



Original Article

# Application of Stable Random Vector with Gaussian Copula

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**Abstract:** More and more real-world datasets have heavy-tailed distribution, while the calculations for these distributions in multi-dimensional cases are complex. This work shows a method to investigate data of multivariate heavy-tailed distributions. The sufficient condition for every  $\alpha$ -stable random vector is that it has  $\alpha$ -stable marginals and Gaussian copula. From that results, we have a procedure testing stable distribution of multi-dimensional data and a formula representing density functions of multivariate stable distribution. Adopted a new tool, datasets about daily returns of 4 stocks on HoSE and 3 grains were analyzed.

**Keywords:** Multivariate stable distribution, Gaussian copula, daily returns data.

## 1. Introduction

Until 1970's, most of the statistical analysis methods were developed under normality assumptions with symmetric and do not allow heavy-tailed distributions. In applications, however, datasets have asymmetric distributions, especially in finance and risk management studies [1-5]. According to the central limit theorem, stable distributions are natural heavy-tailed extensions of normal distributions and have attracted a lot of attention [6-8].

While the univariate stable distributions are now mostly accessible by several methods to estimate stable parameters and reliable programs to compute stable densities, cumulative distribution functions, and quantiles for stable random variables [9-12], the use of the heavy-tailed models in practice has been restricted by the lack of the tools for multivariate stable distributions.

The main challenge of dealing with multivariate data with heavy-tailed distributions is ambiguous dependence between the coordinates of a random vector. Whilst the dependence can be completely

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determined by the covariance matrix for the case of multi-normal data, the covariance matrix does not exist for heavy-tailed data. Fortunately, the problem can be solved by the tool of copula. The term copula was first introduced by Sklar [13] but was not of great interest until recent years. Copula functions describe the dependence structure connecting random variables, allowing separating of the dependence structure and marginal distributions.

Another way of parameterizing multivariate stable distributions is to use the well-known univariate stable distribution results about one-dimensional projections of random vectors. However, in practice, this approach faces challenging computational problems that have not been generally solved for multivariate stable distributions. The problems are caused by the complexity of the possible distributions with an uncountable set of parameters. In recent years, computations have become more accessible for elliptically contoured stable distributions by Nolan [6] which are scale mixtures of multivariate normal distributions. The tools for the very special class of stable distributions were applied in several empirical studies [5, 14]. Although the method is available only for a narrow subclass of symmetric multivariate stable distributions, that approach stimulates researchers to create similar tools for other subclasses of general stable multidimensional distributions. We developed a new method for investigating the data of multivariate distributions with heavy tails, to decrease the complexity of stable copulas downwards to a more practicable case of Gaussian copulas. This method was applied for daily return data on Nasdaq [15]. We continue to use it for daily returns data of stocks on HoSE in Vietnam and grains.

The work is organized as follows. Section 2 presents some auxiliary results on stable distributions and copulas. In Section 3, we give the main results of multivariate stable distributions with Gaussian copulas, demonstrating that Gaussian copulas are also those of some multivariate stable distributions and a random vector is  $\alpha$ -stable if it has Gaussian copula and all its marginals are  $\alpha$ -stable. We formulate the density function of a stable random vector with a Gaussian copula, which can be practically computed. Lastly, the results are applied to the study of stock market data of Vietnam and grains on the world. The conclusion is Section 4.

## 2. Stable Distributions and Copulas

### 2.1. Stable Distributions

A random variable  $X$  is said to have a stable distribution if for any positive numbers  $A, B$ , there is a positive number  $H$  and a real number  $\delta$  such that  $AX_1 + BX_2 \stackrel{D}{=} HX + \delta$ , where  $X_1$  and  $X_2$  are independent copies of  $X$ , and  $\stackrel{D}{=}$  denotes the equality in distribution.

For any stable random variable  $X$ , there is a number  $\alpha \in (0, 2]$  such that the numbers  $A, B$ , and  $C$  in the above definition satisfy the following formula  $A^\alpha + B^\alpha = H^\alpha$ . The number  $\alpha$  is called the index of stability or characteristic exponent. The stable random variable  $X$  with index  $\alpha$  is called  $\alpha$ -stable. The probability densities of  $\alpha$ -stable random variables exist and are continuous but, with a few exceptions, they are not known in closed form [16]. Characteristic function is a useful tool for studying these variables, the following form [17]:

$$\varphi_X(u) = \mathbb{E} \exp(iuX) = \begin{cases} \exp\left(-\gamma^\alpha |u|^\alpha \left[1 - i\beta \left(\tan\left(\frac{\pi\alpha}{2}\right)\right) \text{sign}(u) + i\delta u\right]\right), & \alpha \neq 1 \\ \exp\left(-\gamma |u| \left[1 + i\beta \left(2/\pi\right) \text{sign}(u) \ln|u| + i\delta u\right]\right), & \alpha = 1. \end{cases}$$

with fixed  $\alpha \in (0;2]$ ,  $\beta \in [-1;1]$ ,  $\gamma > 0$ ,  $\delta \in R$  and

$$\text{sign}(u) = \begin{cases} 1, & \text{if } u > 0 \\ 0, & \text{if } u = 0 \\ -1, & \text{if } u < 0 \end{cases}$$

The real parameters  $\alpha, \beta, \gamma,$  and  $\delta$  uniquely determine the distribution of  $X$ , then the symbol  $X \sim S(\alpha, \beta, \gamma, \delta)$  is used to refer to that situation. Usually  $\beta, \gamma,$  and  $\delta$  are named as the skewness, the scale and the location parameters of  $X$ , respectively.

