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Original Article

# Euler Approximation for A Class of Singular Multi-Dimensional SDEs Driven by an Additive Fractional Noise

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**Abstract:** We consider a class of multi-dimensional stochastic differential equations driven by fractional Brownian motion with Hurst index $H > \frac{1}{2}$. In particular, the drift coefficient blows up at 0. We first prove that this equation has a unique positive solution, and then propose a semi-implicit Euler approximation scheme for the equation, and finally show that it is also positive, and study its rate of convergence.

*Keywords:* Euler approximation, Fractional Brownian motion, Fractional stochastic differential equation.

## 1. Introduction

Let $B^H = (B_1^H, \dots, B_m^H)$ be a $m$-dimensional fractional Brownian motion (fBm in short) with the Hurst parameter $H \in (0,1)$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. That is, $B^H$ is a centered Gaussian process with the covariance function $E\big(B_i^H(s)B_j^H(t)\big) = R_H(s,t)\delta_{ij}; i,j = 1, \dots, m$ where

$$R_H(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Notice that if $H = \frac{1}{2}$, the process $B^H$ is a standard Brownian motion. Fractional Brownian motion has the following self - similar property: for any constant $a > 0$, the processes $\{a^H B^H(t), t \geq 0\}$ and $\{B^H(at, t \geq 0\}$ have the same distribution. Fractional Brownian motion has stationary increments, i.e.

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$B^H(t) - B^H(s)$ and $B^H(t-s)$ have the same distribution. But if $H \neq \frac{1}{2}$, fBm does not have independent increments. Moreover, it is neither a semimartingale nor a Markov process for $H \neq \frac{1}{2}$.

Set $X(n) = B^H(n) - B^H(n-1), n \geq 1$. Then $\{X(n), n \geq 1\}$ is a Gaussian stationary sequence with covariance function

$$\rho_H(n) = \frac{1}{2}((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}.$$

Then

$$\lim_{n\to\infty} \frac{\rho_H(n)}{H(2H-1)n^{2H-2}} = 1.$$

Therefore, if $H > \frac{1}{2}$ then the increments of the corresponding fractional Brownian motion are positively correlated and exhibit the long range dependence property ($\sum_{n=1}^{\infty} \rho_H(n) = \infty$). The case $H < \frac{1}{2}$ corresponds to negatively correlated increments and the short range dependence ($\sum_{n=1}^{\infty} |\rho_H(n)| < \infty$). In [1-3], the real financial models are often characterized by the so- called "memory phenomenon". So, this property makes the fractional Brownian motion a suitable model for many applications.

We recall a result on the modulus of continuity of trajectories of fractional Brownian motion (see [4]).

**Lemma 1.1.** Let $B = \{B^H(t), t \geq 0\}$ be a fractional Brownian motion of Hurst parameter $H \in (0,1)$. Then for every $0 < \epsilon < H$ and $T > 0$ there exists an event $\Omega_{\epsilon,T} \subset \Omega$ with $\mathbb{P}(\Omega_{\epsilon,T}) = 1$, and $a$ positive random variable $\eta_{\epsilon,T}$ such that $E\left(|\eta_{\epsilon,T}|^p\right) < \infty$ for all $p \in [1,\infty)$ and for all $s,t \in [0,T]$,

$$|B^H(t,\omega) - B^H(s,\omega)| \leq \eta_{\epsilon,T}(w)|t-s|^{H-\epsilon}, \text{ for any } \omega \in \Omega_{\epsilon,T}.$$

As a consequence, the process $B^H$ has $\beta$ - Hölder continuous paths for all $\beta \in (0,H)$.

In this work, we fix $\frac{1}{2} < H < 1$. We consider a $d$-dimensional process $Y = (Y(t))_{0\leq t\leq T}$ satisfying the following SDEs,

$$dY_i(t) = \left(\frac{k_i}{Y_i(t)} + b_i\big(t, Y(t)\big)\right)dt + \sum_{j=1}^{m} \sigma_{ij} dB_j^H(t),\ 0 \leq t \leq T, \tag{1}$$

where $Y(0) = (Y_1(0), \dots, Y_d(0)), Y_i(0) > 0$ for each $i = 1, \dots, d$, and $B^H = (B_1^H, \dots, B_m^H)$ is a $m$-dimensional fractional Brownian motion with the Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t, t \in [0,T]\}$ satisfying the usual condition.

Our motivation to study Eq. (1) comes from mathematical finance. In [5], Cox, Ingersoll and Ross introduced the so-called CIR process $r = (r(t), t \geq 0)$, which is a solution of the following stochastic differential equation (SDE)

$$r(t) = r(0) + \int_0^t (k - ar(s))ds + \int_0^t \sigma\sqrt{r(s)}dW_s,\ , r(0), k, a, \sigma > 0,$$

where $W$ is a Brownian motion. This process has been widely used in mathematical finance to model the short term interest rates and the log-volatility in the Heston model. Recently, Gatheral, Jaisson, and Rosenbaum [6] stated that the log-volatility is very well modeled by a fractional Brownian motion. This fact raises the question of developing a fractional generalization of the classical CIR process. There have been many approaches to this problem. For example, Euch and Rosenbaum [7] constructed a fractional

CIR process via the rough limit of Hawkes processes. In [8], Mishura and Yurchenko-Tytarenko introduced a fractional Bessel type process

$$dy(t) = \frac{1}{2}\left(\frac{k}{y(t)} - ay(t)\right)dt + \frac{1}{2}\sigma dB_t^H,\ y_0 > 0, \tag{2}$$

where $B^H$ is a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, and then showed that $x(t) = y^2(t)$ satisfied the SDEs

$$dx(t) = (k - ax(t))dt + \sigma\sqrt{x(t)} \circ dB_t^H,\ t \geq 0,$$

where the integral with respect to fractional Brownian motion is considered as the path-wise Stratonovic integral. They also proposed an explicit Euler approximation for $y$. It is worth mentioning that even thought the exact solution $y = (y(t))_{t\geq 0}$ never hits zero, its explicit Euler approximation will hit zero with a positive probability. In [9], Hong et al. introduced a backward Euler scheme to Eq. (2) and show that the scheme strongly converges at the rate of order 1. In this work, we first show that if $b$ is of linear growth and Lipschitz continuous with respect to for all $i = 1, \dots, d$ and $t \in [0, T]$. Next, we propose a semi-implicit Euler approximation $Y^{(n)}$ for $Y$ and show that it converges at some rate, which depends on $H$ and the regularity of $b$ with respect to the time variable, in the path-wise sense. Under a further assumption on the nonnegativity of $b_i$, we show that the approximation converges in $L^p$-norm. A good feature of our approximation is that it is also positive. There are many works on the approximation of multidimensional fractional SDEs, see [10, 11] and the reference therein. However, these papers only deal with fractional SDEs with smooth coefficients. Note that the singular coefficients $\frac{1}{Y_i}$ make the system difficult to deal with. To the best of our knowledge, this is the first paper to discuss the approximation for multi-dimensional fractional SDEs with singular coefficients.

