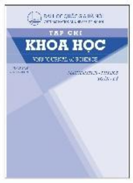
VNU Journal of Science: Mathematics – Physics, Vol. 39, No. 2 (2023) 100-110

VNU Journal of Science: Mathematics - Physics

Journal homepage: https://js.vnu.edu.vn/MaP



Original Article

Bohl Theorem for Volterra Equations

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Received 31 Janurary 2023
Revised 25 March 2023; Accepted 25 March 2023

Abstract: This work deals with the preservation of exponential stability under small perturbations for Volterra differential equations. The so-called Bohl-Perron type stability theorems for these systems are also studied.

Keywords: Hardy inequality, time scales, exponential function.

1. Introduction

The Volterra differential equations play an important role in studying mathematical models because for almost systems in ecology, economy, the evolution of present time depends on the past history of the systems. Therefore, studying the robust stability of systems is important both in theory and practice since the system always operates under the effect of uncertain perturbations. One can deal with the robust stability by many ways. Some research groups measured the robust stability by using the so-called stability radii for linear systems [1, 2] or carried-out some estimates of the perturbation ensuring the stability of perturbed systems [3, 4]. Also, the other method to study robust stability is to consider Bohl-Perron theorem, which establishes a relation between the Liapunov stability of homogeneous differential equations in initial conditions and the boundedness of solutions of inhomogeneous one. We can refer to [5-7] to get some results on this problem for ordinary or delay linear differential equation.

The aim of this work is to study the robust stability and Bohl-Perron theorem for the Volterra integro-differential equation

$$x'(t) = A(t)x(t) + \int_0^t H(t,s)x(s)\,d\mu(s) + f(t), \quad t \geq 0, \tag{1}$$

where μ is a radon measure on $(\mathbb{R}^+, B(\mathbb{R}^+))$ and $A(.)$,$H(.,.)$ are specified later.

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https//doi.org/10.25073/2588-1124/vnumap.4820

Firstly, we deal with the preservation of stability for the linear Volterra integro-differential Eq. (1) under small perturbations and then we study Bohl-Perron theorem. Since the derivative of state process x(t) depends on all past path $x(s), 0 \leq s \leq t,$ we have to use a more general inequality of Gronwall-Bellman type to obtain the upper bound of perturbations. Further, the Cauchy operator of the corresponding homogeneous equation does not have the semi-group property, which implies that the classical argument to solve this problem is no longer valid. To overcome it, we define weighed spaces $L^\gamma([0,\infty), X)$ and $C^\gamma([0,\infty), X)$ (see Definitions below) and consider operators acting between these spaces. The paper is organized as follows. In the next section we recall some basic properties of linear Volterra integro-differential with measures and prove the existence of solutions. In Section 3, we prove that if the linear Volterra equations are exponentially stable, then under small Lipschitz perturbations, the perturbed system is L_p stable. Section 4 presents the famous Bohl-Perron Theorem for Eq. (1). We introduce some weighted spaces and consider the solutions of Eq. (1) as elements of these spaces. Hence, we show that the exponential stability is equivalent to the surjectivity of certain operators. Some examples are introduced to illustrate the results.

2. Linear Volterra Differential Equations

Let X be a Banach space and $L(X)$ be the space of the continuous linear transformations on X. Let $A(.) : [0,\infty) \to L(X)$ is a continuous function valued in $L(X)$ and $H(.,.)$ is a two variable continuous function defined on the set $\{(t,s) : 0 \leq s \leq t < \infty\}$, valued in $L(X)$. For any continuous function $q : [0,\infty) \to X$ we consider the linear integro-differential Volterra system

$$\begin{cases} x'(t) = A(t)x(t) + \int_0^t H(t,s)x(s)d\mu(s) + q(t), \ \ t \geq 0 \\ x(0) = x_0. \end{cases} \tag{2}$$

where μ is a random measure on $(□^+, B(□^+))$. This means that μ is a measure and $\mu(M) < +\infty$ for every compact $M \subset □^+$. We note that although $H(.,.)$ is a two variables continuous function, the mapping $t \to \int_0^t H(t,s)x(s)\,d\mu(s)$ may be discontinuous at some points (at most in a countable set). Therefore, we have to understand the solution x of the initial problem in Eq. (2) in Carathéodory sense, i.e., $x(.)$ is continuous and it is differentiable almost every where with respect to Lebesgue measure on $□^+$. In other words, the function $x(.)$ is a solution of Eq. (2) with the initial condition $x(0) = x_0$ if and only if it satisfies the equation:

$$x(t) = x(0) + \int_0^t A(t)x(t)dt + \int_0^t \int_0^\tau H(\tau,s)x(s)d\mu(s) \ d\tau + \int_0^t q(\tau)d\tau, \ \ t \geq 0. \tag{3}$$

Proposition 2.1 The initial value problem in Eq. (2) has a unique solution.