Similar to the concept of  $\alpha$ -stable random variable in  $\mathbb{R}$ , the stability can be expanded to random vectors in  $\mathbb{R}^d$ . Namely, a random vector  $\mathbf{X}$  is said to have a stable distribution if for any positive numbers  $A, B$ , there is a positive number  $H$  and a vector  $\boldsymbol{\delta}$  in  $\mathbb{R}^d$  such that  $A\mathbf{X}_1 + B\mathbf{X}_2 \stackrel{D}{=} H\mathbf{X} + \boldsymbol{\delta}$ , where  $\mathbf{X}_1$  and  $\mathbf{X}_2$  are independent copies of  $\mathbf{X}$ . Characteristic functions are still used to describe stable distributions in multivariate case, but they are more complicated and determined by three parameters: stable index  $\alpha$ , vector  $\boldsymbol{\delta}$  and spectral measure  $\Lambda$  on the unit sphere  $S_d = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| = 1\}$  of  $\mathbb{R}^d$ :

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \begin{cases} \exp[-\int_{S_d} |\langle \mathbf{t}, \mathbf{s} \rangle|^\alpha (1 - i \text{sign}(\langle \mathbf{t}, \mathbf{s} \rangle) \tan(\pi\alpha/2)) \Lambda(ds) + i\langle \mathbf{t}, \boldsymbol{\delta} \rangle], & \alpha \neq 1 \\ \exp[-\int_{S_d} |\langle \mathbf{t}, \mathbf{s} \rangle| (1 + (2/\pi) i \text{sign}(\langle \mathbf{t}, \mathbf{s} \rangle) \ln|\langle \mathbf{t}, \mathbf{s} \rangle|) \Lambda(ds) + i\langle \mathbf{t}, \boldsymbol{\delta} \rangle], & \alpha = 1 \end{cases}$$

where  $\mathbf{t} = (t_1, \dots, t_d)^T, \mathbf{s} = (s_1, \dots, s_d)^T \in \mathbb{R}^d$  and the inner product  $\langle \mathbf{t}, \mathbf{s} \rangle = t_1s_1 + \dots + t_ds_d$  of  $\mathbf{t}$  and  $\mathbf{s}$  (see [14]). Moreover, any linear combination of the components of  $\mathbf{X} = (X_1, \dots, X_d)^T$  of the type  $Y = \sum_{k=1}^d a_k X_k$  is an  $\alpha$ -stable random variable, where  $a_k \in \mathbb{R}$ .

The spectral measure  $\Lambda$  is a non-negative Borel measure on  $S_d$  of the Euclidean space  $\mathbb{R}^d$ . The dependence structure of a stable distribution on  $\mathbb{R}^d$  is determined by the spectral measure  $\Lambda$ , however, estimating it is difficult and there is no more direct way to observe it. In some special cases, dependence structure of stable random vector is estimated simpler. Stable random vectors with Gaussian copula belong to these special cases.

### 2.2. Copulas

A copula is a multivariate distribution whose marginals have a uniform distribution on  $[0,1]$ . It is used as a model of the dependence structure of random variables. In 1959, copula was first introduced by Abe Sklar [13]. For later years, important basic concepts of copulas were found and became increasingly popular. In 1999, Embrechts, McNeil, Straumann succeeded in applying copulas in finance. In 2006, copulas were utilized as a risk management tool in insurance and bank. Nowadays, copulas have been applied in many fields as sea storm [18], risk evaluation of droughts [19].

Given a random vector  $\mathbf{X} = (X_1, \dots, X_d)^T$  taking values in Euclidean space  $\mathbb{R}^d$ , its cumulative distribution function (CDF hereafter) and probability density function (PDF hereafter) are denoted by  $F_{\mathbf{X}}$  and  $f_{\mathbf{X}}$ , respectively. The coordinates  $X_1, \dots, X_d$  are called marginals, simultaneously  $F_{X_1}, \dots, F_{X_d}$  and  $f_{X_1}, \dots, f_{X_d}$  are called marginal CDF's and marginal PDF's of  $\mathbf{X}$ , respectively.

Let random vector  $\mathbf{X} = (X_1, \dots, X_d)^T$ , a  $d$ -dimensional copula (or  $d$ -copula) of  $\mathbf{X}$  is the function  $C_{\mathbf{X}}: [0, 1]^d \rightarrow [0, 1]$  given by

$$C_{\mathbf{X}}(s_1, \dots, s_d) = F_{\mathbf{X}}\left(F_{X_1}^{\leftarrow}(s_1), \dots, F_{X_d}^{\leftarrow}(s_d)\right)$$

where  $F_{\mathbf{X}}$  is the CDF of  $\mathbf{X}$ , and  $F_{X_1}^{\leftarrow}, \dots, F_{X_d}^{\leftarrow}$  are generalized inverse functions of marginal distribution functions of  $\mathbf{X}$  [20], (If  $F$  is CDF of continuous variable,  $F^{\leftarrow} = F^{-1}$ ). The famous Sklar’s Theorem [15] also confirms the following relationship

$$F_{\mathbf{X}}(x_1, \dots, x_d) = C_{\mathbf{X}}\left(F_{X_1}(x_1), \dots, F_{X_d}(x_d)\right)$$

for  $x_1, \dots, x_d \in \bar{R} = [-\infty; +\infty]$ . From the above equations, we get

$$f_{\mathbf{X}}(x_1, \dots, x_d) = c_{\mathbf{X}}\left(F_{X_1}(x_1), \dots, F_{X_d}(x_d)\right) \times f_{X_1}(x_1) \times \dots \times f_{X_d}(x_d) \tag{1}$$

and

$$c_{\mathbf{X}}(s_1, \dots, s_d) = \frac{f_{\mathbf{X}}(F_{X_1}^{-1}(s_1), \dots, F_{X_d}^{-1}(s_d))}{f_{X_1}(F_{X_1}^{-1}(s_1)) \times \dots \times f_{X_d}(F_{X_d}^{-1}(s_d))} \tag{2}$$

for  $s_1, \dots, s_d \in [0; 1]$  and  $c_{\mathbf{X}}(s_1, \dots, s_d)$  is the PDF of the copula  $C_{\mathbf{X}}(s_1, \dots, s_d)$ , the CDF  $F_{\mathbf{X}}(x_1, \dots, X_d)$  is continuous.