The remainder of the work is organized as follows: The existence and uniqueness of solution Section 3. In Section 4 we introduce an Euler approximation for (1) and study its rate of convergence. A numerical study is presented in Section 5.

## 2. The Existence and Uniqueness of the Solutions

Fix $1 - H < \alpha < \frac{1}{2}$ and $d \in \mathbb{N}^*$. Let's denote by $W_0^{\alpha,\infty}(0, T; \mathbb{R}^d)$ the space of measurable functions $f: [0, T] \to \mathbb{R}^d$ such that

$$\| f \|_{\alpha,\infty} := \sup_{t\in[0,T]} \left(|f(t)| + \int_0^t \frac{|f(t) - f(s)|}{|t - s|^{\alpha+1}} ds\right) < \infty.$$

For any $u < v$, let $C(u, v; \mathbb{R}^d)$ denote the Banach space of continuous functions $f: [u, v] \to \mathbb{R}^d$ equipped with the supremum norm

$$\| f \|_{u,v,\infty} = \sup\{|f(r)|, u < r < v\}.$$

Let $C^\beta(u, v; \mathbb{R}^d)$ denote the space of Hölder continuous functions of order $\beta$ on $[u, v]$. For any $f \in C^\beta(u, v; \mathbb{R}^d)$ the Hölder norm of the function $f$ is defined as follows

$$\| f \|_{u,v,\beta} = \sup\left\{\frac{|f(s) - f(t)|}{|s - t|^\beta}, u \leq s < t \leq v\right\}.$$

For a vector $y = (y_1, y_2, \dots, y_d) \in \mathbb{R}^d$, we define $|y|^2 = \sum_{i=1}^d y_i^2$. We fix a constant $T > 0$ and consider Eq. (1) on the interval $[0, T]$. Throughout this work, we suppose that $k_i > 0$ for all $i = 1, \dots, d$, the coefficients $b_i: [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}$ are mesurable functions, and there exist positive constants $L, L_0, L_1$ such that the following conditions hold:

(A1) $b_i(t,x), i = 1, \dots, d$ are globally Lipschitz continuous with respect to $x$, that is,

$$\sup_{i=1,\dots,d} |b_i(t,x) - b_i(t,y)| \le L|x-y|,$$

for all $x, y \in \mathbb{R}^d$ and $t \in [0,T]$;

(A2) $b_i(t,x), i = 1, \dots, d$ are linearly growth with respect to $x$, that is,

$$\sup_{i=1,\dots,d} |b_i(t,x)| \le L_0(1+|x|),$$

for all $x \in \mathbb{R}^d$ and $t \in [0,T]$;

(A3) There exists $\alpha \in (0,1]$ such that $\sup_{i=1,\dots,d} |b_i(t,x) - b_i(s,x)| \le L_1|t-s|^\alpha$,

for all $x \in \mathbb{R}^d$, $t, s \in [0,T]$.

We denote $\max\{a,b\}$ and $\min\{a,b\}$ by $a \vee b$ and $a \wedge b$, respectively. For each $n \in \mathbb{N}$ and $x = (x_1, \dots, x_d)$,

$$f_i^{(n)}(s,x) = \frac{k_i}{x_i \vee n^{-1}} + b_i(s,x) \vee \frac{-k_i n}{4}.$$

We consider the following fractional SDE

$$\begin{cases} Y_i^{(n)}(t) = Y_i(0) + \int_0^t f_i^{(n)}\left(s, Y^{(n)}(s)\right)ds + \sum_{j=1}^m \sigma_{ij} dB_j^H(t), t \in [0,T], \\ Y_i(0) > 0 \text{ for all } i = 1, \dots, d. \end{cases} \quad (3)$$

**Lemma 2.1.** Under the assumptions (A1), (A2), Eq. (3) has a unique solution $\mathrm{Y}^{(n)} \in \mathrm{L}^0\left(\Omega, \mathcal{F}, \mathbb{P}; \mathrm{W}_0^{\alpha,\infty}\left(0, \mathrm{T}; \mathbb{R}^d\right)\right)$ and for $\mathbb{P}$ - almost all $\omega \in \Omega$

$$\mathrm{Y}^{(n)}(\omega, .) = \left(\mathrm{Y}_i^{(n)}(\omega, .)\right)_{d\times 1} \in \mathrm{C}^{1-\alpha}\left(0, \mathrm{T}; \mathbb{R}^d\right).$$

Moreover,

$$\mathbb{E}\|\mathrm{Y}^{(n)}\|_{\alpha,\infty}^p < \infty, \forall p \ge 1.$$

*Proof.* Using the estimate $|a \vee c - b \vee c| \le |a - b|$ and assumptions (A1), (A2), it is straightforward to verify that

$$\left|f_i^{(n)}(t,x) - f_i^{(n)}(t,y)\right| \le (k_i n^2 + L)|x-y|, \text{ for all } x, y \in \mathbb{R}^d, \text{ and } t \in [0,T],$$

and

$$\left|f_i^{(n)}(t,x)\right| \le \frac{5}{4} k_i n + L_0(1+|x|).$$

Hence it follows from Theorem 2.1 in [12], Eq. (3) has a unique solution $Y^{(n)} \in L^0\left(\Omega, \mathcal{F}, \mathbb{P}; W_0^{\alpha,\infty}\left(0, T; \mathbb{R}^d\right)\right)$ and for $\mathbb{P}$ - almost all $\omega \in \Omega$

$$Y^{(n)}(\omega, .) = \left(Y_i^{(n)}(\omega, .)\right)_{d\times 1} \in C^{1-\alpha}\left(0, T; \mathbb{R}^d\right).$$

Moreover

$$\mathbb{E}\|Y^{(n)}\|_{\alpha,\infty}^p < \infty, \forall p \ge 1. \qquad \square$$

Now, we set

$$\tau_n = \inf\left\{t \in [0, T]: \min_{1\le i\le d} \left|Y_i^{(n)}(t)\right| \le \frac{1}{n}\right\} \wedge T.$$

In order to prove that Eq. (1) has a unique solution on $[0, T]$ we need the following lemma.

**Lemma 2.2.** The sequence $\tau_n$ is non-decreasing, and for almost all $\omega \in \Omega, \tau_n(\omega) = T$ for n large enough.

