(\hat{z}) = m$. Then v and $W^s(\hat{z})$ meet transversely at y. By Lemma 5, we have $h > m$.

Now, let us consider two cases

a) If $h=1$ then m must be zero (i.e. $\hat{z}$ must be a stable equilibrium point), which is a contradiction to the fact that no stable equilibrium point exists on the stability boundary. Consequently, $W^u(\hat{x}) \cap A(x_s) \neq \varnothing$.

b) If $h>1$, without loss of generally, we assume inductively that $W^u(\hat{z}) \cap A(x_s) \neq \varnothing$. Since $W^u(\hat{x})$ and $W^s(\hat{z})$ intersect transversely at y, $W^u(\hat{x})$ contains a m-disk N centered at y, transverse to $W^s(\hat{z})$. Using Lemma 6 with $\hat{\upsilon} = \hat{z}$, we have $\phi(t, N) \cap A(x_s) \neq \varnothing$ for some $t > 0$. Since $A(x_s)$ is invariant, it follows that $N \cap A(x_s) \neq \varnothing$ This means that $W^u(\hat{x}) \cap A(x_s) \neq \varnothing$.

From the above arguments and Theorem 4, the first part of Theorem 7 is followed.

ii) If $W^s(\hat{x}) \subseteq \partial A(x_s)$, we infer that $\hat{x} \in \partial A(x_s)$ because $\hat{x}$ belongs to $W^s(\hat{x})$. We now suppose that $\hat{x} \in \partial A(x_s)$. Let $D \subset W^u(\hat{x}) \cap A(x_s)$ be an m-disk where $m = \dim W^u(\hat{x})$. The set D is well-define and non-empty because $W^u(\hat{x}) \cap A(x_s) \neq \varnothing$. Let $y \in W^s(\hat{x})$. For any $\varepsilon > 0$, let N be an m-disk transversal to $W^s(\hat{x})$ at y, contained in the ε-neighborhood of y. By using Lemma 6 with $\hat{\upsilon} = \hat{x}$, there exists a $t > 0$ such that $\phi(t, N)$ is so close to D and $\phi(t, N)$ contains a point $p \in A(x_s)$. Then $\phi(-t, N) \in N$. As in the first part, $N \cap A(x_s) \neq \varnothing$ because $A(x_s)$ is invariant. Let $\varepsilon \to 0$, $y \in \overline{A(x_s)}$ is inferred. Thus, $W^s(\hat{x}) \subset \overline{A(x_s)}$. Because $W^s(\hat{x})$ is disjoint from A, it implies that $W^s(\hat{x}) \subset \partial A(x_s)$.

□

Note that the transversality condition is essential in Theorem 7. Now, we will give a counterexample which is considered in [2] and [8].

Example 1. In Figure 2, the intersection of the unstable manifold of x_1 and the stable manifold of x_2 is a part of the manifold whose tangent space has a dimension of one. Hence, the transversality condition does not hold. By Theorem 4, the unstable manifold of x_s intersects with the stability boundary of x_s. Of course, it is not the entire stability region. Nevertheless, a part of the stable manifold of x_1 is not in the stability region, which contradicts to Theorem 7.

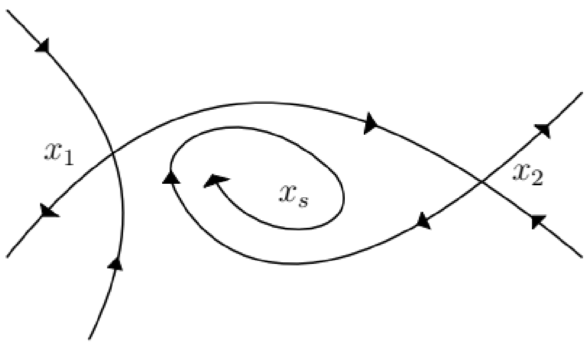

Figure 2. The intersection between the unstable manifold of x_1 and the stable manifold of x_2 does not satisfy the transversality condition.