Proof. Let $T > 0$ fixed. Construct a sequence of Picard approximations

$$x_0(t) = x(0), \ \ 0 \leq t \leq T,$$

$$x_{n+1}(t) = x(0) + \int_0^t A(\tau)x_n(\tau)d\tau + \int_0^t \int_0^\tau H(\tau,s)x_n(s)d\mu(s) \ d\tau + \int_0^t q(\tau)d\tau. \tag{4}$$

It is seen that

$$\|x_{n+1}(t) - x_n(t)\| = \int_0^t \|A(\tau)(x_n - x_{n-1})(\tau)\|d\tau + \int_0^t \int_0^\tau \|H(\tau,s)(x_n - x_{n-1})(s)\|d\mu(s) \ d\tau.$$

Therefore,

$$\sup_{0\leq\tau\leq T} |x_{n+1}(\tau) - x_n(\tau)| = K \int_0^t \sup_{0\leq\tau\leq s} \|(x_n - x_{n-1})(\tau)\|ds,$$

where $K = \sup_{0\le\tau\le T} \|A(\tau)\| + \mu([0,T]) \sup_{0\le s\le t\le T} \le \|H(t,s)\|$. By induction we get

$$\sup_{0\le\tau\le T} \|(x_{n+1}-x_n)(\tau)\| \le K \int_0^t \sup_{0\le\tau\le s} \|(x_n - x_{n-1})(\tau)\| ds$$
$$\le K^2 \int_0^t \left(\int_0^s \sup_{0\le u\le\tau} \|(x_{n-1}-x_{n-2})(u)\| du \right) ds \le \frac{(TK)^{n-1}}{(n-1)!}.$$

Hence, by using Weierstrass criterion for uniform convergence of series we see that

$$x(t) - x_0 = \sum_{k=1}^{\infty} (x_k(\tau) - x_{k-1}(\tau)) = \lim_{n\to\infty} x_n(t) - x_0,$$

is a continuous function. By passing the limit as $n \to \infty$ in Eq. (4) we have

$$x(t) = x(0) + \int_0^t A(t)x(t)dt + \int_0^t \int_0^t H(t,s)x(s)d\mu(s)\ dt + \int_0^t q(t)dt\,,\ \ t \ge 0,$$

i.e., $x(.)$ is the solution of (3) with the initial condition $x(0) = x_0$. The proof is complete.

The homogeneous equation corresponding with Eq. (2), i.e., $q \equiv 0$ is

$$\begin{cases} x'(t) = A(t)x(t) + \int_0^t H(t,s)x(s)d\mu(s)\,,\ \ t \ge 0, \\ x(0) = x_0. \end{cases} \tag{5}$$

We define the Cauchy operator $\Phi(t,s), t \ge s \ge 0$, generated by the system (2.4) as the solution of the matrix equation:

$$\begin{cases} \Phi'(t,s) = A(t)\Phi(t,s) + \int_0^t H(t,s)\Phi(\tau,s)d\mu(\tau)\,,\ \ t \ge s \ge 0 \\ \Phi(s,s) = I \end{cases}. \tag{6}$$

As it is mentioned above, the mapping $t \to \Phi(t,s)$ is continuous and it is differentiable almost every, where $t \in [s,\infty)$. We have the following useful lemma, called the variation of constants formula,