A special copula is independence copula, denoted by  $\Pi_d$ , which is the copula of a random vector  $\mathbf{Y}$  with independent marginals  $X_1, \dots, X_d$ . In this case we have

$$F_{\mathbf{Y}}(x_1, \dots, x_d) = \Pi_d\left(F_{X_1}(x_1), \dots, F_{X_d}(x_d)\right) = F_{X_1}(x_1) \times \dots \times F_{X_d}(x_d), \tag{3}$$

and

$$f_{\mathbf{Y}}(x_1, \dots, x_d) = f_{X_1}(x_1) \times \dots \times f_{X_d}(x_d), \tag{4}$$

The copula was simply the joint distribution function of random variables with uniform marginals. The Gaussian copula is the most popular one in applications. It is simply derived from the correlation matrix  $\Sigma$  and mean vector  $\boldsymbol{\mu}$  of a multivariate Gaussian distribution (without loss of generality, we can assume  $\boldsymbol{\mu} = \mathbf{0}$ ) and is given by the following formula

$$C(s_1, \dots, s_d) = \int_{-\infty}^{\phi^{-1}(s_1)} \dots \int_{-\infty}^{\phi^{-1}(s_d)} (2\pi)^{-d/2} |\Sigma|^{-1/2} \exp\left[\frac{-1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu})\right] dx_d \dots dx_1 \tag{5}$$

where  $s_i \in [0, 1], i = 1, 2, \dots, d, \mathbf{x} = (x_1, \dots, x_d)^T, \phi$  is CDF of standard univariable Gaussian distribution. Moreover, the calculation of Gaussian copula is available on computer softwares, such as R software.

One nice property of copulas is that the copula functions are invariant under strictly increasing transformations. Especially, the following proposition given by Embrechts [5] provides a useful tool for getting the main results of our study.

**Proposition 1.** Let  $(X_1, \dots, X_d)$  be a continuous random vector which has copula  $C$ . If functions  $T_1, \dots, T_d : \mathbb{I} \rightarrow \mathbb{I}$  are strictly increasing on  $\text{ran}(X_1), \dots, \text{ran}(X_d)$ , respectively, then  $(T_1(X_1), \dots, T_d(X_d))$  also has copula  $C$ .

### 3. Results and Discussion

#### 3.1. Stable Random Vector with Gaussian Copula

In applied statistics, sometimes transformations that turn a given data set into a new form need to be used, for example, normalizing transformations. The following result is applied to transformation for continuously distributed data (see [15]).

**Proposition 2.** Let  $X$  and  $Y$  be continuous random variables with probability density functions  $f_X$  and  $f_Y$  which are positive on  $\text{ran}(X)$  and  $\text{ran}(Y)$ , respectively. Then there exists a strictly increasing differentiable function  $g: \text{ran}(X) \rightarrow \text{ran}(Y)$  such that the random variable  $g \circ X$  has the same distribution as  $Y$ , in brief  $g \circ X \sim Y$ .

**Proof.**

By assumption, the probability density functions  $f_X, f_Y$ . Therefore, the cumulative distribution functions  $F_X$  and  $F_Y$  are strictly increasing on the images  $\text{ran}(X)$  and  $\text{ran}(Y)$ , respectively. Then the function  $g: \text{ran}(X) \rightarrow \text{ran}(Y)$  defined by  $g(u) = F_Y^{-1} \circ F_X(u)$  is well-determined as a strictly increasing function. Besides, for each  $u \in \text{ran}(X)$ ,  $g'(u) = f_X(u) / f_Y(g(u))$  is a positive function.

The identity implies  $f_Y(g(u))g'(u)du = f_X(u)du$ , that yields

$$F_Y(g(t)) = \int_{-\infty}^t f_Y(g(u))g'(u)du = \int_{-\infty}^t f_X(u)du = F_X(t), \quad t \in \mathbb{R} \quad (6)$$

On the other hand, for every  $t \in \mathbb{R}$ ,

$$F_{g \circ X}(t) = P(\omega: g(X(\omega)) \leq t) = P(\omega: X \leq g^{-1}(t)) = F_X(g^{-1}(t))$$

Compared the above with (6), putting  $y = g(t)$  implies  $F_Y(y) = F_{g \circ X}(y)$ .

This confirms the random variable  $g \circ X$  has the same distribution as  $Y$ . The proposition is proved.

The above proposition immediately implies the following corollaries.

**Corollary 3.** Let  $X$  be a normal random variable and a positive number  $\alpha \in (0; 2]$ . Then there exists a strictly increasing function  $g: \mathbb{R} \rightarrow \mathbb{R}$  such that the random variable  $g \circ X$  is  $\alpha$ -stable.

**Corollary 4.** Let  $X$  be an  $\alpha$ -stable random variable for a given positive number  $\alpha \in (0; 2]$ . Then there exists a strictly increasing function  $g: \text{ran}(X) \rightarrow \mathbb{R}$  such that the random variable  $g \circ X$  has normal distribution.