In the rest of this section, we always suppose that the stability boundary is non empty. We now extend the characterizations of the stability boundary for a fairly large class of the nonlinear dynamical system (1) in the main theorem as follows.

Theorem 8 ([8])**.** For a nonlinear autonomous dynamical system (1) which satisfies assumptions (A1)-(A3), let $x_i,\ i=1,2,\dots$ be the equilibrium points on the stability boundary $\partial A(x_s)$ of the asymptotically stable point x_s. Then

i) $x_i \in \partial A(x_s)$ if and only if $W^u(x_i)\cap A(x_s)\neq\varnothing$,

ii) $\partial A(x_s)=\bigcup_i W^s(x_i),\ i=1,2,\dots$

Proof.

i) This is the first part in Theorem 7.

ii) Assume that $x_i,\ i=1,2,\dots$ be the equilibrium points on the stability boundary $\partial A(x_s)$. By Theorem 7, $W^s(x_i)\subseteq\partial A(x_s)$ for all i. Consequently,

$$\bigcup_i W^s(x_i)\subseteq\partial A(x_s).$$

Alternatively, every trajectory on the stability boundary approaches one of the equilibrium points as $t\to\infty$ so $\partial A(x_s)\subseteq\cup W^s(x_i)$. Hence, combining the above arguments, the proof is finished.

□

Using the above theorem, we can determine the stability boundary of an asymptotically stable equilibrium point of the nonlinear autonomous dynamical system (1) by an algorithm in [9]. It is convenient for low dimensional systems by finding the stable and unstable manifolds of an equilibrium point. The next theorem will discuss about the number of equilibrium points on the stability boundary. We say that manifold S is a smooth manifold if S is a topological manifold and there exists a smooth structure on S, [6]. By using smooth structure on $\mathbb{R}^n$, we define that $S\subset\mathbb{R}^n$ is a smooth manifold of dimension s if for each point $p\in S$, there exists a neighborhood $U\subset S$ of p and a homeomorphism $h:U\to V$ where V is an open subset in $\mathbb{R}^s$ such that the inverse homeomorphism $h^{-1}:V\to U$ is an immersion of class C^1.

Now, we indicate some concerned concepts in the theory of manifold. It is convenient to show the following theorem. Let M and N be oriented manifolds without boundary and $f:M\to N$ be a smooth map. Furthermore, M is compact and N is connected. Let $x\in M$ be a regular point of f, so that $df_x:T_xM\to T_{f(x)}N$ is a linear isomorphism between oriented vector spaces. Define the sign of df, to be $+1$ or -1 according as df, preserves or reverses orientation. For any regular value $y\in N$ define $\deg(f;y)=\sum\text{sign}df_x, x\in f^{-1}(y)$. Then the degree of f is an integer $\deg(f;y)$ which does not depend on the choice of the regular value $y\in N$. Next, we consider an open set $U\subset\mathbb{R}^m$ and a smooth vector field $v:U\to\mathbb{R}^m$ with an isolated zero at the point $z\in U$. The function $\bar{v}(x)=v(x)/\|v(x)\|$ maps a small sphere centered at z into the unit sphere. The degree of this mapping is called the index of v at the zero z, [10]. According [7], the Euler characteristic of M is the sum of indices of zeros of a vector field on M.

Theorem 9 ([8])**.** Let $A(x_s)$ be the stability region of a stable equilibrium point x_s of nonlinear dynamical system (1). If the stability boundary $\partial A(x_s)$ of a stable equilibrium point is a smooth compact manifold and all the equilibrium points on $\partial A(x_s)$ are hyperbolic then the number of equilibrium points on $\partial A(x_s)$ is even.

Proof. By the Poincaré - Hopf theorem [10], the sum of the indices of equilibrium points that lie on the smooth, compact stability boundary $\partial A(x_s)$ is even. Therefore, the Euler characteristic of the boundary of a compact manifold is even, [7]. Since the index of f at a hyperbolic equilibrium point is either $+1$ or -1, [10] and all the equilibrium points on $\partial A(x_s)$ are hyperbolic, we can conclude that the number of equilibrium points on $\partial A(x_s)$ is even.

