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Original Article

Hardy's Inequality on Time Scale

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Abstract: This work is concerned with the Hardy inequalityon time scales. It finds the condition for the existence of a constant C such that

$$\left[\int_{t_0}^{\infty}\left|U(x)\int_{t_0}^{x} f(t)\Delta t\right|^{p}\Delta x\right]^{\frac{1}{p}} \le C\left[\int_{t_0}^{\infty}\left|V(x)f(x)\right|^{p}\Delta x\right]^{\frac{1}{p}},$$

where p is a fixed number satisfying $1 \le p < \infty$.

Keywords: Hardy inequality, time scales, exponential function.

1. Introduction

The theory of mathematical inequalities is in a process of continuous development state and inequalities have become very effective and powerful tools for studying a wide range of problems in various branches of mathematics, especially in the qualitative and quantitative theory of differential and integral equations. Although this theory has a long history of development but in recent years it still has attracted a lot of attentions of mathematical researchers. In the studying difference or differential equations, one can use the Gronwall-Bellman type inequality to prove the existence and uniqueness of solutions when the coefficients are satisfied the Lipschitz condition. Also, this inequality can be used to estimate the upper bounds or lower bounds of solutions. However, using Gronwall-Bellman type inequality seems to be good tool to study the boundedness and robust stability of solutions [1]. In order to consider the integrality of solutions to a difference/differential equations, one recognize that it had

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better to use Hardy's inequality. That is while Hardy's inequality has evoked the interest of many mathematicians, and large numbers of papers have appeared which deal with new proofs, various extensions, generalizations [2].

So far, the Hardy inequality appears in the discrete time and continuous time. The proofs of Hardy inequality are rather different in each case. The main reason is that the derivative analysis can not use in discrete time and vice versa.

On the other hand, in the last twenty years, the theory of time scales was created by Hilger [3] in order not only to unify the theories of differential equations and of difference equations, but also to extend those theories to other kinds of the so-called dynamic equations. Therefore, the transformation of Hardy inequality from discrete/continuous time to time scale has great meaningfulness.

The main aim of this work is to develop Hardy's inequality on time scales. We extend results of Hardy's inequality in [1, 2] to a dynamic Hardy's inequality on time scales. Since the points of time scales are very diversity, it requires more complex techniques. The paper is organized as follows. In the next section, Section 2, we recall some notions and basic properties of analysis on time scales. Section 3 presents the form of Hardy's inequalities on time scales and its proof. In the last of this section, an application of Hardy inequality is considered to show the relationship between the integrability of solutions of inhomogeneous and the Lyapunov stability for dynamic equations on time scales.

2. Preliminary

In recent years, to unify continuous and discrete analysis or to describe the processing of numerical calculation with non-constant steps, a new theory appeared and became more and more extensively concerned, that is the theory of the analysis on time scales, which was introduced by Stefan Hilger 1988 [12].

A time scale is a nonempty closed subset of the real numbers $\mathbf{R}$, enclosed with the topology inherited from the standard topology on $\mathbf{R}$. We usually denote it by the symbol $\mathbf{T}$.

We define the *forward jump operator* $\sigma(t) = \inf\{s \in \mathbf{T} : s > t\}$ and the graininess $\mu(t) = \sigma(t) - t$. Similarly, the *backward operator* is defined as $\rho(t) = \sup\{s \in \mathrm{T} : s < t\}$ and the backward graininess is $\nu(t) = t - \rho(t)$.

A point $t \in \mathbf{T}$ is said to be *right-dense* if $\sigma(t) = t$, *right*-scattered if $\sigma(t) > t$, *left-dense* if $\rho(t) = t$, *left-scattered* if $\rho(t) < t$ and *isolated* if t is simultaneously right-scattered and left-scattered.

A function f defined on T valuated in $\mathbf{R}$ is *regulated* if there exist the left-sided limit at every left-dense point and right-sided limit at every right-dense point. A regulated function f is *called rd-continuous* if it is continuous at every right-dense point, and ld-continuous if it is continuous at every left-dense point. The set of rd-continuous functions defined on the interval J valued in X will be denoted by $C_{rd}(J, X)$.