It is evident that all marginals of a stable random vector are stable random variables. The inverse statement is not true, a random vector with all stable marginals is not always stable. However, as it is confirmed in the next lemma, the inverse statement is valid if those marginals are independent. Proof of that is quite simple and need not be presented.

**Lemma 5.** Let  $\alpha \in (0; 2]$  and a random vector  $\mathbf{U} = (U_1, \dots, U_d)$  be given. If the marginals  $U_1, \dots, U_d$  are independent  $\alpha$ -stable random variables, then  $\mathbf{U}$  is an  $\alpha$ -stable random vector.

The next lemma is a useful tool for the proof of Theorem 8.

**Lemma 6.** Let  $\alpha \in (0; 2]$  be given and  $\mathbf{X}$  be an  $\alpha$ -stable random vector with marginals  $X_1, \dots, X_d$ . Let  $\mathbf{Y} = \mathbf{Q}\mathbf{X}$ , where  $\mathbf{Q}$  is a  $d \times d$  matrix. Then  $\mathbf{Y}$  also is an  $\alpha$ -stable random vector.

**Proof.** Let  $\mathbf{X}^*$  be an independent copy of  $\mathbf{X}$ , then  $\mathbf{Y}^* = \mathbf{Q}\mathbf{X}^*$  is an independent copy of  $\mathbf{Y}$ . From the stability of  $\mathbf{X}$ , for every pair of positive numbers  $A$  and  $B$ , there is number  $H = [A^\alpha + B^\alpha]^{1/\alpha}$  and a vector  $\delta$  in  $\mathbb{R}^d$  such that  $A\mathbf{X} + B\mathbf{X}^* \stackrel{D}{=} H\mathbf{X} + \delta$ . Then it is obvious that

$$A\mathbf{Y} + B\mathbf{Y}^* = \mathbf{Q}(A\mathbf{X} + B\mathbf{X}^*) \stackrel{D}{=} \mathbf{Q}(H\mathbf{X} + \delta) = H\mathbf{Q}\mathbf{X} + \mathbf{Q}\delta = H\mathbf{Y} + \mathbf{Q}\delta,$$

which confirms the  $\alpha$ -stability of  $\mathbf{Y}^*$  and completes the proof.

The following lemma is an elementary result of probability theory, its proof can be omitted.

**Lemma 7.** Let  $\mathbf{G}$  be a  $d$ -dimensional normally distributed random vector with expectation  $\boldsymbol{\mu}$  and positive definite covariance matrix  $\Sigma$ ,  $\mathbf{G} \sim N_d(\boldsymbol{\mu}; \Sigma)$ . Then there exists an orthogonal  $d \times d$  matrix  $\mathbf{A} = (a_{ij})$  such that the random vector  $\mathbf{G}^* = \mathbf{A}\mathbf{G}$  has independent normally distributed marginals.

The above lemmas are used to prove the next theorem which confirms that Gaussian copula is a copula of a stable random vector.

**Theorem 8.** Let  $C$  be a Gaussian copula of a normally distributed random vector  $\mathbf{G}$  with positive definite covariance matrix  $\Sigma$ . Then for every number  $\alpha \in (0; 2]$ , there exists an  $\alpha$ -stable random vector  $\mathbf{W}$  such that  $C$  is also the copula of  $\mathbf{W}$ .

**Proof.** Because both addition and multiplication by positive numbers are strictly increasing transformations in  $\mathbb{R}$ , by virtue of Proposition 1 it can be supposed that all marginals  $G_1, \dots, G_d$  of the random vector  $\mathbf{G}$  are standard normal random variables,  $G_k \sim N(0;1)$  for  $k = 1, \dots, d$ . Based on the assumption, the covariance matrix  $\Sigma$  of  $\mathbf{G}$  is positive defined, Lemma 7 implies the existence of an orthogonal  $d \times d$  matrix  $\mathbf{A} = (a_{ij})$  such that the normal random vector  $\mathbf{Y} = \mathbf{A}\mathbf{G}$  has independent marginals  $Y_1, \dots, Y_d$ . Now  $\alpha$ -stable random variables  $S_1, \dots, S_d$  with the common distribution  $S(\alpha, 0, 1, 0)$  are concerned. Corollary 3 ensures that, for each  $k = 1, \dots, d$ , there exists a strictly increasing function  $g_k: \mathbb{R} \rightarrow \mathbb{R}$  such that the random variable  $U_k = g_k \circ Y_k$  has the same  $\alpha$ -stable distribution as  $S_k$ . Simultaneously, the independence of marginals  $Y_1, \dots, Y_d$  implies the independence of random variables  $U_1, \dots, U_d$ . Thus, the random vector  $\mathbf{U} = (U_1, \dots, U_d)$  has independent  $\alpha$ -stable marginals, it must be an  $\alpha$ -stable random vector as the conclusion of Lemma 5. Let define a new random vector  $\mathbf{W} = (W_1, \dots, W_d) = \mathbf{A}^{-1}\mathbf{U}$ . Then by virtue of Lemma 6, it is clear that  $\mathbf{W}$  is also an  $\alpha$ -stable random vector. We attempt to point out that  $\mathbf{W}$  has the same copula  $C$  as  $\mathbf{G}$ , that means  $C_{\mathbf{W}} = C_{\mathbf{G}}$ , which is equivalent to

$$c_{\mathbf{W}} = c_{\mathbf{G}}. \tag{7}$$

Firstly, we defined the transformation  $\mathbf{B}: \mathbb{R}^d \rightarrow \mathbb{R}^d$  by  $\mathbf{B}(t_1, \dots, t_d) = (g_1(t_1), \dots, g_d(t_d))$  for  $(t_1, \dots, t_d) \in \mathbb{R}^d$ . Then it is clear that  $\mathbf{W} = \mathbf{A}^{-1}\mathbf{B}\mathbf{A}(\mathbf{G})$ . Denoted by  $J_{\mathbf{K}}$  the Jacobian of the transformation  $\mathbf{K}: \mathbb{R}^d \rightarrow \mathbb{R}^d$ . Then, because  $\mathbf{A}$  is an orthogonal matrix,  $J_{\mathbf{A}} = J_{\mathbf{A}^{-1}} = 1$ , we also have  $J_{\mathbf{B}} = g'_1(t_1) \cdot \dots \cdot g'_d(t_d)$ .