□

Remark 10. If there are at least two stable equilibrium points, then the stability boundary are non-empty. Moreover, the dimension of each stability boundary is $n-1$ in this case.

Note that conditions (A1) is a generic property of C^1 dynamical systems. Assumptions (A2) is also a generic property. Alternatively, they are not easy to check. Assumption (A3) is not a generic property and for higher dimensional systems no general methods for verifying this assumption are known until now, according to [2] and [5].

By expanding the results on the structure of the equilibrium point on the stability boundary, we have below theorem. It presents a necessary condition for the existence of certain types of equilibrium points on a bounded stability boundary.

Theorem 11 ([8])**.** Assume that the nonlinear dynamical system (1) containing two or more stable equilibrium points. If the system satisfies assumptions (A1)-(A3) then the stability boundary $\partial A(x_s)$ of the stable equilibrium point x_s must contain at least one type-one equilibrium point. Furthermore, if the stability region $A(x_s)$ is bounded then $\partial A(x_s)$ must contain at least one type-one equilibrium point and one source.

Proof. Since there are at least two stable equilibrium points, it implies that the dimension of $\partial A(x_s)$ is $n-1$. By Theorem 8, $\partial A(x_s) = \cup W^s(x_j)$, where $x_j \in \partial A(x_s)$, this follows that at least one of the x_j must be a type-one equilibrium point. Hence, the dimension of $\cup W^s(x_j)$ is $n-1$. Suppose that x_1 is a type-one equilibrium point on $\partial A(x_s)$. Repeating the same argument, if $\partial A(x_1)$ is non-empty then the dimension of $\partial W^s(x_1)$ is less than $n-2$ because the dimension of $W^s(x_1)$ is less than or equal to $n-1$ and x_1 is a type-one equilibrium point. Once again, by using Theorem 8, we have $\partial A(x_1) = \cup W^s(x_j)$, where $x_j \in \partial W^s(x_1)$. In order for $\cup W^s(x_j)$ to have dimension $n-k$, at least one of x_j must be a type-k equilibrium point. Since the stability region is bounded as assumption, we will obtain a type-n equilibrium point by repeating this scheme. Hence, there exists a source on $\partial A(x_s)$.

□

Proposition 12. For the nonlinear autonomous dynamical system (1) which satisfies assumptions (A1)-(A3), let x_s be a stable equilibrium point of (1). Furthermore, assume that the stability region $A(x_s)$ is not dense in $\mathbb{R}^n$. Then there is at least one type-one equilibrium point on the stability boundary $\partial A(x_s)$.

Proof. The dimension of the stability boundary $\partial A(x_s)$ is $n-1$ derived from the stability region $A(x_s)$ is not dense in $\mathbb{R}^n$. Since the dimension of the stability boundary of $\partial A(x_s)$ is $n-1$ and $\partial A(x_s) = \cup W^s(x_i)$, where x_i is the equilibrium points on $\partial A(x_s)$, then there exists $x_{i0} \in \partial A(x_s)$ such that $\dim W^s(x_{i0}) = n-1$. Hence, x_{i0} is a type-one equilibrium point on the stability boundary $\partial A(x_s)$.

□

Remark 13. Assume that x_s is a stable equilibrium point of the nonlinear autonomous dynamical system which satisfies assumptions (A1)-(A3).

i) If all other equilibrium points are type-n hyperbolic points (sources), then the stability boundary $\partial A(x_s)$ of x_s has zero measure;

ii) If system (1) is a planar system and the stability region $A(x_s)$ is bounded then there always exists a source point between two type-one equilibrium points which are consecutive on the stability boundary. Furthermore, the number of source point is equal to the number of type-one equilibrium point on the stability boundary in this case.