Lemma 2.2 The solution of the Volterra equation (2.1) can be expressed as

$$x(t) = \Phi(t,0)x_0 + \int_0^t \Phi(\tau,\rho)q(\rho)\, d\rho. \tag{7}$$

Proof. We have

$$\int_0^t \left[A(\tau) \int_0^\tau \Phi(\tau,u)q(u)du + \int_0^\tau H(\tau,s) \left(\int_0^s \Phi(s,u)q(u)du \right) d\mu(s) + q(\tau) \right] d\tau$$
$$= \int_0^t \left[A(\tau) \int_0^\tau \Phi(\tau,u)q(u)du + \int_0^\tau \left(\int_u^\tau H(\tau,s)\Phi(s,u)d\mu(s) \right) q(u)du + q(\tau) \right] d\tau$$
$$= \int_0^t \left[\int_0^\tau \left(A(\tau)\Phi(\tau,u)q(u)du + \int_u^\tau H(\tau,s)\Phi(s,u)d\mu(s) \right) q(u)du + q(\tau) \right] d\tau$$
$$= \int_0^t \left[\int_0^\tau \frac{d}{d\tau}\Phi(\tau,u)\, q(u)du + q(\tau) \right] d\tau = \int_0^t q(u) \left(\int_u^\tau \frac{d}{d\tau}\Phi(\tau,u)d\tau \right) du + \int_0^t q(\tau)d\tau$$
$$= \int_0^t \Phi(t,u)q(u)\, du = x(t).$$

Since the semi-group property of the Cauchy operator does not hold for the Volterra Eq. (5), we have to use another technique to study the Bohl-Perron Theorem for Volterra Equations.

Definition 2.3 i) The Volterra equation (5) is uniformly bounded if there exists a positive number C_0 such that

$$\|\Phi(t,s)\| \le C_0, t \ge s \ge 0.$$

ii) Let $\omega > 0$. The Volterra equation (2.4) is ω -exponentially stable if there exists a positive number M such that

$$\|\Phi(t,s)\| \le Me^{-\omega(t-s)}, t \ge s \ge 0.$$

The conditions ensuring the boundedness or stability of the equation (5) can be referred to [6,8,9] and references therein.

3. Stability of Volterra integro-differential Equation Under Small Perturbations

In this section, we consider the effect of small perturbations to the stability of the Volterra Eq. (5). Let $f(t,s,x)$ and $g(t,x)$ be two continuous functions. Suppose that for every $s \le t$ and $x \in X$, the coefficients $H(t,s)x$ and $A(t)x$ of Eq. (5) are perturbed by f and g. Thus, they become $H(t,s)x \mapsto H(t,s)x + f(t,s,x)$ and $A(t)x \mapsto A(t)x + g(t,x)$. Thus, for any $t_0 \ge 0$, the Cauchy problem for the perturbed Eq. (5) has following form:

$$\begin{cases} x'(t) = A(t)x(t) + \int_0^t H(t,s)x(s)d\mu(s) + \int_0^t f(t,s,x(s))d\mu(s) + g(t,x(t)), \;\; t \ge 0 \\ x(0) = x_0 \end{cases} \tag{8}$$

Suppose further that $f(t,s,x)$ is Lipschitz in x with Lipschitz coefficients $k_{t,s}$ and $g(t,x)$ is Lipschitz with Lipschitz coefficient l_t, where $k_{t,s}, t \ge s \ge 0$, and $l_t, t \ge 0$ are continuous functions. One can suppose that

$$f(t,s,0) = 0, \quad g(t,0) = 0, \;\; t \ge s \ge 0.$$

With these assumptions, Eq. (8) has the trivial solution $x(.) \equiv 0$.

By a similar as in the proof of Proposition 2.1 we can show that for any $x_0 \in X$ and $t_0 \ge 0$, Eq. (8) has a unique solution, namely $x(.,t_0,x_0) \equiv 0$, with the initial condition
$x(t_0,t_0,x_0) = x_0$ and this solution is defined on $t \ge t_0$.

In the following, we write simply $x(.)$ or $x(.,t_0)$ for $x(.,t_0,x_0)$ if there is no confusion. To proceed, we need the following lemma.

Lemma 3.1 (Pachpatte inequality see [10]). Let the functions $u(t), \sigma(t), v(t), \omega(t,r)$ be nonnegative and continuous for $a \le r \le t$, and let c_1 and c_2 be nonnegative. If for $t \in a, \infty)$,

$$u(t) \le c_1 + c_2 \int_a^t \left(v(s)u(s) + \int_a^s \omega(s,r)u(r)d\mu(r) \right) ds,$$

then for $t \ge a$,

$$u(t) \le c_1 \, exp\left\{ c_2 \int_a^t \left(v(s) + \int_a^s \omega(s,r)d\mu(r) \right) ds \right\}.$$

Firstly, we consider the boundedness of solutions {of Eq. (5) under small perturbations.