These equations imply

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{A}(t_1, \dots, t_d)) &= f_{\mathbf{G}}(t_1, \dots, t_d) J_{\mathbf{A}}^{-1}(\mathbf{A}(t_1, \dots, t_d)) = f_{\mathbf{G}}(t_1, \dots, t_d), \\ f_{\mathbf{U}}(\mathbf{B}\mathbf{A}(t_1, \dots, t_d)) &= f_{\mathbf{Y}}(\mathbf{A}(t_1, \dots, t_d)) J_{\mathbf{B}}^{-1}(\mathbf{B}\mathbf{A}(t_1, \dots, t_d)) = f_{\mathbf{G}}(t_1, \dots, t_d) \times [g'_1(t_1) \cdot \dots \cdot g'_d(t_d)]^{-1}, \\ f_{\mathbf{W}}(\mathbf{A}^{-1}\mathbf{B}\mathbf{A}(t_1, \dots, t_d)) &= f_{\mathbf{U}}(\mathbf{B}\mathbf{A}(t_1, \dots, t_d)) J_{\mathbf{A}^{-1}}^{-1}(\mathbf{A}^{-1}\mathbf{B}\mathbf{A}(t_1, \dots, t_d)) = f_{\mathbf{U}}(\mathbf{B}\mathbf{A}(t_1, \dots, t_d)) \\ &= f_{\mathbf{G}}(t_1, \dots, t_d) \times [g'_1(t_1) \cdot \dots \cdot g'_d(t_d)]^{-1}. \end{aligned} \tag{8}$$

Consider  $h_k = F_{W_k}^{-1} \circ F_{G_k}$ , so  $h'_k(u_k) = \frac{f_{G_k}(u_k)}{f_{W_k}(F_{W_k}^{-1} \circ F_{G_k}(u_k))}$ , for  $k = 1, \dots, d$ .

On the other hand,  $W_k \sim S(\alpha, 0, \gamma_k, 0)$ , therefore,  $\frac{1}{\gamma_k} W_k \sim S(\alpha, 0, 1, 0)$ . Suppose that  $W_k = \gamma_k S_k$ , we can be calculated  $\gamma_1 \dots \gamma_d = 1$ .

Moreover,  $U_k = g_k \circ Y_k$ , for  $k = 1, \dots, d$ ,  $g_k = F_S^{-1} \circ F_{Y_k}$ , we have

$$\tag{9}$$

$$g'_k(u_k) = \frac{f_{Y_k}(u_k)}{f_{S_k}(F_{S_k}^{-1} \circ F_{Y_k}(u_k))} = \frac{f_{G_k}(u_k)}{\frac{1}{\gamma_k} f_{W_k}(F_{W_k}^{-1} \circ F_{Y_k}(u_k))} = \gamma_k h'_k(u_k) \tag{10}$$

However, for  $(s_1, \dots, s_d) \in [0;1]^d$ , it implies from (2) that

$$c_{W, s_1, \dots, s_d} = \frac{f_{W_1}(F_{W_1}^{-1}(s_1), \dots, F_{W_d}^{-1}(s_d))}{f_{W_1}(F_{W_1}^{-1}(s_1)) \cdots f_{W_d}(F_{W_d}^{-1}(s_d))}$$

and

$$c_{G, s_1, \dots, s_d} = \frac{f_{G_1}(F_{G_1}^{-1}(s_1), \dots, F_{G_d}^{-1}(s_d))}{f_{G_1}(F_{G_1}^{-1}(s_1)) \cdots f_{G_d}(F_{G_d}^{-1}(s_d))}.$$

The above equations together with (8), (9) and (10) ensure the validity of (7), the proof completes.

In the next theorem we see that owning Gaussian copula is a sufficient condition for a random vector with stable marginals with zero skewness parameters to have stable distribution.

**Theorem 9.** For given  $\alpha \in (0;2]$ , let  $\mathbf{G}$  be a Gaussian random vector with positive definite covariance matrix. Suppose that  $\mathbf{V}$  is a random vector with  $\alpha$ -stable marginals  $V_1 \sim S(\alpha, 0, \gamma_1, \delta_1), \dots, V_d \sim S(\alpha, 0, \gamma_d, \delta_d)$  such that the copula of  $\mathbf{V}$  and of  $\mathbf{G}$  are both equal to each other. Then  $\mathbf{V}$  is an  $\alpha$ -stable random vector.

**Proof.** From Corollary 4, there exist strictly increasing function  $h_k: \mathbb{R} \rightarrow \mathbb{R}, k = 1, \dots, d$ , such that the random variables  $h_k \circ V_k$  has normal distribution  $N(0;1)$ . Repeated the proof of Theorem 8 we have stable random vector  $\mathbf{W} = (W_1, \dots, W_d), W_k \sim S(\alpha, 0, \gamma'_k, 0), k = 1, \dots, d$ . Moreover, we can suppose that  $V_k = \gamma_k W_k + \delta_k$ . Modified Lemma 6 with the  $d \times d$  matrix  $\mathbf{Q}$  replaced by a linear transformation, we can be sure that the random vector  $\mathbf{V} = (V_1, \dots, V_d)$  is an  $\alpha$ -stable random vector. The Theorem is proved

The above theorem together with the special structure of Gaussian copulas allows researchers to combine well - known computational tools for one dimensional stable distributions and Gaussian copulas to compute density functions and cumulative distribution functions of data which follow stable distributions with Gaussian copulas. In particular, from (5) we obtain the following result.