Corollary 14 ([2]). Consider the nonlinear autonomous dynamical system (1) with a stable equilibrium point x_s whose stability boundary is nonempty. If assumptions (A1)-(A3) are satisfied and if $\partial A(x_s)$ contains no source then the stability region $A(x_s)$ is unbounded.

4. Example

In this section, we will present two examples for illustrating the above results. The first example is considered in [2, 8, 11, 12] while the second is introduced [2, 3, 8, 9, 13, 14]. Throughout these examples, we always assume the transversality condition (A2) holds.

Example 2. Consider a planar system as the following

$$\begin{aligned} \dot{x}_1 &= -2x_1 + x_1x_2 \\ \dot{x}_2 &= -x_2 + x_1x_2. \end{aligned} \tag{3}$$

By solving $f(x_1,x_2) = \mathbf{0}$, where $f(x_1,x_2) = (-2x_1 + x_1x_2, -x_2 + x_1x_2)^T$, we obtain two equilibrium points: $(0,0)$ and $(1,2)$ Moreover, we have the following Jacobian matrix

$$J = \begin{pmatrix} -2 + x_2 & x_1 \\ x_2 & -1 + x_1 \end{pmatrix}.$$

Therefore, $(0,0)$ is a stable equilibrium point and $(1,2)$ is a type-one equilibrium point. It means that assumption (A1) holds.

Next, we check assumption (A3). Consider the function

$$V(x_1,x_2) = x_1^2 - 2x_1x_2 + x_2^2.$$

The derivative of $V(x_1,x_2)$ along the trajectory of system (3) is

$$\begin{aligned} \dot{V}(x_1,x_2) &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\ &= -2(x_1 - x_2)(2x_1 - x_2). \end{aligned}$$

Therefore, $\dot{V}(x_1,x_2) < 0$ for $(x_1,x_2) \in B^C := \mathbb{R}^2 - B$

where $B := \{(x_1,x_2) \in \mathbb{R}^2 : 2x_1 - x_2 \geq 0 \text{ and } x_1 - x_2 \leq 0\}$. Let $\tilde{B}$ be as follows $\tilde{B} := B_1 \cup B_2 \cup B_3$, where $B_1 := \{(x_1, x_2) : x_1 < 1, x_2 < 2\}$, $B_2 := \{(x_1,x_2) : x_1 \geq 1, x_2 \leq 2\}$ and $B_3 := \{(x_1,x_2) : x_1 > 1, x_2 > 2\}$. Using the monotony of functions $x_1(t)$ and $x_2(t)$, we see that both $|x_1(t_i)|$ and $|x_2(t_i)|$ are strictly decreasing in B_1. Hence, B_1 is inside the stability region. Consequently, $\partial A(0,0)$ can not belong to B_1 Similarly, we can see that $\dot{x}_1, \dot{x}_2 > 0$ in B_3. It follows that every trajectory in B_3 is unbounded. By checking the vector field in B_2, we see that every trajectory in B_2 will either enter into B_1 or B_3 or converge to $(1,2)$. Thus, assumption (A3) holds.

In Figure 3, we can see that the stability boundary of the stable equilibrium point $(0,0)$ is the bold red line. The stability region of the stable equilibrium point $(0,0)$ is the one to the left and below the bold red line. On the other hand, the bold red line and the bold blue line are the stable and unstable manifolds of the equilibrium point of type-one $(1,2)$, respectively. It is easy to see that there is a part of unstable manifold of $(1,2)$ that lies in the stability region of $(0,0)$ and the stable manifold of $(1,2)$ is the stability boundary of $(0,0)$. This is consistent with the conclusions of Theorem 7 and Theorem 8 since system (3) satisfies assumptions (A1)-(A3). Furthermore, system (3) shows that the conclusions in Theorem 11 and Proposition 12 are completely consistent.