Theorem 3.2 Suppose that the solution of Eq. (5) is uniformly bounded. Then, there exists a constant M_1 such that the following estimate:

$$\|x(t)\| \le M_1 \|x(t_0)\|, \qquad t \ge t_0, \tag{9}$$

holds for solution x(.) of Eq. (8), provided

$$N = \int_{t_0}^{\infty} \left(l_t + \int_{t_0}^t k_{t,u} d\mu(u) \right) dt < \infty.$$

Proof by Eq. (7) we have

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)\left(g(\tau, x(\tau) + \int_{t_0}^{t} f(\tau,u,x(u)d\mu(u) \right) d\tau, \quad t \geq t_0. \tag{10}$$

By the assumptions, $\sup_{t\geq t_0}\|\Phi(t,t_0)\| = C_0 < \infty$ and $f(t,s,x)$, $g(t,x)$ are Lipschitz continuous in x with the Lipschitz coefficients $l_.$ and $k_{.,}$ respectively, we get

$$\|x(t)\| = C_0\|x_0\| + C_0 \int_{t_0}^{t} \left(\|x(\tau)\| + \int_{t_0}^{t} f(\tau,u,x(u)d\mu(u)\right) d\tau.$$

By using Pachpatte inequality in Lemma 3.1 with $c_1 = C_0\|x_0\|$ and $c_2 = C_0$ we have

$$\|x(t)\| = C_0\|x_0\| \, exp\left\{ C_0 \int_{t_0}^{t} \left(l_\tau + \int_{t_0}^{t} k_{\tau,u}\|x(u)\|d\mu(u)\right) d\tau \right\}.$$

Thus, we get the estimate $x(t) \leq C_0 e^{C_0 N}\|x_0\|$, for all $t \geq t_0$. The proof is complete.

Example 3.3 Consider the equation

$$x'(t) = -x(t) + \frac{1}{t^2}\int_0^t sx(s)d\mu(s), \quad t \geq 0, \tag{11}$$

where $\mu(.) = \sum_{n=1}^{\infty} \delta_n(.)$. Thus $x_0 > 0$

i) When $0 \leq t \leq 1$, we have $x(t) = x_0 e^{-t}$.

ii) When$1 < t < 2$, we have $x'(t) = -x(t) + \frac{1}{t^2}x(1) \leq -x(t) + x(1)$, therefore

$$x(t) \leq x(1) = x(0)e^{-1}.$$

iii) When $2 \leq t < 2$, we have

$$x'(t) = -x(t) + \frac{1}{t^2}(2x(2) + x(1)) \leq -x(s) + \frac{1}{2^2}(2x(2) + x(1)) \leq -x(s) + x(1),$$

which implies that $x(t) \leq x(1) = x(0)e^{-1}$.

iv) Continuing this way we have $x(t) \leq x(0)e^{-1}$, $\forall t \geq 0$. When x(0)<0 we can prove by a similar way that $x(0)e^{-1} \leq x(t)$, $\forall t \geq 0$. This means that the solution of (3.4) is bounded. Consider a perturbed equation

$$x'(t) = -x(t) + \frac{1}{t^2}\int_0^t sx(s)d\mu(s) + \int_0^t \frac{s}{1+t^4} \sin x(s)d\mu(s), \quad t \geq 0.$$

The function $f(t,s,x) = \frac{s \sin x(s)}{1+t^4}$ is Lipschitz continuous with the Lipschitz coefficient $k_{t,s} = \frac{s}{1+t^4}$. It is clear that

$$\int_0^t k_{t,s}d\mu(s) = \sum_{n=1}^{[t]} \frac{n}{1+t^4} \leq \frac{t(t+1)}{2(1+t^4)} \leq \frac{2}{1+t^4}, \quad and \quad N = \int_0^\infty \int_0^t k_{t,s}d\mu(s) \leq \int_0^t \frac{2}{1+t^2}dt = \pi.$$

Thus, from theorem 3.2, it follows that the solution of Eq. (8) is bounded by $e^{\pi e^{-1}}$.

Next, we prove that the exponential stability implies L_p stability under under small perturbations.