**Corollary 10.** For given  $\alpha \in (0;2]$ , let  $\mathbf{G}$  be a Gaussian random vector with positive definite correlation matrix  $\Sigma$ , let  $\mathbf{S}$  be an  $\alpha$ -stable random vector with  $\alpha$ -stable marginals  $S_k$  with zero skewness parameters,  $k = 1, \dots, d$ . Suppose that the Gaussian copula  $C_G$  is the copula of  $\mathbf{S}$ . Then the density function of  $\mathbf{S}$  can be calculated by the formula

$$f_{\mathbf{S}}(y_1, \dots, y_d) = \frac{\exp(\frac{-1}{2}(\phi^{-1}(F_{S_1}(y_1)), \dots, \phi^{-1}(F_{S_d}(y_d)))^T \Sigma^{-1} (\phi^{-1}(F_{S_1}(y_1)), \dots, \phi^{-1}(F_{S_d}(y_d))))}{(2\pi)^{\frac{d}{2}} |\Sigma|^{\frac{1}{2}} \varphi(\phi^{-1}(F_{S_1}(y_1))) \cdots \varphi(\phi^{-1}(F_{S_d}(y_d)))} \times f_{S_1}(y_1) \cdots f_{S_d}(y_d),$$

where  $\varphi$  and  $\phi$  are PDF and CDF of univariable standard Gaussian distribution.

Based on Theorem 9, we suggest a procedure to check whether a data set can be fitted to multi-dimensional stable distribution or not, with details as follows:

*Step 1.* To estimate stable parameters of data marginals and to check if all marginals have  $\alpha$ -stable distributions with zero skewness parameters and suitably chosen common stable index  $\alpha$ .

There are several available software packages (eg. R package) that serve to estimate stable parameters of univariate data. Having the estimated marginal stable indexes  $\alpha_i$ , it is most practicable to take their average as the common stable index  $\alpha$  for checking the  $\alpha$ -stability of all marginals. Then, the Kolmogorov-Smirnov test, which is available in almost all statistical packages, can be applied to check the  $\alpha$ -stability of each marginal. If any marginal is rejected to have  $\alpha$ -stable distribution, the procedure can be immediately stopped and one can conclude the concerned data set cannot be fitted to any multi-dimensional stable distribution. In the case when all marginals are accepted to have  $\alpha$ -stable distribution with zero skewness parameters, the procedure can be continued to the next step.

*Step 2.* i) To apply Corollary 4 for transforming the concerned multivariate data to the data which have all normal distributed marginals; ii) To estimate the covariance/correlation matrix  $\Sigma$  of the transformed data and to check if the matrix is positive definite; and iii) To test hypothesis for checking if the transformed data are fitted to the Gaussian copula determined by (5).

By virtue of Proposition 1, the transformed data in point: i) Has the same copula as the copula of the original data. Besides, it is clear that the matrix in point; ii) Is positive definite if its determinant differs from 0. Moreover, the hypothesis test in point; and iii) Can be conducted by using an available statistical package (R package for instance).

If the hypothesis test in Step 2 gets the acceptance, Theorem 9 allows us to conclude that the concerned data are fitted to a random vector with stable distribution. However, if the hypothesis remains rejected, one cannot conclude that the data are not extracted from a random vector with stable distribution, because the class with all stable random vectors is much broader than the class with stable random vectors having Gaussian copulas.

### 3.2. Application of Stable Random Vector with Gaussian Copula for Real Data

In this section, we analyze data of stable distribution with Gaussian copula by using the results given in the previous section.

#### 3.2.1. Dataset of Stocks in Vietnam Stock Market

Dataset includes daily close prices of 4 stocks: BID (BIDVbank), VCB (Vietcombank), FLC (FLC Group), VNM (Vinamilk), with a sample from July 24, 2017 to October 14, 2019 to imply observations downloaded from Bao Viet Securities website (<https://bvsc.com.vn/>). These are stocks on Ho Chi Minh Stock Exchange (abbreviated HoSE).

Continuously compounded percentage returns are considered, i.e. daily returns are measured by log-differences of closing pricing multiplied by 100. Descriptive statistics together with Kolmogorov-Smirnov test (KS test) for normal distribution of the univariate series are shown in Table 1 and the result for the univariate stable model estimation are presented in Table 2.

Table 1. Normal distribution test for HoSE daily returns

Stock	Size	Mean	SD	Sknew	Kurtosis	p-value
BID	497	0.0015	0.03	0.42	3.74	0.00455
VCB	497	0.0021	0.02	0.34	3.85	0.00068
FLC	497	-0.0015	0.02	-1.18	13.55	0.00014
VNM	497	-0.0003	0.02	-2.47	43.59	0.00018

In Table 1, all p-values smaller than 5% confirm the significant divergence from normal distribution of the 4-dimensional vector. Simultaneously, the greater than 5% p-values of KS tests in Table 2 are crucial arguments to conclude all the daily returns series of BID, VCB, FLC and VNM have



univariate stable distributions with common stable index  $\bar{\alpha} = 1.416$  (the average number of  $\alpha_i$ 's of those returns series).

