Original Article

Scalar-tensor Theory $L_G = \phi R - 2\gamma\phi^\eta$ and the Problem of Dark Matter

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Abstract: Searching for the origin of the dark matter and the dark energy is among the problems that the modern physics, including the general theory of relativity, is facing to. Various theories or models have been proposed to solve these problems without satisfactory success yet. In this work, a new model based on the scalar-tensor theory with Lagrangian $L_G = \phi R - 2\gamma\phi^\eta$ is suggested to solve some aspects of the above-mentioned problems. In this model the dark matter is interpreted as geometric background of the space-time.

Keywords: Scalar-tensor theory, modified gravity, dark energy, dark matter, FLRW cosmology.

1. Introduction

The general theory of relativity (GR) based on the Einstein equation [1, 2],

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu},$$

(1)

has been shown to be a good theory in explaining different phenomena of the ordinary matter such as that of stellar objects including our Sun system. This equation can be derived from the Hilbert-Einstein action built on the Lagrangian

$$L_{GR} = R.$$

(2)

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The GR, however, fails to explain the Universe at large spacetime scales. Two major issues are called dark matter (DM) [3-7] and dark energy (DE) [8-11], which account for about 27% and 68% of the matter-energy content of the Universe, respectively. These terminologies, "dark matter" and "dark energy" respectively, were coined for "missing mass" and "missing energy" in explaining unexpected observations of unusual rotations of the disks of galaxies or galaxy clusters and the accelerated expansion of the Universe [12-15]. At early days of the GR, Einstein also stumbled upon the question if the Universe was static or not. He introduced the so-called cosmological constant [2], denoted usually as Λ, in order to "keep" the Universe static as he believed. Later, after knowing theoretical works by Friedmann [16] and Lemaˆıtre [17], especially, after the observation of the Universe's extension by Hubble [18], he rejected the idea of a static Universe and said that the introduction of the cosmological constant was a big mistake. The subsequent development of cosmology, however, revived the idea of the cosmological constant expected by some people to unravel the riddle of the Universe's accelerated expansion. However, the cosmological constant is also treated as (or related to) the vacuum energy which according to the quantum field theory is too big (about 120 orders of magnitude bigger) in comparison with the value necessary for examinationof the Universe's observed accelerated expansion [19, 20]. It is the largest discrepancy ever between theory and experiment/observation in physics and that is the so-called cosmological constant problem for solving of which no way has found until now. Another difficulty of the GR is the DM problem [6, 7]. The discrepancy between the observed and the theoretically-expected rotation curves of a spiral galaxy, such as Messier 33 (see Figure 1), cannot be explained within the GR. The observed rotation speed becomes distance-independent (or weakly dependent of the distance, at least) after an increasing part, unlike the theoretical one increasing and then decreasing with distance from the galaxy center. Both the DM and the DE can be considered in the point of view of particle physics or that of the GR (based on the very geometry of the spacetime). In general, the GR with or without the cosmological constant is unable to describe consecutive stages of the Universe's evolution, namely, the inflationary- and the radiation eras and the era of dark energy (including the present time). So far, the results obtained in the two approaches have not always been compatible with each other. Here, we will follow the scalar-tensor approach combining the advantages of particle physics and the GR to propose a modified model of gravitation with the hope to overcome some of the difficulties of the GR and particle physics.
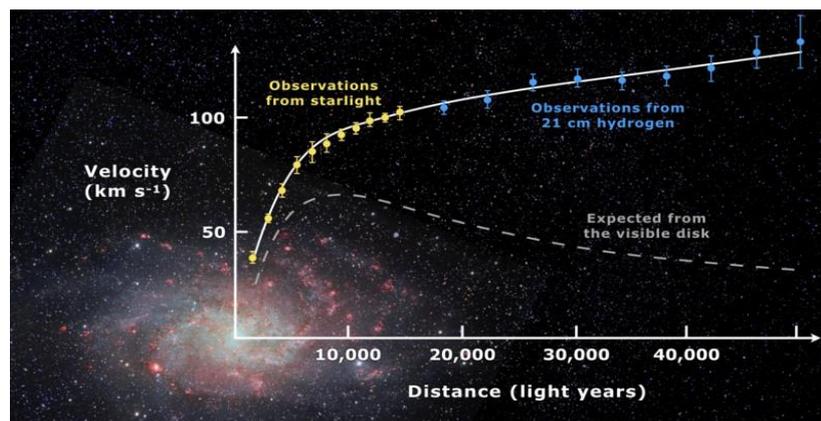
Figure 1. The discrepancy between the observed and the theoretically-expected rotation curves of Messier 33 (a spiral galaxy) can be explained by assuming the presence of a huge "invisible mass". (Source: Mario De Leo, Wikipedia).