Definition 3.4 (See [2]) The trivial solution $x \equiv 0$ of Eq. (8) is said to be uniformly L_p-stable if there exist constants M_1, M_2 such that

$$\|x(t,t_0,x_0)\|_{\mathbb{R}^n} \leq M_1\|x_0\|_{\mathbb{R}^n}, t \geq t_0, \tag{12}$$

$$\|x(t,t_0,x_0)\|_{L_p(t_0,\infty)} \leq M_2\|x_0\|_{\mathbb{R}^n}. \tag{13}$$

Lemma 3.5 Let $1\leq p,q<\infty, \frac{1}{p}+\frac{1}{q}=1$ and U(x), V(x) are positive functions. There is a finite C for which

$$\left[\int_{t_0}^{\infty}\left|U(x)\int_{t_0}^{x}f(t)\Delta t\right|^p\Delta x\right]^{\frac{1}{p}}\leq C\left[\int_{t_0}^{\infty}|V(x)f(x)|^p\Delta x\right]^{\frac{1}{p}}, \tag{14}$$

is true for real f, where $B=\sup_{r>0}\left[\int_{r}^{\infty}|U(x)|^p\Delta x\right]^{\frac{1}{p}}\left[\int_{t_0}^{r}|V(x)|^{-q}\Delta x\right]^{\frac{1}{q}}$, (with the convention $0^{\infty}=\infty^{0}$ $=1$). Furthermore, if C is the least constant for which (14) holds, then $B\leq C\leq p^{\frac{1}{p}}q^{\frac{1}{q}}B$, for which $1<p<\infty$ and B=C if $p=1$ or ∞.

Remark 3.6 If we use $U(x)=V(x)=e^{\alpha x}$, then

$$B=\sup_{r\in T_{t_0}}\left[\int_{r}^{\infty}(e^{\alpha s})^p ds\right]^{\frac{1}{p}}\left[\int_{t_0}^{r}(e^{-\alpha s})^{-q}ds\right]^{\frac{1}{q}}\leq\frac{1}{\alpha p^{\frac{1}{p}}q^{\frac{1}{q}}}\ \text{ and }\ \frac{1}{\alpha p^{\frac{1}{p}}q^{\frac{1}{q}}}\leq C\leq\frac{1}{\alpha}.$$

where $\eta_\alpha=\frac{\alpha_1}{1+\alpha_2\mu^*}$ and $\mu^*=\max\mu(t)$, t ∈ **T**.

$$\left[\int_{t_0}^{\infty}\left|e_{\Theta\alpha}(t,t_0)\int_{t_0}^{t}f(s)\Delta s\right|^p\Delta t\right]^{\frac{1}{p}}\leq\frac{p^{\frac{1}{p}}q^{\frac{1}{q}}}{\eta_\alpha}\left[\int_{t_0}^{\infty}|e_{\Theta\alpha}(t,t_0)f(t)|^p\Delta x\right]^{\frac{1}{p}}.$$

Theorem 3.7 Assume that Eq. (5) is exponentially stable and

$$\sup_{t\geq t_0}l_t+\sup_{t\geq t_0}\left(\int_{t}^{\infty}k_{t,u}^p d\tau\right)^{\frac{1}{p}}=m<\frac{1}{MC}$$

with α, M to be defined in Definition 2.3, and $C=C(\alpha)$ is defined in Remark 3.6 corresponding to the function $U(x)=V(x)=e^{\alpha x}$. Then, the solution θ of the perturbed equation

$$\begin{cases}x'(t)=A(t)x(t)+\int_{t_0}^{t}H(t,s)x(s)d\mu(s)+\int_{t_0}^{t}f(t,s,x(s))ds+g(t,x(t)),\ \ t\geq t_0\\ x(t_0)=x_0\in X,\end{cases}$$

is uniformly L_p −stable.

Proof: By the variation of constants formula in Eq. (7), when $t\geq t_0$ one has

$$x(t)=\Phi(t,t_0)x_0+\int_{t_0}^{t}\Phi(t,\rho)\left(\int_{t_0}^{t}f(\rho,u,x(s)ds+g(\rho,x(\rho))\right)d\rho.$$

$$\|x(t)\|\ \leq\ Me^{-\omega(t-t_0)}\|x_0\|+M\int_{t_0}^{t}e^{-\omega(t-\rho)}\left(\int_{t_0}^{t}k_{\rho,s}\|x(s)\|ds+l_\rho\|x(\rho)\|\right)d\rho.$$