*Proof.* We will adapt the contradiction method in [8]. It follows from Lemma 1.1 that for any $\epsilon \in \left(0, H - \frac{1}{2}\right)$, there exists a finite random variable $\eta_{\epsilon,T}$ and an event $\Omega_{\epsilon,T} \in \mathcal{F}$ which do not depend on $n$, such that $\mathbb{P}(\Omega_{\epsilon,T}) = 1$, and

$$\sup_{1\le i\le d}\left|\sum_{j=1}^m \sigma_{ij}\left(B_j^H(t,\omega) - B_j^H(s,\omega)\right)\right| \le \eta_{\epsilon,T}(\omega)|t - s|^{H-\epsilon}, \tag{4}$$

for any $\omega \in \Omega_{\epsilon,T}$ and $0 \le s < t \le T$. Assume that for some $\omega_0 \in \Omega_{\epsilon,T}, \tau_n(\omega_0) < T$ for all $n \in \mathbb{N}$. Thanks to the continuity of sample paths of $Y^{(n)}$, it follows from the definition of $\tau_n$ that there exists an index $i_0 \in \{1, \dots, d\}$ such that $Y_{i_0}^{(n)}(\tau_n(\omega_0), \omega_0) = \frac{1}{n}$. Set

$$\kappa_n(\omega_0) = \sup\left\{t \in [0, \tau_n(\omega_0)]: Y_{i_0}^{(n)}(t, \omega_0) \ge \frac{2}{n}\right\}.$$

We have $\frac{1}{n} \le Y_{i_0}^{(n)}(t, \omega_0) \le \frac{2}{n}$, for all $t \in [\kappa_n(\omega_0), \tau_n(\omega_0)]$. In order to simplify our notation, we will omit $\omega_0$ in brackets in further formulas. We have

$$Y_{i_0}^{(n)}(\tau_n) - Y_{i_0}^{(n)}(\kappa_n) = -\frac{1}{n} = \int_{\kappa_n}^{\tau_n} f_{i_0}^{(n)}\left(s, Y^{(n)}(s)\right) ds + \sum_{j=1}^m \sigma_{i_0 j}\left(B_j^H(\tau_n) - B_j^H(\kappa_n)\right).$$

This implies

$$\left|\sum_{j=1}^m \sigma_{i_0 j}\left(B_j^H(\tau_n) - B_j^H(\kappa_n)\right)\right| = \left|\frac{1}{n} + \int_{\kappa_n}^{\tau_n}\left(\frac{k_{i_0}}{Y_{i_0}^{(n)}(s)\vee n^{-1}} + b_{i_0}\left(s, Y^{(n)}(s)\right) \vee \frac{-k_{i_0}n}{4}\right)ds\right|. \tag{5}$$

Note that for all $s \in [\kappa_n, \tau_n]$

$$\frac{k_{i_0}}{Y_{i_0}^{(n)}(s) \vee n^{-1}} + b_{i_0}\left(s, Y^{(n)}(s)\right) \vee \frac{-k_{i_0}n}{4} \ge \frac{k_{i_0}}{Y_{i_0}^{(n)}(s)} - \frac{k_{i_0}n}{4} \ge \frac{k_{i_0}n}{4}.$$

Thus for all $n > n_0 = \frac{2}{\min_{1\le i\le d} Y_i(0)}$, it follows from (5) that

$$\left|\sum_{j=1}^m \sigma_{i_0 j}\left(B_j^H\tau_n) - B_j^H(\kappa_n)\right)\right| \ge \frac{1}{n} + \frac{k_{i_0}n}{4}(\tau_n - \kappa_n).$$

This fact together with (4) implies that

$$\eta_{\epsilon,T}|\tau_n - \kappa_n|^{H-\epsilon} \ge \frac{1}{n} + \frac{k_{i_0}n}{4}(\tau_n - \kappa_n), \quad \text{for all } n \ge n_0. \tag{6}$$

Applying Hölder's inequality, for any $x > 0$, it holds

$$\frac{k_{i_0}n}{4}x + \frac{1}{n} \ge \left(\frac{k_{i_0}}{4(H-\epsilon)}\right)^{H-\epsilon}\left(\frac{1}{1-H+\epsilon}\right)^{1-H+\epsilon} n^{2(H-\epsilon)-1}x^{H-\epsilon}.$$

Then it follows from (6) that

$$\eta_{\epsilon,T} \geq \left(\frac{k_{i_0}}{4(H-\epsilon)}\right)^{H-\epsilon} \left(\frac{1}{1-H+\epsilon}\right)^{1-H+\epsilon} n^{2(H-\epsilon)-1}, \text{ for all } n \geq n_0.$$

This is a contradiction since $2(H-\epsilon)-1>0$ for any $\epsilon \in \left(0, H-\frac{1}{2}\right)$. Therefore $\tau_n(\omega_0)=T$ for $n$ large enough. □

**Lemma 2.3.** If $(Y(t))_{0\leq t\leq T}$ is a solution of Eq. (1) then $Y_i(t)>0$ for all $t\in[0,T]$ almost surely.

*Proof.* Assume that for some $\omega_0\in\Omega$ and $i_0\in\{1,\dots,d\}$, $\inf_{t\in[0,T]} Y_{i_0}(t,\omega_0)=0$. Denote $M=\sup_{t\in[0,T]}|Y(t,\omega_0)|$ and $\tau=\inf\{t: Y_{i_0}(t,\omega_0)=0\}$. For each $n\geq 1$, we denote $\nu_n=\sup\{t<\tau : Y_{i_0}(t,\omega_0)=\frac{1}{n}\}$. Since $Y$ has continuous sample paths, $0<\nu_n<\tau\leq T$ and $Y_{i_0}(t,\omega_0)\in\left(0,\frac{1}{n}\right)$ for all $t\in(\nu_n,\tau)$. Note that

$$-\frac{1}{n}=Y_{i_0}(\tau)-Y_{i_0}(\nu_n)=\int_{\nu_n}^{\tau}\left(\frac{k_{i_0}}{Y_{i_0}(s)}+b_{i_0}(s,Y(s))\right)ds+\sum_{j=1}^{d}\sigma_{i_0 j}\left(B_j^H(\tau)-B_j^H(\nu_n)\right).$$

If $n>\frac{2L_0(1+M)}{k_{i_0}}$ then $\left|b_{i_0}(s,Y(s,\omega_0))\right|\leq L_0(1+|Y(s,\omega_0)|)\leq L_0(1+M)\leq\frac{k_{i_0}n}{2}$, and

$$\left|\sum_{j=1}^{m}\sigma_{i_0 j}\left(B_j^H(\tau,\omega_0)-B_j^H(\nu_n,\omega_0)\right)\right|\geq\frac{1}{n}+\frac{k_{i_0}n}{2}(\tau-\nu_n). \tag{7}$$

Using the same argument as in the proof of Lemma 2.2 we see that the inequality (7) fails for all $n$ large enough. This contradiction completes the lemma. □

**Theorem 2.4.** For each $T>0$ Eq. (1) has a unique solution $Y\in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$. Moreover the solution is $(1-\alpha)$-Hölder continuous and

$$\mathbb{E}\,\|Y\|_{\alpha,\infty}^p<\infty, \forall p\geq 1.$$

*Proof.* We first show the existence of a positive solution. From Lemma 2.2, there exists a finite random variable $n_0$ such that $Y_i^{(n)}(t)\geq\frac{1}{n_0}>0$ almost surely for any $t\in[0,T]$ and $i=1,\dots,d$. Since $\left|x\vee\frac{-k_i n}{4}\right|\leq|x|$, for all $n>n_0$ we have

$$\left|Y_i^{(n)}(t)\right|\leq|Y_i(0)|+n_0k_iT+\sup_{s\in[0,T]}\left|\sum_{j=1}^{m}\sigma_{ij}B_j^H(s)\right|+\int_0^t\left|b_i(s,Y^{(n)}(s))\right|ds$$