There have been a number of modified theories of gravitation proposed but the so-called *f(R)*-theory of gravitation (or just *f(R)*-theory or *f(R)*-gravitation for short) [13-15] may be one of the simplest modifications of the GR. The *f(R)*-theory may help us in explaining different stages of the cosmological evolution (see, for example, [21]). The scalar-tensor itself is also a modified gravitational theory. It is clear that there is a bridge (conformal transformation) between the scalar-tensor theory and the *f(R)* theory [13]. Based on the cosmological principles we assume that the distribution of DM (dominating the Universe's matter content) at large scale is homogeneous and isotropic. Here, as in [21], the isotropy allows us to work in a geometry with a spherically symmetric metric.

2. Dark Matter as Spacetime Geometric Background

As the dark matter, overwhelming the ordinary matter in the Universe, is not involved in any interaction other than the gravitational one with the ordinary matter, the former can be treated as a spacetime geometric background which according to the cosmological principles can be assumed to be homogeneous and isotropic. This background naturally adopts a geometry with a spherically symmetric metric [1]

$$ds^2 = e^{a(r, t)}(dx^0)^2 - e^{b(r, t)}dr^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2). \tag{3}$$

The motion of a freely moving particle in the spacetime satisfies the equation [1]

$$Du^\mu \equiv du^\mu + \Gamma^\mu_{\alpha\beta} u^\alpha dx^\beta = 0, \tag{4}$$

where D is the covariant differential and u^μ is the four-velocity, $u^\mu = \frac{dx^\mu}{d\tau} = c\frac{dx^\mu}{ds}$. Inserting the latter in (4) we get

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0. \tag{5}$$

This equation in the spacetime with metric (3) and $x^0 = ct$, $d\tau = \sqrt{g_{00}}dt = \sqrt{e^a}dt$ has the form

$$\frac{d^2 r}{d\tau^2} + \frac{b'}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{c\dot{b}}{\sqrt{e^a}}\frac{dr}{dt} + \frac{a'c^2}{2e^b} - \frac{r}{e^b}\left(\frac{d\theta}{d\tau}\right)^2 - \frac{r\sin^2\theta}{e^b}\left(\frac{d\varphi}{d\tau}\right)^2 = 0, \tag{6}$$

Where $b' = \frac{\partial b}{\partial r}$ and $\dot{b} = \frac{\partial b}{\partial ct}$. Let us consider a circular free motion ($dr = 0, \theta = \pi/2$) of a particle in a static ($\dot{b} = 0$) central field, the latter equation becomes

$$\frac{a'c^2}{2} - r\left(\frac{d\varphi}{d\tau}\right)^2 = 0. \tag{7}$$

The particle's orbital velocity can be calculated as

$$v = \frac{dl}{d\tau} = \frac{dl}{\sqrt{g_{00}}dt} = \frac{cdl}{\sqrt{g_{00}}dx^0} = c\frac{rd\varphi}{\sqrt{g_{00}}dx^0} = r\frac{d\varphi}{d\tau}. \tag{8}$$

where $dl = [e^{b(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)]^{1/2} = rd\varphi$. Replacing (8) in (7) we get

$$\frac{a'c^2}{2} - \frac{v^2}{r} = 0. \tag{9}$$

In the case of a constant velocity ($v = $ const.$:= v_{tg}$)[1] the solution of (9) is simply