A function f from $\mathbf{T}$ to $\mathbf{R}$ is *regressive* (resp., *positively regressive*) if for every $t \in \mathbf{T}$, then $1 + \mu(t) \neq 0$ (resp., $1 + \mu(t) > 0$). We denote by $\Re = \Re(\mathbf{T}, \mathbf{R})$ (resp., $\Re^+ = \Re^+(\mathbf{T}, \mathbf{R})$) the set (resp., positively regressive) regressive functions, and $C_{rd}\Re(\mathbf{T}, \mathbf{R})$ (resp., $C_{rd}\Re^+(\mathbf{T}, \mathbf{R})$) the set of rd-continuous (resp., positively regressive) regressive functions from $\mathbf{T}$ to $\mathbf{R}$.

For all $p, q \in \mathbf{T}$ the addition (*circle plus*) $\oplus$ and subtraction (*circle minus*) Θ of $p, q \in \mathbf{T}$ are defined

$$p \oplus q = p + q + \mu(t)pq, \qquad p\Theta q = \frac{p-q}{1+\mu(t)q}, q \in \Re.$$

It is easy to verify that, for all $p, q \in \Re,\ p \oplus q,\ p\ \Theta\ q \in \Re$. Element $(\Theta q)(.)$ is called the inverse element of element $q(.) \in \Re$. Hence, the set $\Re(\mathbf{T}, \mathbf{R})$ with the calculation $\oplus$ forms an Abelian group.

Definition 2.1 ([2])**.** (Delta Derivative)**.** A function $\varphi : \mathbf{T} \to \mathbf{R}^d$ is called delta differentiable at t if there exists a vector $\varphi^{\Delta}(t)$ such that for all ε >0,

$$\varphi(\sigma(t)) - \varphi(s)\left(\sigma(t) - s\right) \le \varepsilon \left|\sigma(t) - s\right|,$$

for all $s \in (t-\delta, t+\delta) \cap \mathbf{T}$ and for some δ >0. The vector $\varphi^{\Delta}(t)$ is called the delta-derivative of φ at t.

Theorem 2.2 ([2])**.** *If p is regressive and fix* $t_0 \in \mathbf{T}$*, then the only solution of the initial value problem*

$$y^{\Delta}(t) = p(t), \quad y(t_0) = 1,$$

on $\mathbf{T}$ *is defined by* $e_p(t, t_0)$, where

$$e_p(t, t_0) = \exp\left\{ \int_{t_0}^{t} \lim_{u \to \mu(\tau)} \frac{Ln(1 + up(\tau))}{u} \Delta\tau \right\}.$$

Theorem 2.3 ([2]) (Properties of the Exponential function)**.** *If* $\mathrm{p}, \mathrm{q} : \mathbf{T} \to \mathbf{R}$ *are regressive, rd-continuous functions and* $t, r, s \in \mathbf{T}$*, then the following hold*

$$e_p(t, \mathrm{t}) = 1, \ e_0(t, s) = 1 \qquad e_p(\sigma(t), \mathrm{s}) = (1 + \mu(\mathrm{t})\mathrm{p}(t))e_p(t, \mathrm{s}),$$

$$\frac{e_p(t, \mathrm{s})}{e_q(t, \mathrm{s})} = e_{p\Theta q}(t, \mathrm{s}), \qquad \frac{1}{e_p(t, \mathrm{s})} = e_{\Theta p}(t, \mathrm{s}) = e_p(\mathrm{s}, t),$$

$$e_p(t, \mathrm{s})e_q(t, \mathrm{s}) = e_{p \oplus q}(t, \mathrm{s}), \qquad e_p(t, \mathrm{s})e_q(s, r) = e_p(t, r).$$

.

Let T be a time scale. For any $\mathrm{a}, b \in \mathbf{T}$, the notation $[a, b]$ or (a, b) means the segment on $\mathbf{T}$, that is $[a, b] = \{t \in \mathrm{T} : a \le t \le b\}$ *or* $(a, b) = \{t \in \mathrm{T} : a < t < b\}$. and $\mathrm{T}_a = \{t \in \mathrm{T} : t \ge a\}$. We can define a measure $\Delta_{\mathbf{T}}$ on $\mathbf{T}$ by considering the Caratheodory construction of measures when we put $\Delta_{\mathbf{T}}[a, b) = b - a.$ The Lebesgue integral of a measurable function f with respect to $\Delta_{\mathbf{T}}$ is denoted by $\int_a^b f(\mathrm{s}) \Delta_{\mathbf{T}} s$ or $\int_a^b f(\mathrm{s}) \Delta s.$ For more details of analysis on time scales we can refer to [*2*]. In this paper, we suppose that the time scales T is unbounded above, i.e., $\sup \mathbf{T} = \infty.$