Thanks to condition $(A2)$,

$$\begin{aligned}\sum_{i=1}^d\left|Y_i^{(n)}(t)\right|\leq\sum_{i=1}^d|Y_i(0)|+n_0T\sum_{i=1}^d k_i &+\sum_{i=1}^d\sum_{j=1}^m|\sigma_{ij}|\sup_{s\in[0,T]}\left|B_j^H(s)\right|\\ &+L_0d\int_0^t\left(1+\sum_{i=1}^d\left|Y_i^{(n)}(s)\right|\right)ds\end{aligned} \tag{8}$$

Applying Gronwall's inequality, we get $\sum_{i=1}^d\left|Y_i^{(n)}(t)\right|\leq C_1e^{dL_0T}$, for any $t\in[0,T]$, where

$$C_1=\sum_{i=1}^d|Y_i(0)|+n_0T\sum_{i=1}^d k_i+\sum_{i=1}^d\sum_{j=1}^m|\sigma_{ij}|\sup_{s\in[0,T]}\left|B_j^H(s)\right|+dL_0T<\infty.$$

Note that $C_1$ is a finite random variable which does not depend on $n$. Therefore,

$$\sup_{0\leq t\leq T}\left|b_i(t,Y^{(n)}(t))\right|\leq L_0\left(1+\sup_{0\leq t\leq T}\left|Y^{(n)}(t)\right|\right)\leq L_0\left(1+C_1e^{dL_0T}\right).$$

This implies that for any $n \geq n_0 \vee \frac{4L_0(1+C_1e^{dL_0T})}{k_i}$, $\inf_{t\in[0,T]} b_i(t, Y^{(n)}(t)) > \frac{-k_i n}{4}$. Therefore the process $Y_i^{(n)}(t)$ converges almost surely to a positive limit, called $Y_i(t)$ when $n$ tends to infinsolution $Y \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ and is $(1-\alpha)$ - Hölder continuous. Moreover

$$\mathbb{E} \parallel Y \parallel_{\alpha,\infty}^p < \infty, \forall p \geq 1.$$

Next, we show that Eq. (1) has a unique solution in path-wise sense. Let $Y(t)$ and $\hat{Y}(t)$ be

$$\left|Y_i(t,\omega) - \hat{Y}_i(t,\omega)\right| = \left|\int_0^t \left(\frac{k_i}{Y_i(s,\omega)} + b_i(s, Y(s,\omega)) - \frac{k_i}{\hat{Y}_i(s,\omega)} - b_i(s, \hat{Y}(s,\omega))ds\right|\right.$$

$$\leq \int_0^t \left|\frac{k_i}{Y_i(s,\omega)} - \frac{k_i}{\hat{Y}_i(s,\omega)}\right| ds + \int_0^t \left|b_i(s, Y(s,\omega)) - b_i\left(s, \hat{Y}(s,\omega)\right)\right| ds.$$

Using continuous property of the sample paths of $Y(t)$ and $\hat{Y}(t)$ and Lemma 2.3, we have

$$m_0 = \min_{i=1,\ldots,d} \min_{t\in[0,T]} \left\{Y_i(t,\omega), \hat{Y}_i(t,\omega)\right\} > 0.$$

Together with the Lipschitz condition of $b$ we obtain

$$\left|Y_i(t,\omega) - \hat{Y}_i(t,\omega)\right| \leq \int_0^t \frac{k_i\left|Y_i(s,\omega) - \hat{Y}_i(s,\omega)\right|}{m_0^2} ds + \int_0^t L|Y(s,\omega) - \hat{Y}(s,\omega)|ds.$$

Set $K = \max_{1\leq i\leq d} k_i$, we have $\sum_{i=1}^d \left|Y_i(t,\omega) - \hat{Y}_i(t,\omega)\right| \leq \left(\frac{K}{m_0^2} + Ld\right) \int_0^t \sum_{i=1}^d |Y_i(s,\omega) - (s,\omega)|ds$. By Gronwall's inequality, $\sum_{i=1}^d \left|Y_i(t,\omega) - \hat{Y}_i(t,\omega)\right| = 0$, for all $t \in [0,T]$. Therefore, $Y(t,\omega) = \hat{Y}(t,\omega)$ for all $t \in [0,T]$. The uniqueness has been concluded. □

## 3. Moment and Inverse Moment

Fix $\beta \in (1/2, H)$. The next result provides an estimate for the supremum norm of the solution in terms of the Hölder norm of the fractional Brownian motion $B^H$.

**Theorem 3.1.** Assume that conditions (A1) − (A2) are satisfied, and Y(t) is the solution of Eq. (1). Then for any $\gamma > 2$, and for any $T > 0$,

$$\parallel Y \parallel_{0,T,\infty} \leq C_{1,\gamma,\beta,T,K,C,d}(|y_0| + 1)\exp\left\{C_{2,\gamma,\beta,T,K,C,d,\sigma}\left(\|B^H\|_{0,T,\beta}^{\frac{\gamma}{\beta(\gamma-1)}} + 1\right)\right\},$$

Where $K = \max_{1\leq i\leq d} k_i, \sigma = \max_{1\leq i\leq d}\left(\sum_{j=1}^m |\sigma_{ij}|\right)$.

*Proof.* Fix a time interval $[0,T]$. For each $i = 1, \ldots, d$, let $z_i(t) = Y_i^\gamma(t)$. Applying the chain rule for Young integral, we have

$$z_i(t) = Y_i^\gamma(0) + \gamma \int_0^t \left(\frac{k_i}{z_i^{\frac{1}{\gamma}}(s)} + b_i(s, Y(s))\right) z_i^{1-\frac{1}{\gamma}}(s)ds + \gamma \int_0^t \sum_{j=1}^m \sigma_{ij} z_i^{1-\frac{1}{\gamma}}(s) dB_j^H(s).$$

Then

$$|z_i(t) - z_i(s)| = \left|\gamma \int_s^t \left(\frac{k_i}{z_i^{1/\gamma}(u)} + b_i(u, Y(u))\right) z_i^{1-\frac{1}{\gamma}}(u)du + \gamma \int_s^t \sum_{j=1}^m \sigma_{ij} z_i^{1-\frac{1}{\gamma}}(u) dB_j^H(u)\right|. \quad (9)$$

Applying Hölder's inequality

$$|Y(u)|^2 \leq \left(\sum_{i=1}^{d} |Y_i(u)|^{2\gamma}\right)^{1/2\gamma} \left(\sum_{i=1}^{d} |Y_i(u)|^{\frac{2\gamma}{2\gamma-1}}\right)^{\frac{2\gamma-1}{2\gamma}}$$
$$\leq |z(u)|^{1/\gamma} \left(\sum_{i=1}^{d} |Y_i(u)|\right) \leq |z(u)|^{1/\gamma}\sqrt{d}|Y(u)|.$$