$$a(r) = \frac{2v_{tg}^2}{c^2} \left(\ln \frac{c^2 r}{2v_{tg}^2} + \ln a_0 \right),$$
(10)

with a_0 being an integration constant. Therefore,

$$g_{00} = e^{a(r)} = \left(\frac{r}{r_0} \right)^{\frac{2v_{tg}^2}{c^2}},$$
(11)

where, $r_0 = \frac{2v_{tg}^2}{c^2 a_0} = $ const. Thus, the metric (3) now has the following form [12]

$$ds^2 = \left(\frac{r}{r_0} \right)^{\frac{2v_{tg}^2}{c^2}} dx^{0^2} - e^{b(r)} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$
(12)

The geometric origin of the dark matter lies in the metric (12) encoding the presence of dark matter in spacetime.

3. The Equivalence between the $f(R)$-gravitation and the Scalar-tensor Gravitation

Einstein's theory of general relativity (GR) is built on the Hilbert-Einstein action

$$S_{GR} = \frac{1}{2kc} \int d^4x \sqrt{-g} \mathcal{L}_{GR},$$
(13)

where $k = \frac{8\pi G}{c^4}$. The fact that there are a number of GR-beyond problems calls for extending of the GR. The $f(R)$-theory of gravitation is one of the first and simplest extensions of the GR. This modified gravitation theory is based on the action [13, 14, 15]

$$S = \frac{1}{2kc} \int d^4x \sqrt{-g} f(R),$$
(14)

where $f(R)$ is a regular general function of the scalar curvature R. This theory generalizes the GR and returns to the latter at $f(R) = R$ (or $f(R) = R - 2\Lambda$ if the cosmological constant Λ is included). By some manipulation, and then, substitution

$$R = \varphi,$$
(15)

in some term, we can prove that the action (14) is equivalent to the action of the scalar-tensor theory [13]

$$S = \frac{1}{c} \int d^4x \sqrt{-g} \left[\frac{1}{2k} \phi R - V(\phi) \right],$$
(16)

With $V(\phi) = \frac{\phi\varphi(\phi) - f(\varphi(\phi))}{2k}$, where

$$\phi = f'(R).$$
(17)

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[1] The hypothesis of the existence of dark matter is based on the observation that the rotation velocity of luminous matter objects near the edges of galaxies or galaxy clouds is constant

Here, to save the length of the paper, we do not go into the details of a proof of this claim, but instead refer readers to Ref. [13].

The action (16) is an action of a scalar field interacting with gravitation (or just a free field in a curved spacetime) without kinetic term, $(\partial\phi)^2 = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 0$, which can be recovered by using $f(R,(\nabla R)^2,\Box R)$ instead of $f(R)$ (see [23] for more details). The scalar-tensor theory represents an interface between particle physics and gravitation theories and has different variations depending on what problems to be solved. Hence, this theory is very interesting for consideration. It is namely the subject of the next section.

4. Scalar-tensor Theory $L_{ST} = \phi R - 2\gamma\phi^\eta$

The action of a general scalar-tensor theory has the form

$$S = \int d^4x\sqrt{-g}\left[p(\varphi)R + \frac{\omega(\varphi)}{\varphi}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - V(\varphi)\right].$$

(18)

After some rescaling and transformation it can be re-written in a simpler form

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}\varphi R + \frac{\omega(\varphi)}{\varphi}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - V(\varphi)\right]$$

(19)