3. Hardy Inequality on Time Scales

Theorem 3.1 Let $1 \le p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $t_0 \in \mathrm{T}$, and U(x), V(x) are positive functions. Then,

$$\left[\int_{t_0}^{\infty} \left| U(x) \int_{t_0}^{x} f(t)\Delta t \right|^p \Delta x\right]^{\frac{1}{p}} \le p^{\frac{1}{p}} q^{\frac{1}{q}} B \left[\int_{t_0}^{\infty} \left| V(x) f(x)\right|^p \Delta x\right]^{\frac{1}{p}},$$

is true for real f, where $B = \sup_{r>0} \left[\int_{r}^{\infty} |U(x)|^p \Delta x\right]^{\frac{1}{p}} \left[\int_{t_0}^{r} |V(x)|^{-q} \Delta x\right]^{\frac{1}{q}}$, (with the convention $0^{\infty} = \infty^0 = 1$).

Proof

i) In case $1 < p < \infty$, let $h(\mathrm{x}) = \int_{t_0}^{x} V^{-q}(\mathrm{t})\Delta t$ and $g(x) = h^{\frac{1}{pq}}(x)$. Using H¨older inequality gets

$$\int_{t_0}^{\infty} | U(x) \int_{t_0}^{x} f(t)\Delta t |^p \Delta x = \int_{t_0}^{\infty} |U(x)|^p \left| \int_{t_0}^{x} V(\mathrm{t}) f(t) \mathrm{g}_{\sigma}(\mathrm{t}) V^{-1}(\mathrm{t}) \mathrm{g}_{\sigma}^{-1}(\mathrm{t}) \Delta t \right|^p \Delta x$$

$$\le \int_{t_0}^{\infty} |U(x)|^p \int_{t_0}^{x} |V(\mathrm{t}) f(t) \mathrm{g}_{\sigma}(\mathrm{t})|^p \Delta t \left[\int_{t_0}^{x} V^{-q}(\mathrm{t}) \mathrm{g}_{\sigma}^{-q}(\mathrm{t}) \Delta t\right]^{\frac{p}{q}} \Delta x$$

$$\le \int_{t_0}^{\infty} \left(|U(x)|^p \left[\int_{t_0}^{x} V^{-q}(\mathrm{t}) \mathrm{g}_{\sigma}^{-q}(\mathrm{t}) \Delta t\right]^{\frac{p}{q}} \int_{t_0}^{x} |V(\mathrm{t}) f(t) \mathrm{g}_{\sigma}(\mathrm{t})|^p \Delta t \right) \Delta x.$$

From Fubini theorem, we get

$$\int_{t_0}^{\infty} | U(x) \int_{t_0}^{x} f(t)\Delta t |^p \Delta x \le \int_{t_0}^{\infty} |V(\mathrm{t}) f(t)|^p \left(\mathrm{g}_{\sigma}^{p}(\mathrm{t}) \int_{\sigma(t)}^{\infty} |U(x)|^p \left[\int_{t_0}^{x} V^{-q}(\mathrm{s}) \mathrm{g}_{\sigma}^{-q}(s) \Delta s\right]^{\frac{p}{q}} \Delta x \right) \Delta t.$$

If x is right-dense then $\left(g^p(x)\right)^{\Delta} = \frac{1}{q} V^{-q}(x) g^{-q}(x)$. When x right-scattered, by using the finite-increments formula we have

$$\left(g^p(x)\right)^{\Delta} = \left(h^{\frac{1}{q}}(x)\right)^{\Delta} = \frac{h^{\frac{1}{q}}(\sigma(x)) - h^{\frac{1}{q}}(x)}{\mu(x)} = \frac{\left[h(x) + \mu(x) V^{-q}(x)\right]^{\frac{1}{q}} - h^{\frac{1}{q}}(x)}{\mu(x)}$$

$$\ge \frac{1}{q} V^{-q}(x) \left(\int_{t_0}^{\sigma(x)} V^{-q}(s)\Delta s \right)^{\frac{-1}{p}} = \frac{1}{q} V^{-q}(x) g^{-q}(\sigma(x)) = \frac{1}{q} V^{-q}(x) g_{\sigma}^{-q}(x).$$