It leads to $|Y(u)| \leq \sqrt{d}|z(u)|^{1/\gamma}$. Together with the condition $(A2)$ we obtain

$$I_{1i} := \left|\int_s^t \left(\frac{k_i}{z_i^{1/\gamma}(u)} + b_i(u, Y(u))\right) z_i^{1-\frac{1}{\gamma}}(u)du\right|$$
$$\leq \int_s^t \left(k_i \left|z_i^{1-\frac{2}{\gamma}}(u)\right| + L_0\left(1 + \sqrt{d}|z(u)|^{1/\gamma}\right)\left|z_i^{1-\frac{1}{\gamma}}(u)\right|\right) du.$$

Since $\gamma > 2$ then we have

$$I_{1i} \leq \left[k_i \parallel z \parallel_{s,t,\infty}^{1-\frac{2}{\gamma}} + L_0 \parallel z \parallel_{s,t,\infty}^{1-\frac{1}{\gamma}} + L_0\sqrt{d} \parallel z \parallel_{s,t,\infty}\right](t-s). \tag{10}$$

For each $i = 1, \dots, d$, let $I_{2ij} = \left|\int_s^t z_i^{1-\frac{1}{\gamma}}(u)dB_j^H(u)\right|, j = 1, \dots, m$. Following the argument in the proof of Theorem 2.3 in [13] we have

$$I_{2ij} \leq R\|B_j^H\|_{0,T,\beta}\left(\parallel z \parallel_{s,t,\infty}^{1-\frac{1}{\gamma}} (t-s)^{\beta} + \parallel z \parallel_{s,t,\beta}^{1-\frac{1}{\gamma}} (t-s)^{\beta\left(2-\frac{1}{\gamma}\right)}\right).$$

where $R$ is a generic constant depending on $\alpha, \beta$ and $T$. Then for any $j = 1, \dots, m$,

$$I_{2ij} \leq R\|B^H\|_{0,T,\beta}\left(\parallel z \parallel_{s,t,\infty}^{1-\frac{1}{\gamma}} (t-s)^{\beta} + \parallel z \parallel_{s,t,\beta}^{1-\frac{1}{\gamma}} (t-s)^{\beta\left(2-\frac{1}{\gamma}\right)}\right). \tag{11}$$

Substituting (10) and (11) into (9), we obtain

$$|z_i(t) - z_i(s)| \leq \gamma \left[k_i \parallel z \parallel_{s,t,\infty}^{1-\frac{2}{\gamma}} + L_0 \mid z \left\|_{s,t,\infty}^{1-\frac{1}{\gamma}} + L_0\sqrt{d}\right\| z \parallel_{s,t,\infty}\right](t-s) +$$
$$+\left(\sum_{j=1}^{d} |\sigma_{ij}|\right)\gamma R\|B^H\|_{0,T,\beta}\left[\parallel z \parallel_{s,t,\infty}^{1-\frac{1}{\gamma}} (t-s)^{\beta} + \parallel z \parallel_{s,t,\beta}^{1-\frac{1}{\gamma}} (t-s)^{\beta\left(2-\frac{1}{\gamma}\right)}\right].$$

Therefore, for any $i = 1, \dots d$, we obtain

$$|z_i(t) - z_i(s)| \leq \gamma \left[K \parallel z \parallel_{s,t,\infty}^{1-\frac{2}{\gamma}} + L_0 \parallel z \parallel_{s,t,\infty}^{1-\frac{1}{\gamma}} + L_0\sqrt{d} \parallel z \parallel_{s,t,\infty}\right](t-s) +$$
$$+\sigma\gamma R\|B^H\|_{0,T,\beta}\left[\parallel z \parallel_{s,t,\infty}^{1-\frac{1}{\gamma}} (t-s)^{\beta} + \parallel z \parallel_{s,t,\beta}^{1-\frac{1}{\gamma}} (t-s)^{\beta\left(2-\frac{1}{\gamma}\right)}\right], \tag{12}$$

where $K = \max_{1\leq i\leq d} k_i, \sigma = \max_{1\leq i\leq d}\left(\sum_{j=1}^{m} |\sigma_{ij}|\right)$. So

$$\parallel z \parallel_{s,t,\beta} \leq \sqrt{d}\gamma \left[K \parallel z \parallel_{s,t,\infty}^{1-\frac{2}{\gamma}} + L_0 \parallel z \parallel_{s,t,\infty}^{1-\frac{1}{\gamma}} + L_0\sqrt{d} \parallel z \parallel_{s,t,\beta}\right](t-s)^{1-\beta} +$$
$$+\sqrt{d}\sigma\gamma R\|B^H\|_{0,T,\beta}\left[\parallel z \parallel_{s,t,\infty}^{1-\frac{1}{\gamma}} + \left(1 + \parallel z \parallel_{s,t,\infty}\right)(t-s)^{\beta\left(1-\frac{1}{\gamma}\right)}\right].$$

We choose $\Delta$ such that

$$\Delta = \left[\frac{1}{2\sqrt{d}\sigma\gamma R\|B^H\|_{0,T,\beta}}\right]^{\frac{\gamma}{\beta(\gamma-1)}} \wedge \frac{1}{8\sqrt{d}\gamma(K+L_0)+8d\gamma L_0} \wedge \left(\frac{1}{8\sqrt{d}\sigma\gamma R \parallel B \parallel_{0,T,\beta}}\right)^{1/\beta}.$$

By following similar arguments in the proof of Theorem 2.3 in [13], for all $s,t \in [0,T], s \le t$ such that $t-s \le \Delta$, we have $\parallel z \parallel_{s,t,\infty} \le 2|z(s)| + 4\sqrt{d}\gamma(K+L_0)T + 4T^\beta$. It leads to

$$\parallel z \parallel_{0,T,\infty} \le 2^{T\left[\left(2\sqrt{d}\sigma\gamma R\|B^H\|_{0,T,\beta}\right)^{\frac{\gamma}{\beta(\gamma-1)}} \vee \left(8\sqrt{d}\gamma(K+L_0)+8d\gamma L_0\right) \vee \left(8\sqrt{d}\sigma\gamma R\|B\|_{0,T,\beta}\right)^{1/\beta}\right]+1} \times$$
$$\times \left(|z(0)| + 4\sqrt{d}\gamma(K+L_0)T + 4T^\beta\right).$$

This fact together with the estimate $\parallel Y \parallel_{0,T,\infty} \le \sqrt{d} \parallel z \parallel_{0,T,\infty}^{1/\gamma}$, implies that

$$\parallel Y \parallel_{0,t,\infty} \le C_{1,\gamma,\beta,T,K,L_0,d}(|y_0|+1)\exp\left\{C_{2,\gamma,\beta,T,K,L_0,d,\sigma}\left(\|B^H\|_{0,T,\beta}^{\frac{\gamma}{\beta(\gamma-1)}} + 1\right)\right\},$$

which concludes the proof. □

In the following we show that the inverse moment of $Y$ exists when $b$ is non-negative.