By choosing different $\omega(\varphi)$ and/or $V(\varphi)$ in (19) we get different scalar-tensor theories. Choosing ω = const. and $V(\varphi) = 0$ leads to the well-known Brans-Dicke theory (in vacuum), also called sometimes the Jordan–Brans–Dicke theory. It is an early and simple version of the scalar-tensor theory which has a long research history [2] and admits more solutions, in particular, the vacuum ones, than the GR [24]. Here, we will work with the model

$$S = \frac{1}{2kc}\int d^4x\sqrt{-g}\left[\phi R - V(\phi)\right],$$

(20)

which is the model (19) with $\omega = 0$. We also choose

$$V(\phi) = 2\gamma\phi^\eta,$$

(21)

where η is a dimensionless constant, while the constant γ has the same dimension with R ($[\gamma] = [R]$). It is not difficult to find the extended Einstein equation in this case

$$R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = \frac{1}{\phi}\left[\delta^\mu_\nu\Box\phi - \nabla^\mu\nabla_\nu\phi - \gamma\delta^\mu_\nu\phi^\eta\right].$$

(22)

via a variation of the action (20) with respect to the metric $g_{\mu\nu}$, where $\Box = \nabla^\mu\nabla_\mu$ with ∇_μ being the covariant derivative. On the other hand, the variation of (20) with respect to ϕ leads to the equation

$$R = 2\gamma\eta\phi^{\eta-1}.$$

(23)

Two equations (22) and (23) are independent and can be solved for $a(r,t)$, $b(r,t)$ and $\phi(r,t)$. Indeed, the trace of (22) and (23) give

$$\Box\phi = \frac{2\gamma(2-\eta)}{3}\phi^\eta.$$

(24)

Replacing this equation in (22) we get

$$R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = \frac{1}{\phi}\left[\frac{\gamma(1-2\eta)}{3}\delta^\mu_\nu\phi^\eta - \nabla^\mu\nabla_\nu\phi\right].$$

(25)

Now we solve the last equation in several variants of the spherically symmetric metric (3),

$$ds^2 = e^{a(r,\,t)}(dx^0)^2 - e^{b(r,\,t)}dr^2 - r^2\,(d\theta^2 + \sin^2\theta d\varphi^2). \tag{3}$$

With this metric, the non-vanishing Christoffel elements are [1]

$$\Gamma^1_{11} = \frac{b'}{2}, \quad \Gamma^0_{10} = \frac{a'}{2}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta,$$

$$\Gamma^0_{11} = \frac{\dot{b}}{2}e^{b-a}, \quad \Gamma^1_{22} = -re^{-b}, \quad \Gamma^1_{00} = \frac{a'}{2}e^{a-b}$$

$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}, \quad \Gamma^3_{23} = \cot\theta, \quad \Gamma^0_{00} = \frac{\dot{a}}{2},$$

$$\Gamma^1_{10} = \frac{\dot{b}}{2}, \quad \Gamma^1_{33} = -r\sin^2\theta e^{-b}. \tag{26}$$

where $\dot{a} = \dfrac{\partial a}{\partial ct}$ and $a' = \dfrac{\partial a}{\partial r}$. Then, the non-zero elements of the Einstein tensor

$$G^\mu_{\ \nu} = R^\mu_{\ \nu} - \frac{1}{2}\delta^\mu_{\ \nu}R \tag{27}$$

are [1]

$$G^0_{\ 0} = R^0_{\ 0} - \frac{1}{2}R = -e^{-b(r,t)}\left[\frac{b'(r,t)}{r} - \frac{1}{r^2}\right] - \frac{1}{r^2}, \tag{28}$$

$$G^1_{\ 1} = R^1_{\ 1} - \frac{1}{2}R = e^{-b(r,t)}\left[\frac{a'(r,t)}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2}, \tag{29}$$

$$G^1_{\ 0} = R^1_{\ 0} = \frac{e^{-b(r,t)}}{r}\dot{b} = \frac{e^{-b(r,t)}}{r}\frac{\partial b(r,t)}{\partial ct}, \tag{30}$$

$$G^0_{\ 1} = R^0_{\ 1} = -\frac{e^{-a(r,t)}}{r}\dot{b} = -\frac{e^{-a(r,t)}}{r}\frac{\partial b(r,t)}{\partial ct}, \tag{31}$$

$$G^2_{\ 2} = G^3_{\ 3} = \frac{1}{2}e^{-b}\left(a'' + \frac{a'^2}{2} + \frac{a'-b'}{r} - \frac{a'b'}{2}\right) - \frac{1}{2}e^{-a}\left(\ddot{b} + \frac{\dot{b}^2}{2} - \frac{\dot{a}\dot{b}}{2}\right). \tag{32}$$