Denote $G(x) = \int_{x}^{\infty} |U(s)|^p \Delta s.$ We obtain

$$\int_{t_0}^{\infty}|U(x)\int_{t_0}^{x} f(t)\Delta t\,|^p\ \Delta x \le q^{\frac{p}{q}}\int_{t_0}^{\infty}\left|V(t)f(t)\right|^p\left(g_\sigma^p(t)\int_{\sigma(t)}^{\infty}\left|U(x)\right|^p\left[\int_{t_0}^{x}V^{-q}(s)g_\sigma^{-q}(s)\Delta s\right]^{\frac{p}{q^2}}\Delta x\right)\Delta t$$

$$\le q^{\frac{p}{q}}B^{\frac{p}{q}}\int_{t_0}^{\infty}\left|V(t)f(t)\right|^p\left(g_\sigma^p(t)\int_{\sigma(t)}^{\infty}\left|U(x)\right|^p\left[\int_{x}^{\infty}\left|U(s)\right|^p\Delta s\right]^{\frac{-1}{q}}\Delta x\right)\Delta t$$

$$\le q^{\frac{p}{q}}B^{\frac{p}{q}}\int_{t_0}^{\infty}\left|V(t)f(t)\right|^p\left(g_\sigma^p(t)\int_{\sigma(t)}^{\infty}\left|U(x)\right|^p G^{\frac{-1}{q}}(x)\Delta x\right)\Delta t\ .$$

It is seen that when x is right-scattered, by using the finite-increments formula we have

$$\left(G^{\frac{1}{p}}(x)\right)^{\Delta}=\frac{G^{\frac{1}{p}}(x+\mu(\mathrm{x}))-G^{\frac{1}{p}}(x)}{\mu(\mathrm{x})}=\frac{\left[G(x)+\mu(\mathrm{x})\,\mathrm{U}^p(\mathrm{x})\right]^{\frac{1}{p}}-G^{\frac{1}{p}}(x)}{\mu(\mathrm{x})}\le\frac{-1}{p}\mathrm{U}^p(\mathrm{x})G^{\frac{-1}{q}}(x).$$

If x is right-dense $(G^{\frac{1}{p}}(x))^{\Delta}=\dfrac{-1}{p}\mathrm{U}^p(\mathrm{x})G^{\frac{-1}{q}}(x).$ Thus,

$$\int_{\sigma(t)}^{\infty}\left|U(\mathrm{s})\right|^p G^{\frac{-1}{q}}(s)\ \Delta s\le p\ G^{\frac{1}{p}}(\sigma(t))=\mathrm{p}\int_{\sigma(t)}^{\infty}\left|U(s)\right|^p\Delta s.$$

Summing up we have

$$\int_{t_0}^{\infty}|U(x)\int_{t_0}^{x} f(t)\Delta t\,|^p\ \Delta x \le pq^{\frac{p}{q}}B^{\frac{p}{q}}\int_{t_0}^{\infty}\left|V(t)f(t)\right|^p\left(g_\sigma^p(t)\int_{\sigma(t)}^{\infty}\left|U(s)\right|^p\Delta s\right)\Delta t\ .$$

Since $g_\sigma^p(t)\left(\int_{\sigma(t)}^{\infty}\left|U(s)\right|^p\Delta s\right)^{\frac{1}{p}}=\left(\int_{t_0}^{\sigma(t)}|V^{-q}(s)|\,\Delta s\right)^{\frac{1}{q}}\left(\int_{\sigma(t)}^{\infty}\left|U(s)\right|^p\Delta s\right)^{\frac{1}{p}}\le B,$ it follows that

$$\int_{t_0}^{\infty}|U(x)\int_{t_0}^{x} f(t)\Delta t\,|^p\ \Delta x \le pq^{\frac{p}{q}}B^p\int_{t_0}^{\infty}\left|V(t)f(t)\right|^p\Delta t\ .$$

In the other word,

$$\left[\int_{t_0}^{\infty}\left|U(x)\int_{t_0}^{x}f(t)\Delta t\right|^p\Delta x\right]^{\frac{1}{p}}\le p^{\frac{1}{p}}q^{\frac{1}{q}}B\left[\int_{t_0}^{\infty}\left|V(x)f(x)\right|^p\Delta x\right]^{\frac{1}{p}}.$$

ii) In case p=1, let us prove that

$$\int_{t_0}^{\infty}\left|U(x)\int_{t_0}^{x}f(t)\Delta t\right|\Delta x\le B\int_{t_0}^{\infty}\left|V(x)f(x)\right|\Delta x.$$