**Proposition 3.2.** Let $(Y(t))_{t\ge0}$ be the solution to Eq. (1). We assume that $b_i(t,x) \ge 0$ for all $i = 1,2,\dots,d, x \in \mathbb{R}^d$ and $t \ge 0$. Then, for all $p \ge 1$

$$\underset{1\le i\le d}{sup}\ \underset{t\in[0,T_p]}{sup}\ \mathbb{E}(Y_i(t)^{-p}) < \infty$$

where $T_p = \left(\frac{K_0}{(p+1)H\sigma^2}\right)^{\frac{1}{2H-1}}, K_0 = min\{k_i, 1 \le i \le d\}$, and $\sigma = max\left\{\sum_{j=1}^m \sigma_{ij}, 1 \le i \le d\right\}$.

*Proof.* We consider the following equations

$$X_i(t) = Y_i(0) + \int_0^t \frac{k_i}{X_i(s)} ds + \sum_{j=1}^m \sigma_{ij} dB_j^H(t),\ 1 \le i \le d. \tag{13}$$

By Section 2, Eq. (13) has a unique positive solution in the sense that $X_i(t) > 0$ for all $i = 1,2,\dots,d$. Moreover, using the argument of the Proposition 3.4 in [13], we have

$$\sup_{1\le i\le d}\ \sup_{t\in[0,T_p]} \mathbb{E}(X_i(t)^{-p}) < \infty, \tag{14}$$

for all $p \ge 1$.

Since $Y_i(t) - X_i(t) = \int_0^t \left(\frac{-k_i(Y_i(s)-X_i(s))}{X_i(s)Y_i(s)} + b_i(s,Y(s))\right) ds$, we have

$Y_i(t) - X_i(t) = e^{\int_0^t a(s)ds} \int_0^t b_i(s,Y(s)) e^{-\int_r^s a(u)du} ds$, where $a(t) = \frac{-k_i}{X_i(t)Y_i(t)}$. Since $b_i$ is nonnegative, $Y_i(t) \ge X_i(t) \ge 0$ for all $i = 1,2,\dots,d$ and $t > 0$. This fact together with (14) implies the desired result. □

## 4. Euler Approximation Scheme

For each positive integer $n$, we consider an uniform partition of the interval $[0,T], t_i = \frac{iT}{n}, i = 0,1,\dots,n$, we define $\eta_n(t) = t_i$ when $t_i \le t < t_i + \frac{T}{n}$ and $\kappa_n(t) = t_i + \frac{T}{n}$ when $t_i \le t < t_i + \frac{T}{n}$.

The classical Euler approximation scheme for (1) is defined as follows: $\tilde{Y}^{(n)}(0) := Y(0)$, and for each $k = 0,\dots,n-1$ and $i = 1,\dots,d$,

$$\tilde{Y}_i^{(n)}(t_{k+1}) = \tilde{Y}_i^{(n)}(t_k) + \left(\frac{k_i}{\tilde{Y}_i^{(n)}(t_k)} + b_i\left(t_k, \tilde{Y}^{(n)}(t_k)\right)\right)\frac{T}{n} + \sum_{j=1}^{m} \sigma_{ij}\left(B_j^H(t_{k+1}) - B_j^H(t_k)\right). \quad (15)$$

Since $B^H(t_{k+1}) - B^H(t_k)$ has a normal distribution, $\tilde{Y}_i^{(n)}(t_{k+1}) < 0$ with a positive probability as long as $\sigma_{ij} \neq 0$ for some $j = 1, \dots, d$. Therefore, the classical Euler approximate solution does not preserve the positivity of the true solution.

In this section we introduce a modified Euler approximation scheme for the Eq. (1), which can preserve the positivity of the true solution. That new scheme is defined as follows: $Y^{(n)}(0) := Y(0)$, and for each $k = 0, \dots, n-1$, $Y^{(n)}(t_{k+1}) = \left(Y_1^{(n)}(t_{k+1}), \dots, Y_d^{(n)}(t_{k+1})\right)$ where $Y_i^{(n)}(t_{k+1}), i = 1, \dots, d$, are the unique positive solutions of the following equations

$$Y_i^{(n)}(t_{k+1}) = Y_i^{(n)}(t_k) + \left(\frac{k_i}{Y_i^{(n)}(t_{k+1})} + b_i\left(t_k, Y^{(n)}(t_k)\right)\right)\frac{T}{n} + \sum_{j=1}^{m} \sigma_{ij}\left(B_j^H(t_{k+1}) - B_j^H(t_k)\right) \quad (16)$$

It is straightforward to rewrite (16) as a quadratic equation of $Y_i^{(n)}(t_{k+1})$, then we can obtain an explicit formula for $Y_i^{(n)}(t_{k+1})$.

**Theorem 4.1.** For any $\epsilon \in (0, H)$, there exists a finite random variable $\theta$, which does not depend on $n$, such that

$$\sup_{k=1,\dots,n} \left|Y(t_k) - Y^{(n)}(t_k)\right| \leq \frac{\theta}{n^{(H-\epsilon)\wedge\alpha}}, \text{ a.s, for all } n \geq 1.$$

*Proof.* We denote $e_i(t_k) = Y_i(t_k) - Y_i^{(n)}(t_k)$ for $k = 0, 1, \dots, n-1$. Then

$$e_i(t_{k+1}) = e_i(t_k) + \left(\frac{k_i}{Y_i(t_{k+1})} - \frac{k_i}{Y_i^{(n)}(t_{k+1})}\right)\frac{T}{n} + \left(b_i(t_k, Y(t_k)) - b_i\left(t_k, Y^{(n)}(t_k)\right)\right)\frac{T}{n} + r_i(k),$$

where

$$r_i(k) = \int_{t_k}^{t_{k+1}} \left(\frac{k_i}{Y_i(s)} - \frac{k_i}{Y_i(t_{k+1})}\right) ds + \int_{t_k}^{t_{k+1}} \left(b_i(s, Y(s)) - b_i(t_k, Y(t_k))\right) ds. \quad (17)$$

Hence,

$$|e(t_{k+1})|^2 = \sum_{i=1}^{d} e_i(t_{k+1})e_i(t_k) + \sum_{i=1}^{d} e_i(t_{k+1})\left(\frac{k_i}{Y_i(t_{k+1})} - \frac{k_i}{Y_i^{(n)}(t_{k+1})}\right)\frac{T}{n}$$
$$+ \sum_{i=1}^{d} e_i(t_{k+1})\left(b_i(t_k, Y(t_k)) - b_i\left(t_k, Y^{(n)}(t_k)\right)\right)\frac{T}{n} + \sum_{i=1}^{d} e_i(t_{k+1})r_i(k).$$

Note that

$$e_i(t_{k+1})\left(\frac{k_i}{Y_i(t_{k+1})} - \frac{k_i}{Y_i^{(n)}(t_{k+1})}\right)\Bigg) = \left(Y_i(t_{k+1}) - Y_i^{(n)}(t_{k+1})\right)\left(\frac{k_i}{Y_i(t_{k+1})} - \frac{k_i}{Y_i^{(n)}(t_{k+1})}\right) \leq 0.$$