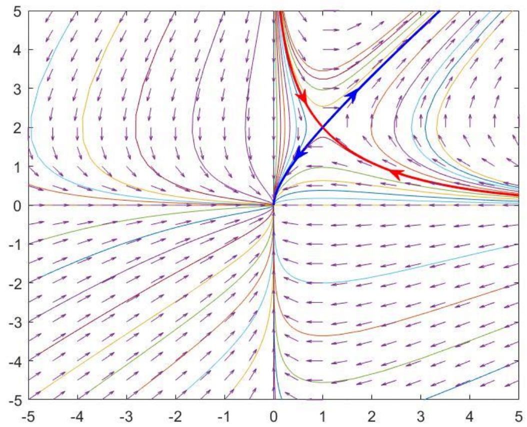

Figure 3. The phase portrait and the stability boundary of the system (3).

In the following, we will consider an example whose the stability region is bounded. Furthermore, we assume that assumption (A3) holds.

Example 3. Consider an example as the follows

$$\begin{aligned}\dot{x}_1 &= -\sin x_1 - 0.5\sin(x_1 - x_2) + 0.01\\ \dot{x}_2 &= -0.5\sin x_2 - 0.5\sin(x_2 - x_1) + 0.05.\end{aligned} \quad (4)$$

Although there are many stable equilibrium points, we only focus on the stable equilibrium point closest to the origin. Solving $f(x_1, x_2) = \mathbf{0}$, we obtain $(0.02801, 0.06403)$ as a stable equilibrium point. Besides, we also obtain other equilibrium points, all of which are hyperbolic equilibrium points and are on the stability boundary of $(0.02801, 0.06403)$. Among them, there are six type-one hyperbolic points and six sources as shown in Table 1.

Table 1. All unstable equilibrium points on the stability boundary

Type	x_1	x_2	Type	x_1	x_2
1	0.0467	3.1149	2	2.6081	4.2548
1	0.0467	-3.1683	2	3.5972	1.5753
1	3.0407	3.2232	2	-2.6860	1.5753
1	3.2458	0.3341	2	2.6081	-2.0284
1	-3.0374	0.3341	2	-3.6751	-2.0284
1	-3.2425	-3.0600	2	-2.6860	-4.7097

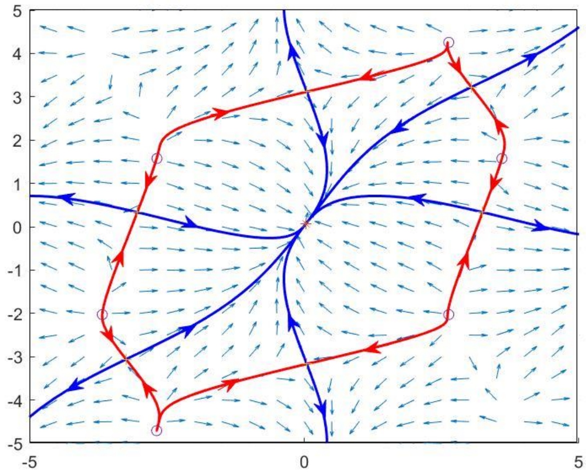

Figure 4. The stability boundary of the stable equilibrium point $(0.02801,\ 0.06403)$.

In Figure 4, the stability boundary is the bold red line and all equilibrium points in Table 1 are defined on the stability region. Furthermore, the stability boundary is the union of the stable manifolds of equilibrium point of type one. Figure 4 also shows that conclusions in Theorem 8, Proposition 12, Remark 13 and Corollary 14 are consistent since the stability boundary is bounded with six type-one equilibrium points and six source points.

5. Conclusion

In this work, we indicated several main characteristics of the stability boundary of a stable equilibrium point which are based on hyperbolic equilibrium points for a (autonomous) nonlinear dynamical system. Besides, we have also presented remarkable properties for predicting the stability boundary and the stability region of a stable equilibrium point. Our results can be extended to the critical points instead of hyperbolic equilibrium points and be applied for finding the stability region of nonlinear dynamical systems.

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