Following the Fubini theorem, it yields that

$$\int_{t_0}^{\infty}\left|U(x)\int_{t_0}^{x} f(t)\Delta t\right|\Delta x=\int_{t_0}^{\infty}\left(|f(t)|\int_{\sigma(t)}^{\infty}|\mathrm{U}(x)|\Delta x\right)\Delta t=\int_{t_0}^{\infty}\left(|V(\mathrm{t})f(t)|\frac{1}{V(\mathrm{t})}\int_{\sigma(t)}^{\infty}|\mathrm{U}(x)|\Delta x\right)\Delta t$$

$$\le\int_{t_0}^{\infty}\left(|V(\mathrm{t})f(t)|\sup_{0\le s\le\sigma(\mathrm{t})}\frac{1}{V(\mathrm{t})}\int_{\sigma(\mathrm{s})}^{\infty}|\mathrm{U}(x)|\Delta x\right)\Delta t\le B\int_{t_0}^{\infty}|V(\mathrm{t})f(t)|\Delta t.$$

iii) The case $p=\infty$ can be proved by a similar way. This completes the proof.

Corollary 3.2 Let $\alpha(.)$ be a function defined on **T** such that $0<\alpha_1=\min\alpha(\mathrm{t})\le\max\alpha(\mathrm{t}):=\alpha_2<\infty.$ Let $U(t)=V(t)=e_\alpha(t,t_0)$. Then

$$B=\sup_{r\in\mathrm{T}_{t_0}}\left[\int_r^{\infty}\left(e_{\Theta\alpha}(t,t_0)\right)^p\Delta x\right]^{\frac{1}{p}}\left[\int_{t_0}^{r}\left(e_{\Theta\alpha}(t,t_0)\right)^{-q}\Delta x\right]^{\frac{1}{q}}\le\frac{1}{\eta_\alpha},$$

where $\eta_\alpha=\dfrac{\alpha_1}{1+\alpha_2\mu^*}$ and $\mu^*=\max\mu(\mathrm{t}),\ \mathrm{t}\in\mathbf{T}.$

Proof

i) Consider the function $\left(e_{\Theta\alpha}(t,t_0)\right)^p$:

When $\mu(t)=0,\ \left(e^p_{\Theta\alpha}(t,t_0)\right)^\Delta=\Theta\alpha(s)e^p_{\Theta\alpha}(t,t_0)$, when $\mu(t)>0,$

$$\left(e^p_{\Theta\alpha}(t,t_0)\right)^\Delta=\frac{[1+\Theta\alpha(s)\mu(s)]^p-1}{\mu(t)}e^p_{\Theta\alpha}(t,t_0)=\frac{[1+\Theta\alpha(s)\mu(s)]^p-1}{\Theta\alpha(s)\mu(t)}\Theta\alpha(s)e^p_{\Theta\alpha}(t,t_0).$$

Moreover, $\Theta\alpha(s)\mu(t)=\dfrac{-\alpha(s)\mu(s)}{1+\alpha(s)\mu(t)}\in[-1,0]$,it is easy to prove that $1\le\dfrac{(1+x)^p-1}{x}\le p$, $x\in[-1,0].$

Therefore,

$$\Theta\alpha(s)p\le\frac{(1+\Theta\alpha(s)\mu(s))^p-1}{\Theta\alpha(s)\mu(t)}\le\Theta\alpha(s)=\frac{-\alpha(\mathrm{s})}{1+\alpha(\mathrm{s})\mu(\mathrm{s})}\le\frac{-\alpha_1}{1+\alpha_2\mu^*}-\eta_\alpha.$$

And $\left(e^p_{\Theta\alpha}(t,t_0)\right)^\Delta\le-\eta_\alpha e^p_{\Theta\alpha}(t,t_0).$ Hence

$$-e^p_{\Theta\alpha}(r,t_0)=\int_r^\infty\left(e^p_{\Theta\alpha}(s,t_0)\right)^\Delta\Delta s\le-\int_r^\infty\eta_\alpha e^p_{\Theta\alpha}(s,t_0)\Delta s.$$

Thus,

$$\int_r^\infty e^p_{\Theta\alpha}(s,t_0)\Delta s\le\frac{e^p_{\Theta\alpha}(r,t_0)}{\eta_\alpha}. \qquad (1)$$

ii) Consider the function $\left(e_{\Theta\alpha}(t,t_0)\right)^{-q}$:

When $\mu(t)=0,\ \left(e^{-q}_{\Theta\alpha}(t,t_0)\right)^\Delta=\Theta\alpha(s)e^{-q}_{\Theta\alpha}(t,t_0)$, when $\mu(t)>0,$

$$\left(e^{-q}_{\Theta\alpha}(t,t_0)\right)^\Delta=\frac{[1+\Theta\alpha(s)\mu(s)]^{-q}-1}{\mu(t)}e^{-q}_{\Theta\alpha}(t,t_0).$$