This fact together with Young's inequality $ab \leq \frac{a^2+b^2}{2}$ and the Lipschitz continuity of $b$ yields

$$|e(t_{k+1})|^2 \leq \frac{1}{2}|e(t_{k+1})|^2 + \frac{1}{2}|e(t_k)|^2 + \sum_{i=1}^{d} L|e_i(t_{k+1})||e(t_k)|\frac{T}{n} + \sum_{i=1}^{d} |e_i(t_{k+1})||r_i(k)|.$$

This implies

$$|e(t_{k+1})|^2 \leq \frac{2LT}{n}\sum_{l=0}^{k}\sum_{i=1}^{d} |e_i(t_{l+1})||e(t_l)| + 2\sum_{l=0}^{k}\sum_{i=1}^{d} |e_i(t_{l+1})||r_i(l)|.$$

By taking the supremum with respect to $k$, we obtain for any $p = 1, \dots, n$,

$$\sup_{k=1,\dots,p} |e(t_k)|^2 \leq \frac{2LT}{n}\sum_{l=0}^{p-1}\sum_{i=1}^{d} |e_i(t_{l+1})||e(t_l)| + 2\sum_{l=0}^{p-1}\sum_{i=1}^{d} |e_i(t_{l+1})||r_i(l)|$$

$$\leq \sup_{k=1,\dots,p} |e(t_k)| \sum_{l=0}^{p-1} \sup_{j=1,\dots,l} |e(t_j)| \frac{2dLT}{n} + 2\sup_{k=1,\dots,p} |e(t_k)| \sum_{l=0}^{p-1}\sum_{i=1}^{d} |r_i(l)|$$

So,

$$\sup_{k=1,\dots,p} |e(t_k)| \leq \sum_{l=0}^{p-1} \sup_{k=1,\dots,l} |e(t_k)| \frac{2dLT}{n} + 2\sum_{l=0}^{p-1}\sum_{i=1}^{d} |r_i(l)|.$$

By using discrete Gronwall's inequality, we obtain

$$\sup_{k=1,\dots,p} |e(t_k)| \leq 2\left(1 + 2dLTe^{2dLT}\right)\sum_{l=0}^{p-1}\sum_{i=1}^{d} |r_i(l)|. \tag{18}$$

On the other hand, for any $i = 1, \dots, d$,

$$\begin{aligned}\sum_{l=0}^{p-1} |r_i(l)| \leq\ & \int_0^T \left|\frac{k_i}{Y_i(s)} - \frac{k_i}{Y_i(\kappa_n(s))}\right| ds + \int_0^T \left|b_i(s, Y(s)) - b_i\big(\eta_n(s), Y(\eta_n(s))\big)\right| ds \\ \leq\ & \int_0^T \frac{k_i |Y_i(s) - Y_i(\kappa_n(s))|}{\inf_{0\leq s\leq T} Y_i(s)^2} ds + \int_0^T \left|b_i(s, Y(s)) - b_i\big(s, Y(\eta_n(s))\big)\right| ds \\ & + \int_0^T \left|b_i\big(s, Y(\eta_n(s))\big) - b_i\big(\eta_n(s), Y(\eta_n(s))\big)\right| ds.\end{aligned} \tag{19}$$

Since $b_i(t, x)$ is Lipschitz continuous with respect to $x$ and $\alpha$-Hölder continuous with respect to $t$,

$$\begin{aligned}&\int_0^T \left|b_i(s, Y(s)) - b_i\big(s, Y(\eta_n(s))\big)\right| ds + \int_0^T \left|b_i\big(s, Y(\eta_n(s))\big) - b_i\big(\eta_n(s), Y(\eta_n(s))\big)\right| ds \\ &\leq L\int_0^T \left|Y(s) - Y\big(\eta_n(s)\big)\right| ds + L_1\int_0^T |s - \eta_n(s)|^\alpha ds.\end{aligned} \tag{20}$$

Set $K = \max\{k_i, 1 \leq i \leq d\}$ and $\| \sigma \| = \sum_{ij} |\sigma_{ij}|$. Thanks to Lemma 1.1, for any $\epsilon \in (0, H)$, there exists a finite random variable $\eta_{\epsilon,T} > 0$ such that, for any $0 \leq s \leq t \leq T$,

$$\begin{aligned}|Y_i(t) - Y_i(s)| &\leq \int_s^t \left|\frac{k_i}{Y_i(u)} + b_i(u, Y(u))\right| du + \sum_{j=1}^{m} |\sigma_{ij}||B_j^H(t) - B_j^H(s)| \\ &\leq \left(\frac{K}{\inf_{0\leq u\leq T} Y_i(u)} + L_0 + L_0 \sup_{0\leq u\leq T} |Y(u)|\right)(t - s) + \| \sigma \| \eta_{\epsilon,T} |t - s|^{H-\epsilon}, \text{ a.s.}\end{aligned} \tag{21}$$

If follows from (18) - (21) that for any $\epsilon \in (0, H)$, there exists a finite random variable $\theta$ which does not depend on $n$ such that $\sup_{1\leq k\leq n} |e(t_k)| \leq \frac{\theta}{n^{(H-\epsilon)\wedge\alpha}}$, a.s, which concludes the proof. □

In the next result we want to show the convergence of the numerical approximation in $L_p$– norm. We need the following hypothesis.

**Hypothesis 4.2.** There exists positive constants $\hat{p}, L_2, L_3$ such that

$$\sup_{t\in[0,T]} \mathbb{E}\{|Y(t)|^{\hat{p}}\} + \max_{0\leq i<d} \sup_{t\in[0,T]} \mathbb{E}\{|Y_i(t)|^{-\hat{p}}\} < L_3,$$

and for any $i = 1, \ldots, d$,

$$\mathbb{E}\{|Y_i(t) - Y_i(s)|^{\hat{p}}\} \le L_4|t-s|^{\hat{p}/2}, \textit{for all } 0 \le s < t \le T.$$

**Remark 4.3.** It follows from Proposition 3.2 that Hypothesis 4.2 satisfies for $T = T_p$ when $b_i, 1 \le i \le d$, are nonnegative.

With these assumptions, we can estimate the rate convergence of the numerical approximation in $L_p$ - norm. It is stated by the following theorem.