Consequently,

$$G = -R = e^{-b}\left[a'' + \frac{a'^2}{2} + \frac{2(a'-b')}{r} - \frac{a'b'}{2} + \frac{2}{r^2}\right] - \frac{2}{r^2} - e^{-a}\left(\ddot{b} + \frac{\dot{b}^2}{2} - \frac{\dot{a}\dot{b}}{2}\right). \tag{33}$$

It follows

$$\nabla^0\nabla_0\phi = e^{-a}\left(\ddot{\phi} - \frac{\dot{a}\dot{\phi}}{2} - \frac{a'\phi'e^{a-b}}{2}\right), \tag{34}$$

$$\nabla^1\nabla_1\phi = -e^{-b}\left(\phi'' - \frac{\dot{b}\dot{\phi}e^{b-a}}{2} - \frac{b'\phi'}{2}\right) \tag{35}$$

$$\nabla^2\nabla_2\phi = \nabla^3\nabla_3\phi = -\frac{\phi'e^{-b}}{r}, \tag{36}$$

$$\nabla^1\nabla_0\phi = -e^{-b}\left(\dot{\phi}' - \frac{a'\dot{\phi}}{2} - \frac{\dot{b}\phi'}{2}\right), \tag{37}$$

$$\nabla^0\nabla_1\phi = e^{-a}\left(\dot{\phi}' - \frac{a'\dot{\phi}}{2} - \frac{\dot{b}\phi'}{2}\right). \tag{38}$$

Therefore, the equation (25) become:

$$-e^{-b}\left[\frac{b'}{r} - \frac{1}{r^2}\right] - \frac{1}{r^2} = \frac{1}{\phi}\left[\frac{\gamma(1-2\eta)}{3}\phi^\eta + \frac{a'\phi'e^{-b}}{2}\right], \tag{39}$$

$$e^{-b}\left[\frac{a'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2} = \frac{1}{\phi}\left[\frac{\gamma(1-2\eta)}{3}\phi^\eta + e^{-b}\left(\phi'' - \frac{b'\phi'}{2}\right)\right], \tag{40}$$

$$\frac{1}{2}e^{-b}\left(a'' + \frac{a'^2}{2} + \frac{a'-b'}{r} - \frac{a'b'}{2}\right) = \frac{1}{\phi}\left[\frac{\gamma(1-2\eta)}{3}\phi^\eta + \frac{\phi'e^{-b}}{r}\right] \tag{41}$$

The equation (24) in a central static field gets the form

$$\Box\phi = -e^{-b}\left[\phi'' + \left(\frac{a'-b'}{2} + \frac{2}{r}\right)\phi'\right] = \frac{2\gamma(2-\eta)}{3}\phi^\eta, \tag{42}$$

while Eq. (23) now is

$$-e^{-b}\left[a'' + \frac{a'^2}{2} + \frac{2(a'-b')}{r} - \frac{a'b'}{2} + \frac{2}{r^2}\right] + \frac{2}{r^2} = 2\gamma\eta\phi^{\eta-1} \tag{43}$$

Combining (39) with (40) we get

$$\left(\frac{1}{r} + \frac{\phi'}{2\phi}\right)(a' + b') = \frac{\phi''}{\phi}. \tag{44}$$

A combination of (41) and (43) gives

$$-e^{-b}\left(\frac{a'-b'}{r} + \frac{2}{r^2} + \frac{2\phi'}{r\phi}\right) + \frac{2}{r^2} = \frac{2\gamma(\eta+1)\phi^{\eta-1}}{3}. \tag{45}$$

With the substitution

$$b(r) = -\ln q(r). \tag{46}$$

the equations (39) – (45) becomes respectively.