Table 2. KS test for univariate stability of HoSE daily returns

Stock	$\alpha$	$\bar{\alpha}$	$\beta$	$\gamma$	$\delta$	p-value
BID	1.464	1.416	0.008	0.0143	-0.0015	0.5046
VCB	1.446	1.416	0.164	0.010	-0.0021	0.2884
FLC	1.256	1.416	0.02	0.0093	-0.0007	0.2233
VNM	1.498	1.416	0.02	0.0082	0.0002	0.1952

After the above conclusion, we guess the 4-component vector of series of BID, VCB, FLC and VNM has multivariate stable distribution. To check this, we will use model of multivariate stable distribution with Gaussian copula. In the first step, the correlation matrix of daily returns (after normalizing by respective functions determined in Proposition 2 and Corollary 4) was calculated, with results given in Table 3.

Table 3. Correlation matrix of HoSE daily returns

	BID	VCB	FLC	VNM
BID	1	0.05001	0.00232	0.01380
VCB	0.05001	1	0.04964	0.05349
FLC	0.00232	0.04964	1	0.00167
VNM	0.01380	0.05349	0.00167	1

Then, the function named `gofCopula` in the copula package of R software was used to test the hypotheses of having Gaussian copula for 4-coordinates vector of daily returns, p-value is 0.2552. The result shows that the copula of daily returns vector is significantly Gaussian copula. Thus, according to Theorem 8, the daily returns vector of 4 stocks BID, VCB, FLC and VNM has multivariate stable distribution with stable index  $\bar{\alpha} = 1.416$ .

Consequently, by virtue of (1), (2), and (5), Corollary 9 can be applied to determine the density functions of 4-coordinate vector of daily returns. In particular, the density functions are defined by the following explicit form:

$$f_{\mathbf{X}}(x_1, \dots, x_4) = \frac{\exp\left(\frac{-1}{2}\left(\phi^{-1}(F_{X_1}(x_1)), \dots, \phi^{-1}(F_{X_4}(x_4))\right)^T \Sigma^{-1} \left(\phi^{-1}(F_{X_1}(x_1)), \dots, \phi^{-1}(F_{X_4}(x_4))\right)\right)}{(2\pi)^2 |\Sigma|^{1/2} \varphi\left(\phi^{-1}(F_{X_1}(x_1))\right) \dots \varphi\left(\phi^{-1}(F_{X_4}(x_4))\right)} \\ \times f_{X_1}(x_1) \times \dots \times f_{X_4}(x_4)$$

where  $\mathbf{X} = (X_1, \dots, X_4)^T$  with 4-coordinate valued daily returns of stocks BID, VCB, FLC, and VNM, respectively. Moreover, all linear combinations of  $\alpha$ -stable random vector are  $\alpha$ -stable variables. Therefore, we can consider choice investment portfolio when invests in four these stocks.

### 3.2.2. Dataset of Grains

With the advantage of nutrients, the grain market is increasingly active. Grain prices are also highly volatile, while the margins for some grains do not follow a normal distribution. This article will analyze daily returns of three kinds of grains: oat, US corn, US soybean meal. Dataset was downloaded from "Investing.com website", with 502 observations sample from January 22<sup>nd</sup>, 2021 to December 30<sup>th</sup>, 2022.

Descriptive statistics together with KS test for normal distribution of the univariate series are shown in Table 4 and the result for the univariate stable model estimation are presented in Table 5. In Table 4,

all p-values smaller than 5% confirm the significant divergence from normal distribution of the 3-dimensional vector. Simultaneously, the greater than 5% p-values of KS tests in Table 5 are crucial arguments to conclude all the daily returns series of Oat, Corn, and Soybean have univariate stable distributions with common stable index  $\bar{\alpha} = 1.598667$ .

Table 4. Normal distribution test for grains daily returns

Grains	Size	Mean	SD	Sknew	Kurtosis	p-value
Oat	502	0.0015	0.03	0.42	3.74	0.00455
Corn	502	0.0021	0.02	0.34	3.85	0.00068
Soybean	502	-0.0015	0.02	-1.18	13.55	0.00014

Table 5. KS test for univariate stability of grains daily returns

Stock	$\alpha$	$\bar{\alpha}$	$\beta$	$\gamma$	$\delta$	p-value
Oat	1.648	1.598667	-0.225	1.080558	0.1388824	0.7207
Corn	1.486	1.598667	-0.067	1.42446101	0.02137818	0.3315
Soybean	1.662	1.598667	-0.22	1.09887852	0.09673249	0.9960

Next, we check multivariate stable distribution with Gaussian copula of the 3-component vector of series Oat, Corn and Soybean. The correlation matrix of daily returns (after normalizing by respective functions determined in Proposition 2 and Corollary 4) was calculated, with results given in Table 6.

Table 6. Correlation matrix of grains daily returns

	Oat	Corn	Soybean
Oat	1	-0.05413226	0.06138928
Corn	-0.05413226	1	0.05633384
Soybean	0.06138928	0.05633384	1

Then, the function named `gofCopula` in the copula package of R software was used to test the hypotheses of having Gaussian copula for 3-coordinate vector of daily returns, p-value is 0.5559. The result shows that the copula of daily returns vector is significantly Gaussian copula. Thus, according to Theorem 9, the daily returns vector of 3 kinds of grains Oat, Corn and Soybean has multivariate stable distribution with stable index  $\bar{\alpha} = 1.598667$ .