Using the inequality $-q\le\dfrac{(1+x)^{-q}-1}{x}\le-1$, for $x\in[-1,0],$ we have

$$\left(e_{\Theta\alpha}^{-q}(t,t_0)\right)^{\Delta}=\frac{\left[1+\Theta\alpha(s)\mu(s)\right]^{-q}-1}{\mu(t)}e_{\Theta\alpha}^{-q}(t,t_0)\geq-\Theta\alpha(s)e_{\Theta\alpha}^{-q}(t,t_0)\geq\eta_\alpha e_{\Theta\alpha}^{-q}(t,t_0).$$

Which implies that

$$e_{\Theta\alpha}^{-q}(r,t_0)-1=\int_{t_0}^{r}\left(e_{\Theta\alpha}^{-q}(s,t_0)\right)^{\Delta}\Delta s\geq\int_{r}^{\infty}\eta_\alpha e_{\Theta\alpha}^{-q}(s,t_0)\Delta s.$$

Thus,

$$\int_{r}^{\infty}e_{\Theta\alpha}^{-q}(s,t_0)\Delta s\leq\frac{e_{\Theta\alpha}^{-q}(r,t_0)-1}{\eta_\alpha}\leq\frac{e_{\Theta\alpha}^{-q}(r,t_0)}{\eta_\alpha}. \tag{2}$$

Combining (1) and (2) obtains the proof

$$B=\sup_{r\in T_{t_0}}\left[\int_{r}^{\infty}\left(e_{\Theta\alpha}(t,t_0)\right)^{p}\Delta x\right]^{\frac{1}{p}}\left[\int_{t_0}^{r}\left(e_{\Theta\alpha}(t,t_0)\right)^{-q}\Delta x\right]^{\frac{1}{q}}\leq\sup_{r\in T_{t_0}}\left[\frac{e_{\Theta\alpha}^{p}(r,t_0)}{\eta_\alpha}\right]^{\frac{1}{p}}\left[\frac{e_{\Theta\alpha}^{-q}(r,t_0)}{\eta_\alpha}\right]^{\frac{1}{q}}\leq\frac{1}{\eta_\alpha}.$$

Moreover, in this case, we have another form of Hardy inequality,

$$\left[\int_{t_0}^{\infty}\left|e_{\Theta\alpha}(t,t_0)\int_{t_0}^{t}f(s)\Delta s\right|^{p}\Delta t\right]^{\frac{1}{p}}\leq\frac{p^{\frac{1}{p}}q^{\frac{1}{q}}}{\eta_\alpha}\left[\int_{t_0}^{\infty}\left|e_{\Theta\alpha}(t,t_0)f(\mathrm{t})\right|^{p}\Delta x\right]^{\frac{1}{p}}.$$

Theorem 3.3 Let $1\leq p,q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, $t_0\in T$, and U(x), V(x) are positive functions. Then,

$$\left[\int_{t_0}^{\infty}\left|U(x)\int_{x}^{\infty}f(t)\Delta t\right|^{p}\Delta x\right]^{\frac{1}{p}}\leq C\left[\int_{t_0}^{\infty}\left|V(x)f(x)\right|^{p}\Delta x\right]^{\frac{1}{p}}, \tag{3}$$

is true for real f, where $B=\sup_{r>0}\left[\int_{t_0}^{r}\left|U(x)\right|^{p}\Delta x\right]^{\frac{1}{p}}\left[\int_{t_0}^{\infty}\left|V(x)\right|^{-q}\Delta x\right]^{\frac{1}{q}}$. Furthermore, if C is the least constant for which (3) true, then $B\leq C\leq p^{\frac{1}{p}}q^{\frac{1}{q}}B$.

Proof

Let $g(\mathrm{x})=\left|U(x)\int_{x}^{\infty}f(t)\Delta t\right|^{p-1}$, the p-th power of the left side of (5) becomes

$$\int_{t_0}^{\infty}\left|U(x)\int_{x}^{\infty}f(t)\Delta t\right|^{p}\Delta x=\int_{t_0}^{\infty}\left(U(x)\int_{x}^{\infty}f(t)\Delta t\right)g(x)\Delta x.$$