$$-q\left[-\frac{q'}{rq} - \frac{1}{r^2}\right] - \frac{1}{r^2} = \frac{1}{\phi}\left[\frac{\gamma(1-2\eta)}{3}\phi^\eta + \frac{a'\phi'q}{2}\right], \tag{47}$$

$$q\left[\frac{a'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2} = \frac{1}{\phi}\left[\frac{\gamma(1-2\eta)}{3}\phi^\eta + q\left(\phi'' + \frac{q'\phi'}{2q}\right)\right], \tag{48}$$

$$\frac{q}{2}\left(a'' + \frac{a'^2}{2} + \frac{a'}{r} + \frac{q'}{rq} + \frac{a'q'}{2q}\right) = \frac{1}{\phi}\left[\frac{\gamma(1-2\eta)}{3}\phi^\eta + \frac{\phi'q}{r}\right], \tag{49}$$

$$\Box\phi = -q\left[\phi'' + \left(\frac{a'}{2} + \frac{q'}{2q} + \frac{2}{r}\right)\phi'\right] = \frac{2\gamma(2-\eta)}{3}\phi^\eta, \tag{50}$$

$$-q\left[a'' + \frac{a'^2}{2} + \frac{2}{r}\left(a' + \frac{q'}{q}\right) + \frac{a'q'}{2q} + \frac{2}{r^2}\right] + \frac{2}{r^2} = 2\gamma\eta\phi^{\eta-1}, \tag{51}$$

$$\left(\frac{1}{r} + \frac{\phi'}{2\phi}\right)\left(a' - \frac{q'}{q}\right) = \frac{\phi''}{\phi}, \tag{52}$$

$$-q\left(\frac{a'}{r} + \frac{q'}{rq} + \frac{2}{r^2} + \frac{2\phi'}{r\phi}\right) + \frac{2}{r^2} = \frac{2\gamma(\eta+1)\phi^{\eta-1}}{3}, \tag{53}$$

Note that, not all but only three of Eqs. (47)–(53) are independent. It is enough for solving for three functions $a(r)$, $q(r)$ and $\phi(r)$. For illustration we consider several cases following.

4.1. The Case $\phi = 1$

In this case, setting $V = 2\gamma$ conveys the model (20) into Einstein's GR with cosmological constant $\Lambda = \gamma$. Then, Eq. (42) means $\eta = 2$ and Eq. (44) gives

$$a' + b' = 0, \tag{54}$$

while (39) becomes

$$-e^{-b}\left[\frac{b'}{r} - \frac{1}{r^2}\right] - \frac{1}{r^2} = -\gamma. \tag{55}$$

In the next subsection we will consider the condition (54) with more arbitrary ϕ. To solve (55) we substitute

$$b(r) = -\ln\left[1 + \frac{C(r)}{r}\right], \tag{56}$$

in (55) and get

$$\frac{C'(r)}{r^2} = -\gamma. \tag{57}$$

Consequently,

$$C(r) = C_1 - \frac{\gamma}{3}r^3, \tag{58}$$

where C_1 is an intergration constant. Putting Eqs. (58), (56) and (54) all together we find the metric

$$ds^2 = \left(1 - \frac{kc^2 M}{4\pi r} - \frac{\gamma r^2}{3}\right)dx^{0^2} - \left(1 - \frac{kc^2 M}{4\pi r} - \frac{\gamma r^2}{3}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{59}$$

where the constant C_1 is fixed at $C_1 = -\frac{kc^2 M}{4\pi}$ with M being the mass of the gravitational source, to obtain Newton's gravitation at $\gamma = 0$. Metric (59) is exactly a spherically symmetric metric in the GR with cosmological constant $\Lambda = \gamma$.