**Theorem 4.4.** Assume that assumptions $(A1) - (A2)$ hold and the hypothesis (4.2) holds for some $\hat{p} = 3p \ge 3$. Then there exists $\hat{C} > 0$ which does not depend on $n$ such that for any $n \in \mathbb{N}$,

$$\mathbb{E}\left[\sup_{k=1,\ldots,n}\left|Y(t_k) - Y^{(n)}(t_k)\right|^p\right]^{1/p} \le \frac{\hat{C}}{n^{\alpha\wedge 1/2}}.$$

*Proof.* The proof is adapted from that of Theorem 2.8 in [14] (see also [15]). By (18), for any $q = 1, \ldots, n$,

$$\sup_{k=1,\ldots,q}|e(t_k)| \le 2\left(1 + 2dLTe^{2dLT}\right)\sum_{l=0}^{q-1}\sum_{i=1}^{d}|r_i(l)|. \quad (22)$$

But it follows from Eq. (17) for any $i = 1, \ldots, d$,

$$\sum_{l=0}^{q-1}\sum_{i=1}^{d}|r_i(l)| \le \sum_{i=1}^{d}\int_0^{t_q}\left|\frac{k_i}{Y_i(s)} - \frac{k_i}{Y_i(\kappa_n(s))}\right|ds + \sum_{i=1}^{d}\int_0^{t_q}\left|b_i\left(s, Y_i(s)\right) - b_i\left(\eta_n(s), Y_i\left(\eta_n(s)\right)\right)\right|ds. \quad (23)$$

Applying Höder's inequality we have

$$\mathbb{E}\left[\left(\int_0^{t_q}\left|\frac{k_i}{Y_i(s)} - \frac{k_i}{Y_i(\kappa_n(s))}\right|ds\right)^p\right] \le T^{p-1}k_i^p\int_0^T\mathbb{E}\left[\frac{|Y_i(s) - Y_i(\kappa_n(s))|^p}{|Y_i(s)|^p \mid Y_i(\kappa_n(s)|^p}\right]ds$$

$$\le T^{p-1}k_i^p\int_0^T(\mathbb{E}[|Y_i(s) - Y_i(\kappa_n(s))|^{3p}])^{1/3}(\mathbb{E}[|Y_i(s)|^{-3p}])^{1/3}(\mathbb{E}[|Y_i(\kappa_n(s))|^{-3p}])^{1/3}ds.$$

This fact and Hypothesis 4.2 lead to

$$\mathbb{E}\left[\left(\int_0^{t_q}\left|\frac{k_i}{Y_i(s)} - \frac{k_i}{Y_i(\kappa_n(s))}\right|ds\right)^p\right] \le \frac{\hat{C}_1}{n^{\frac{p}{2}}}, \quad (24)$$

for some constant $\hat{C}_1 > 0$ which does not depend on $n$. Moreover, we have

$$\int_0^{t_q}\left|b_i(s, X(s)) - b_i\left(\eta_n(s), X(\eta_n(s))\right)\right|ds$$

$$\le \int_0^{t_q}\left|b_i\left(s, X(s)\right) - b_i\left(s, X\left(\eta_n(s)\right)\right)\right|ds + \int_0^{t_q}\left|b_i\left(s, X\left(\eta_n(s)\right)\right) - b_i\left(\eta_n(s), X\left(\eta_n(s)\right)\right)\right|ds. \quad (25)$$

Applying Lipschitz continuous condition with respect to $x$ of $b_i(t,x)$, we have

$$\mathbb{E}\left[\left(\int_0^{t_q}\left|b_i(s, X(s)) - b_i\left(s, X(\eta_n(s))\right)\right|ds\right)^p\right] \le L^pT^{p-1}\int_0^T\mathbb{E}\left[\left|X(s) - X\left(\eta_n(s)\right)\right|^p\right]ds \le \frac{\hat{C}_2}{n^{\frac{p}{2}}}, \quad (26)$$

Using $\alpha -$ Höder continuous condition with respect to $t$ of $b_i(t,x)$, we obtain

$$\mathbb{E}\left[\left(\int_0^{t_q}\left|b_i\left(s, X\left(\eta_n(s)\right)\right) - b_i\left(\eta_n(s), X\left(\eta_n(s)\right)\right)\right|ds\right)^p\right]$$

$$\le L_1^pT^{p-1}\int_0^T\mathbb{E}[|s - \eta_n(s)|^{\alpha p}]ds \le \frac{\hat{C}_3}{n^{\alpha p}}. \quad (27)$$

for some constant $\hat{C}_3 > 0$ which does not depend on $n$.Estimates (22)- (27) together with Minkowski's inequality, we conclude the proof. □

## 5. Simulation

In this section we consider the following system of two dimensional fractional differential equations

$$\begin{cases} X_1(t) = 1 + \int_0^t \left( \frac{1}{X_1(s)} - X_1(s) - X_2(s) \right) ds + \frac{1}{2} dB_1^H(t) + 2dB_2^H(t), \\ X_2(t) = 1 + \int_0^t \left( \frac{2}{X_2(s)} + \cos(X_1(s)) - \cos(X_2(s)) \right) ds + dB_1^H(t) + 2dB_2^H(t). \end{cases}$$

It follows from Theorem 2.4 that the system has a unique solution which always stay in the domain $(0, +\infty)^2$.

We use Cholesky method to generate the fraction Brownian motions (see [16, 17]) with $H = 0.52$ at $n = 10^3$ points on the interval [0,1]. We generate $(X_1, X_2)$ by both classical Euler scheme (15), and the modified Euler scheme (16). The result is shown in Figure 1.

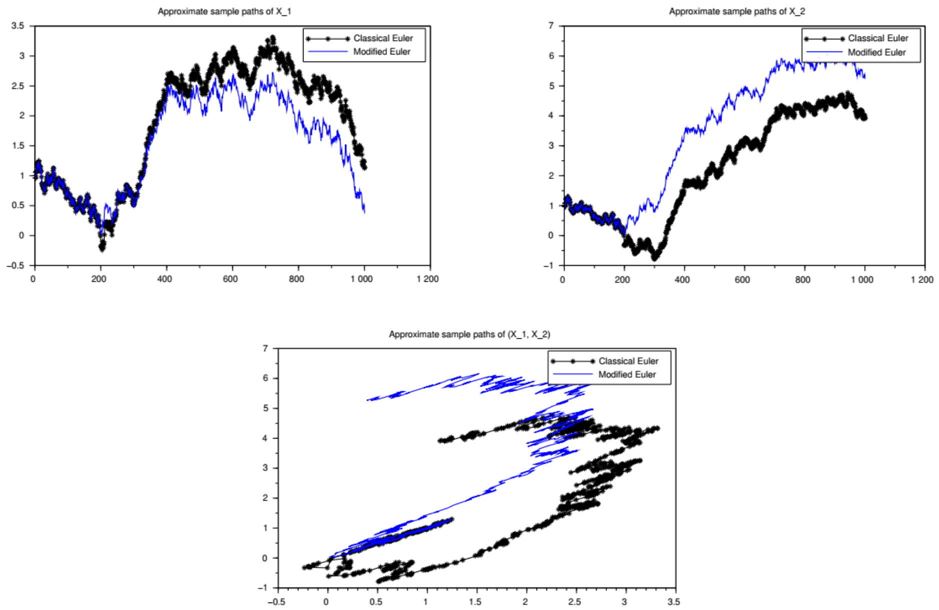


Figure 1. Approximate solutions of $(\boldsymbol{X_1}, \boldsymbol{X_2})$ by classical Euler and modified Euler schemes.

We can see that the both approximate solutions are almost coincide until they are close to axes. At that time, the classical Euler approximate solution exits the domain $(0, +\infty)^2$ while the modified Euler one still stays inside that domain.

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