By using Fubini theorem and Holder inequality, we get

$$\int_{t_0}^{\infty}\left|U(x)\int_{x}^{\infty} f(t)\Delta t\right|^p \Delta x=\int_{t_0}^{\infty} f(t)\left(\int_{t_0}^{x}U(x)g(x)\Delta x\right)\Delta t=\int_{t_0}^{\infty}V(t)f(t)\left(\frac{1}{V(t)}\int_{t_0}^{x}U(x)g(x)\Delta x\right)\Delta t$$

$$\leq\left(\int_{t_0}^{\infty}\left|V(t)f(t)\right|^p\Delta t\right)^{\frac{1}{p}}\left(\int_{t_0}^{\infty}\left|\frac{1}{V(t)}\int_{t_0}^{x}U(x)g(x)\Delta x\right|^q\Delta t\right)^{\frac{1}{q}}.$$

Theorem 1 shows that

$$\left(\int_{t_0}^{\infty}\left|V^{-1}(t)\int_{t_0}^{x}U(x)g(x)\Delta x\right|^q\Delta t\right)^{\frac{1}{q}}\leq p^{\frac{1}{p}}q^{\frac{1}{q}}B\left(\int_{t_0}^{\infty}\left|U^{-1}(x)U(x)g(x)\right|^q\Delta x\right)^{\frac{1}{q}}=p^{\frac{1}{p}}q^{\frac{1}{q}}B\left(\int_{t_0}^{\infty}\left|g(x)\right|^q\Delta x\right)^{\frac{1}{q}},$$

where $$B=\sup_{r>0}\left[\int_{t_0}^{\infty}\left|V^{-1}(x)\right|^q\Delta x\right]^{\frac{1}{q}}\left[\int_{t_0}^{r}\left|U^{-1}(x)\right|^{-p}\Delta x\right]^{\frac{1}{p}}=\sup_{r>0}\left[\int_{t_0}^{r}\left|U(x)\right|^p\Delta x\right]^{\frac{1}{p}}\left[\int_{t_0}^{\infty}\left|V(x)\right|^{-q}\Delta x\right]^{\frac{1}{q}}.$$

And

$$\left(\int_{t_0}^{\infty}\left|g(x)\right|^q\Delta x\right)^{\frac{1}{q}}=\left(\int_{t_0}^{\infty}\left|U(x)\int_{x}^{\infty}f(t)\Delta t\right|^{(p-1)q}\Delta x\right)^{\frac{1}{q}}=\left(\int_{t_0}^{\infty}\left|U(x)\int_{x}^{\infty}f(t)\Delta t\right|^{p}\Delta x\right)^{\frac{1}{q}}.$$

Summing up, we get

$$\int_{t_0}^{\infty}\left|U(x)\int_{x}^{\infty}f(t)\Delta t\right|^p\Delta x\leq p^{\frac{1}{p}}q^{\frac{1}{q}}B\left(\int_{t_0}^{\infty}\left|V(t)f(t)\right|^p\Delta t\right)^{\frac{1}{p}}\left(\int_{t_0}^{\infty}\left|U(x)\int_{x}^{\infty}f(t)\Delta t\right|^p\Delta x\right)^{\frac{1}{q}}.$$

Therefore, $\left(\int_{t_0}^{\infty}\left|U(x)\int_{x}^{\infty}f(t)\Delta t\right|^p\Delta x\right)^{\frac{1}{p}}\leq p^{\frac{1}{p}}q^{\frac{1}{q}}B\left(\int_{t_0}^{\infty}\left|V(t)f(t)\right|^p\Delta t\right)^{\frac{1}{p}}$. The theorem is proved.

Application 3.4 Consider the dynamic equation on time scales $\mathbf{T}_{t_0}$,

$$x^{\Delta}(t)=A(t)x(t)+f(t),\qquad t\geq t_0. \tag{4}$$

Assume that $\Phi(t,s)$ is Cauchy operator generated by the homogeneous equation

$$x^{\Delta}(t)=A(t)x(t),\qquad t\geq t_0. \tag{5}$$

The equation (5) is called ω-exponentially stable if there exist numbers $M,\ \omega>0$ such that

$$\left\|\Phi(t,s)\right\|\leq Me_{\Theta\omega}(t,s),\quad \forall t\geq s\geq t_0.$$

For any $\beta>0$, we define the weighted spaces $\mathbf{L}_{\boldsymbol{\beta}}^{\mathbf{p}}(t_0)=\left\{f\in L_p^{loc}(\mathrm{T}_{t_0},\mathbf{R}^n):\ \int_{t_0}^{\infty}\left\|e_{\beta}(t,t_0)f(t)\right\|^p\Delta t<\infty\right\}$, with $\|f\|_{\mathbf{L}^{\mathbf{p}}_{\boldsymbol{\beta}}(t_0)} = \left(\int_{t_0}^{\infty}\|e_\beta(\mathrm{t},\mathrm{t}_0)f(\mathrm{s})\|^p\Delta t\right)^{\frac{1}{p}}$. It is easy to see that $\mathbf{L}^{\mathbf{p}}_{\boldsymbol{\beta}}(t_0)$ is a Banach space.