4.2. The Case $a' + b' = 0$ for More General ϕ

It is not difficult to show that Einstein's GR (with or without the cosmological constant) can be derived from the present model as a special case under the condition

$$a' + b' = 0. \tag{54}$$

It follows from (44) that $\phi'' = 0$, or

$$\phi(r) = C_2 r + C_3, \tag{60}$$

where C_2 and C_3 are constants. Replacing (60) in (42), (45) and (39) we get, respectively, the following equations

$$-e^{-b}\left(-b' + \frac{2}{r}\right)C_2 = \frac{2\gamma(2-\eta)}{3}\left(C_2 r + C_3\right)^\eta, \tag{61}$$

$$-e^{-b}\left[-\frac{2b'}{r}+\frac{2}{r^2}+\frac{2C_2}{r(C_2r+C_3)}\right]+\frac{2}{r^2}=\frac{2\gamma(\eta+1)(C_2r+C_3)^{\eta-1}}{3}, \tag{62}$$

$$-e^{-b}\left[b'\left(\frac{1}{r}-\frac{C_2}{2(C_2r+C_3)}\right)-\frac{1}{r^2}\right]-\frac{1}{r^2}=\frac{\gamma(1-2\eta)(C_2r+C_3)^{\eta-1}}{3} \tag{63}$$

which all together, after some intermediate calculations, lead to the following result

$$\frac{4C_2\gamma(2-\eta)(C_2r+C_3)^{\eta-1}}{3r}\left[\frac{2\gamma(2-\eta)(C_2r+C_3)^{\eta}}{3r}-\frac{C_2\gamma(\eta+1)(C_2r+C_3)^{\eta-1}}{3}+\frac{C_2}{r^2}\right]=0. \tag{64}$$

Since the latter is valid for any r the condition $C_2\gamma(2-\eta)=0$ must be held. It follows the only solution is that in (59) as it has to be proved. The above condition is fulfilled in three cases, either $C_2 = 0$ or $\gamma = 0$ or $\eta = 2$. The case $C_2 = 0$ (that is, $\phi = const.$) corresponds to the GR with cosmological constant (with Largangian $L_G = R - 2\Lambda$). The case $\gamma = 0$ (with $L_G = \phi R$) and, especially, the case $\eta = 2$ with $L_G = \phi R - 2\gamma\phi^2$ give rise to GR-extended models but the corresponding solutions still have spherically symmetric forms similar to (59) as the GR with cosmological constant. Thus, hopefully, the models $L_G = \phi R$ and $L_G = \phi R - 2\gamma\phi^2$ can describe an accelerating expanding of Universe (problem of dark energy). Therefore, the dark energy can be treated geometrically.

4.3. The Case $e^a = \alpha r^\beta$

Let us now to look for a dark matter solution of (20). Referring to (12), one could do this by setting $e^a = \alpha r^\beta$ or, equivalently,

$$a = \ln\alpha + \beta\ln r, \tag{65}$$

where α and β are constants (but $\alpha > 0$). Therefore, $a' = \frac{\beta}{r}$. Then, the equation (47), (48), (49), (50), (51), (52) and (53) becomes

$$-q\left[-\frac{q'}{rq}-\frac{1}{r^2}\right]-\frac{1}{r^2}=\frac{1}{\phi}\left[\frac{\gamma(1-2\eta)}{3}\phi^\eta+\frac{\beta\phi'q}{2r}\right], \tag{66}$$

$$q\left[\frac{\beta}{r^2}+\frac{1}{r^2}\right]-\frac{1}{r^2}=\frac{1}{\phi}\left[\frac{\gamma(1-2\eta)}{3}\phi^\eta+q\left(\phi''+\frac{q'\phi'}{2q}\right)\right], \tag{67}$$

$$\frac{q}{2}\left(\frac{\beta^2}{2r^2}+\frac{q'}{rq}+\frac{\beta q'}{2rq}\right)=\frac{1}{\phi}\left[\frac{\gamma(1-2\eta)}{3}\phi^\eta+\frac{\phi'q}{r}\right], \tag{68}$$

$$\Box\phi=-q\left[\phi''+\left(\frac{\beta}{2r}+\frac{q'}{2q}+\frac{2}{r}\right)\phi'\right]=\frac{2\gamma(2-\eta)}{3}\phi^\eta, \tag{69}$$