Similar, Corollary 10 can be applied to determine the density functions of 3-coordinate vector of daily returns grains. In particular, the density functions are defined by the following explicit form:

$$f_{\mathbf{X}}(x_1, x_2, x_3) = \frac{\exp\left(\frac{-1}{2}\left(\phi^{-1}(F_{X_1}(x_1)), \dots, \phi^{-1}(F_{X_3}(x_3))\right)^T \Sigma^{-1} \left(\phi^{-1}(F_{X_1}(x_1)), \dots, \phi^{-1}(F_{X_3}(x_3))\right)\right)}{(2\pi)^{3/2} |\Sigma|^{1/2} \varphi(\phi^{-1}(F_{X_1}(x_1))) \dots \varphi(\phi^{-1}(F_{X_3}(x_3)))} \\ \times f_{X_1}(x_1) \times f_{X_2}(x_2) \times f_{X_3}(x_3)$$

where  $\mathbf{X} = (X_1, X_2, X_3)^T$  with 3-coordinate valued daily returns of grains Oat, Corn, and Soybean, respectively. Therefore, we can consider choosing investment portpolio when investing in three these products.

#### 4. Conclusion

Stable random vectors are increasingly being applied to model real-world data, especially for heavy-tailed multivariate data, while the calculations on this kind of data are very complicated. Copula is a connection mechanism that relates marginal distributions together to determine the multivariate distribution of a random vector. The paper represents the theoretical base of the Gaussian copula role in connection of univariate stable marginal distributions into multivariate stable distributions. This base allows to create a tool for testing the stable multivariate distribution of random vector. The tool can be applied widely for real-world data.

#### References

- [1] R. M. Kunst, Apparently Stable Increments in Finance Data: Could ARCH Effects Be the Cause?, *J. Statist. Comput. Simulation*, Vol. 45, No. 1-2, 1993, pp. 121-127.
- [2] K. J. Palmer, M. S. Ridout, B. J. T. Morgan, Modelling Cell Generation Times Using The Tempered Stable Distribution, *J. R. Stat. Soc. Ser. C. Appl. Stat.*, Vol. 57, No. 4, 2008, pp. 379-397.
- [3] P. Samuelson, Efficient Portfolio Selection for Pareto - Lévy Investments, *J. Financ. Quant. Anal.*, Vol. 2, No. 2, 1967, pp.107-117.
- [4] M. S. Taqqu, The Modeling of Ethernet Data and of Signals That Are Heavy-tailed with Infinite Variance, *Scand. J. Stat.*, Vol. 29, No. 2, 2002, pp. 273-295.
- [5] P. Embrechts, F. Lindskog, A. McNeil, Modelling Dependence with Copulas and Applications to Risk Management, *Handbook of Heavy Tailed Distributions in Finance*, Elsevier, 2009, Chapter 8, pp. 329-384.
- [6] J. P. Nolan, Multivariate Elliptically Contoured Stable Distributions: Theory and Estimation, *Comput. Statist.*, Vol. 28, No. 5, 2013, pp. 2067-2089.
- [7] N. B. Quang, On Stable Probability Distributions and Statistical Application, Thesis, Academy of Military Science and Technology, Hanoi, 2016.
- [8] J. H. McCulloch, Financial Applications of Stable Distributions, *Handbook of Statistics 14*, ed. G. Maddala and C. Rao, Elsevier Science Publishers, North-Holland, 1996, pp. 393-425.
- [9] J. H. McCulloch, Simple Consistent Estimators of Stable Distribution Parameters, *Comm. Statist. Simulation Comput.*, Vol. 15, No. 4, 1986, pp. 1109-1136.
- [10] R. J. Adler, R. E. Feldman, M. S. Taqqu, *A Practical Guide to Heavy Tailed Data*, Birkhäuser, Boston, 1998.
- [11] S. M. Kogon, D. B. Williams, Characteristic Function Based Estimation of Stable Parameters, In Adler R., Feldman R. and Taqqu M. (eds.) *A Practical Guide to Heavy Tailed Data*, Birkhäuser, Boston, MA, 1998, pp. 311-335.
- [12] J. P. Nolan, *Stable Distributions - Models for Heavy Tailed Data*, Birkhauser: Boston, MA, USA, <https://edspace.american.edu/jpnolan/wp-content/uploads/sites/1720/2020/09/Chap1.pdf/>, 2018 (accessed on: December 15<sup>th</sup>, 2018).
- [13] A. Sklar, Fonctions De Répartition à n Dimensions et Leurs Marges, *Publications de l'Institut de Statistique de l'Université de Paris*, Vol. 8, No. 3, 1959, pp. 229-231.
- [14] F. Lamantia, S. Ortobelli, S. Rachev, VaR, CVaR and Time Rules with Elliptical and Asymmetric Stable Distributed Returns, *Investment Management and Financial Innovations*, Vol. 3, No. 4, 2006, pp. 19-39.
- [15] H. D. Phuc, V. T. T. Giang, Gaussian Copula of Stable Random Vectors and Application, *Hacet. J. Math. Stat.*, Vol. 49, No. 2, 2020, pp. 887-901.
- [16] J. Zolotarev, Empirical Characteristic Function Estimation and Its Applications, *Econometric Reviews*, Vol. 23, No. 2, 2004, pp. 93-123.
- [17] G. Samorodnitsky, M. S. Taqqu, *Stable Non-Gaussian Random Processes*, New York, NY, Chapman & Hall, 1994.
- [18] S. Corbella, D. D. Stretch, Simulating a Multivariate Sea Storm using Archimedean Copulas, *Coast. Eng.*, Vol. 76, 2013, pp. 68-78.
- [19] Q. Zhang, M. Xiao, V. P. Singh, X. Chen, Copula-based Risk Evaluation of Droughts Across the Pearl River basin, China, *Theoretical and Applied Climatology*, Vol. 111, No. 1-2, 2013, pp. 119-131.
- [20] R. P. Nelsen, *An Introduction to Copulas*, New York, Springer, 2006.