Denote also, $L^p_\beta(t_0) = \left\{ y \in L^{loc}_p(T_{t_0}, \mathbf{R}^n) : \int_{t_0}^{\infty} e_\beta(t,t_0)\|y(t)\|^p\Delta t < \infty \right\}$.

Then, we have conclusion as follows.

4. Conclusion

If the system (5) is α-exponentially stable, then for every $f \in \mathbf{L}^{\mathbf{p}}_{\boldsymbol{\beta}}(t_0)$, the solution x(t) of (4) with the initial $x(\mathrm{t}_0) = 0$ is in $L^p_\beta(t_0)$.

Indeed, by the variation of constants formula, the solution of the dynamic equation (3.2) with the initial $x(\mathrm{t}_0) = 0$ can be formulated as

$$x(\mathrm{t}) = \int_{t_0}^{t} \Phi(\mathrm{t},\sigma(\mathrm{s}))f(\mathrm{s})\Delta\mathrm{s}.$$

Since $e_{\Theta\omega}(\mathrm{t},\sigma(\mathrm{s})) = e_\omega(\sigma(\mathrm{s}),\mathrm{t}) = \left(1+\omega\mu(\mathrm{s})\right)e_\omega(\mathrm{s},\mathrm{t}) \le \left(1+\omega\mu^*\right)e_{\Theta\omega}(\mathrm{t},s)$, from the exponential stability of (3.5) we have

$$\|x(\mathrm{t})\|_{L^\beta_{t_0}} = \left(\int_{t_0}^{\infty} e_\beta(\mathrm{t},\mathrm{t}_0)\left\|\int_{t_0}^{t}\Phi(\mathrm{t},\sigma(\mathrm{s}))f(\mathrm{s})\Delta\mathrm{s}\right\|^p \Delta t\right)^{\frac{1}{p}} = M\left(\int_{t_0}^{\infty} e_\beta(\mathrm{t},\mathrm{t}_0)\left\|\int_{t_0}^{t}e_{\Theta\omega}(\mathrm{t},\sigma(\mathrm{s}))f(\mathrm{s})\Delta\mathrm{s}\right\|^p \Delta t\right)^{\frac{1}{p}}$$

$$\le M\left(1+\omega\mu^*\right)\left(\int_{t_0}^{\infty} e_\beta(\mathrm{t},\mathrm{t}_0)\left\|\int_{t_0}^{t}e_{\Theta\omega}(\mathrm{t},\mathrm{s})f(\mathrm{s})\Delta\mathrm{s}\right\|^p \Delta t\right)^{\frac{1}{p}} = M\left(1+\omega\mu^*\right)\left(\int_{t_0}^{\infty} e_{\beta\Theta\omega}(\mathrm{t},\mathrm{t}_0)\left\|\int_{t_0}^{t}e_{\omega}(\mathrm{s},\mathrm{t}_0)f(\mathrm{s})\Delta\mathrm{s}\right\|^p \Delta t\right)^{\frac{1}{p}}.$$

By using again Hardy's inequality with $U(t) = V(t) = e_{\beta\Theta\omega}(t,t_0)$ we obtain,

$$\left(\int_{t_0}^{\infty} e_\beta(\mathrm{t},\mathrm{t}_0)\left\|\int_{t_0}^{t}\Phi(\mathrm{t},\sigma(\mathrm{s}))f(\mathrm{s})\Delta\mathrm{s}\right\|^p \Delta t\right)^{\frac{1}{p}} \le M\left(1+\omega\mu^*\right)\left(\int_{t_0}^{\infty}\|e_\beta(\mathrm{t},\mathrm{t}_0)f(\mathrm{s})\|^p\Delta t\right)^{\frac{1}{p}}$$

$$= M\left(1+\omega\mu^*\right)\left(\int_{t_0}^{\infty} e^p_\beta(\mathrm{t},\mathrm{t}_0)\|f(\mathrm{s})\|^p\Delta t\right) = M\left(1+\omega\mu^*\right)\|f\|_{\mathbf{L}\boldsymbol{\beta}} < \infty.$$

Hence, $x(\mathrm{t}) \in L^p_\beta(t_0)$, we have above mentioned proofs.

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