$$-q\left[\frac{-\beta}{r^2}+\frac{\beta^2}{2r^2}+\frac{2}{r}\left(\frac{\beta}{r}+\frac{q'}{q}\right)+\frac{\beta q'}{2rq}+\frac{2}{r^2}\right]+\frac{2}{r^2}=2\gamma\eta\phi^{\eta-1}, \tag{70}$$

$$\left(\frac{1}{r}+\frac{\phi'}{2\phi}\right)\left(\frac{\beta}{r}-\frac{q'}{q}\right)=\frac{\phi''}{\phi}, \tag{71}$$

$$-q\left(\frac{\beta}{r^2}+\frac{q'}{rq}+\frac{2}{r^2}+\frac{2\phi'}{r\phi}\right)+\frac{2}{r^2}=\frac{2\gamma(\eta+1)\phi^{\eta-1}}{3}, \tag{72}$$

respectively. Among these equations (66)–(72) only three are independent. To solve them in general is quite complicated. Let us choose a relatively simple case where

$$q = \lambda = \text{constant.} \tag{73}$$

Putting (73) in (70) we immediately find

$$\phi(r) = \left[\frac{4 - \lambda(\beta^2 + 2\beta + 4)}{4\gamma\eta} \right]^{\frac{1}{\eta-1}} r^{\frac{-2}{\eta-1}}. \tag{74}$$

Using the latter and (71), (66) and (68) we obtain

$$\beta = \frac{2(\eta + 1)}{(\eta - 1)(\eta - 2)}, \tag{75}$$

$$\lambda = \frac{1}{1 + \frac{\beta - 2}{\eta - 1} - \frac{\beta^2}{4}}. \tag{76}$$

Finally, we get a spherically symmetric static metric

$$ds^2 = \alpha r^\beta dx^{0^2} - \left(1 + \frac{\beta - 2}{\eta - 1} - \frac{\beta^2}{4} \right) dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{77}$$

One of the parameters β or η can be determined by (75) if the other one is given. Starting with the value

$$\beta = \frac{2v_{tg}^2}{c^2}. \tag{78}$$

we get the metric (77) which resembles the metric (12) describing the dark matter. This is a geometric interpretation of dark matter. The solution (77) is similar the dark-matter solution obtained in [12] but the former is an exact solution, while the latter is a perturbative solution. To see the geometric meaning of this approach, let us go back to the metric (77) where we identify the metric element g_{00},

$$g_{00} = \left(\frac{r}{r_0} \right)^{\frac{2v_{tg}^2}{c^2}}, \tag{79}$$

With $\alpha = (1/r_0)^{\frac{2v_{tg}^2}{c^2}}$. In the first order Taylor expansion (Newtonian approximation) it becomes

$$g_{00} = 1 + \frac{2\phi_N}{c^2} \tag{80}$$

where

$$\phi_N = v_{tg}^2 \ln\left(\frac{r}{r_0} \right). \tag{81}$$

is a Newtonian potential. This potential keeps a material object to rotate at distance r around a gravitational source of mass M at a constant speed v_{tg} related to the mass M as follows $GM/r = v_{tg}^2$. It means that, to guarantee a constant rotation speed around a mass at distance r, the mass density should

be $\rho = \frac{v_{tg}^2}{4\pi G r^2}$. It is easy to show that (81) satisfies Poinsson equation for an isotropic mass distribution $\rho = \frac{v_{tg}^2}{4\pi G r^2}$, namely, $\Delta\phi_N = 4\pi G\rho$, or more explicitly,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi_N}{dr}\right) = \frac{v_{tg}^2}{r^2}.$$

(82)

5. Conclusion

Following the cosmological principles and the fact that the dark matter distribution dominates the Universe's matter content we have worked out a dark matter model based on the scalar-tensor theory with Lagrangian $L_{ST} = \phi R - 2\gamma\phi^\eta$. We have obtained an exact solution (77) consistent with a perturbative solution obtained elsewhere by other authors. This solution can be reduced to the GR solution under a particular condition. The solution means that the dark-matter is a geometric behaviour of the space-time rather than a mystery substance in the space-time.

Acknowledgement

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