By using Hardy inequality first and then using Minkowski inequality we get

$$\int_{t_0}^{T}\left(\int_{t_0}^{t}e^{-\omega(t-\rho)}\left(\int_{t_0}^{\rho}k_{\rho,s}\|x(s)\|ds\right)d\rho\right)^{\frac{1}{p}}\ \leq\ C\int_{t_0}^{T}\left(\left(\int_{t_0}^{\rho}k_{\rho,s}\|x(s)\|ds\right)^p d\rho\right)^{\frac{1}{p}}$$

$$\leq C\left(\int_{t_0}^{T}\|x(s)\|^p\int_{t_0}^{T}\left(k_{\rho,s}\right)^p d\rho ds\right)^{\frac{1}{p}}\ \leq\ C\sup_{t\geq t_0}\left(\int_{t}^{\infty}\left(k_{\tau,t}\right)^p d\tau\right)^{\frac{1}{p}}\left(\int_{t_0}^{T}\|x(s)\|^p ds\right)^{\frac{1}{p}}.$$

Thus, from Eq. (7) we get

$$\|x(t)\|_{L_p[t_0,T]} \leq \frac{M\|x_0\|}{(p\omega)^{1/p}} + M\left[\int_{t_0}^{T}\left(\int_{t_0}^{t} e^{-\omega(t-\tau)}\left(\int_{t_0}^{\tau} k_{\tau,s}\|x(s)\|ds\right)d\tau\right)^{p} dt\right]^{\frac{1}{p}}$$

$$+ M\left[\int_{t_0}^{T}\left(\int_{t_0}^{t} e^{-\omega(t-\tau)}\, l_\tau\|x(\tau)\|d\tau\right)^{p} dt\right]^{\frac{1}{p}}$$

$$\leq \frac{M\|x_0\|}{(p\omega)^{1/p}} + MC\sup_{t\geq t_0}\left(\int_{t}^{\infty} k_{\tau,t}\, d\tau\right)^{\frac{1}{p}} \|x(.)\|_{L_p[t_0,T]} + \sup_{t\geq t_0} l_t\|x(.)\|_{L_p[t_0,T]}$$

$$\leq \frac{M\|x_0\|}{(p\omega)^{1/p}} + MCm\|x(.)\|_{L_p[t_0,T]},$$

where $m = \sup_{t\geq t_0} l_t + \sup_{t\geq t_0}\left(\int_t^\infty k_{\tau,t}\, d\tau\right)^{\frac{1}{p}}$. Hence,

$$\|x(t)\|_{L_p[t_0,T]} \leq \frac{M\|x_0\|}{(1-MCm)(p\omega)^{1/p}}.$$

Letting $T \to \infty$ obtains $\|x(t)\|_{L_p[t_0,\infty]} \leq \frac{M\|x_0\|}{(1-MCm)(p\omega)^{1/p}}$. Thus we get (13).

We pass to prove (13). Using Holder inequality and Minkowski inequality we see that

$$\int_{t_0}^{t} e^{-\omega(t-\tau)}\left(\int_{t_0}^{\tau} k_{\tau,s}\|x(s)\|ds\right)d\tau \leq e^{-\omega t}\left(\int_{t_0}^{t} e^{q\omega\tau}\, d\tau\right)^{\frac{1}{q}}\left(\int_{t_0}^{t}\left(\int_{t_0}^{\tau} k_{\tau,s}\,\|x(s)\|ds\right)^{p} d\tau\right)^{\frac{1}{p}}$$

$$\leq \frac{1}{(q\omega)^{\frac{1}{q}}}\left(\int_{t_0}^{t}\|x(s)\|^p\left(\int_{s}^{t} k_{\tau,s}\, d\tau\right)ds\right)^{\frac{1}{p}}$$

$$\leq \frac{1}{(q\omega)^{\frac{1}{q}}}\sup_{t\geq t_0}\left(\int_{t}^{\infty} k_{\tau,t}\, d\tau\right)^{\frac{1}{p}} \|x(.)\|_{L_p[t_0,t]}$$

$$\leq \frac{M_2}{(q\omega)^{\frac{1}{q}}}\sup_{t\geq t_0}\left(\int_t^\infty k_{\tau,t}\, d\tau\right)^{\frac{1}{p}}\|x(t_0)\|$$

Similarly,

$$\int_{t_0}^{t} e^{-\omega(t-\tau)}\, l_\tau\|x(\tau)\|d\tau \leq \frac{1}{(q\omega)^{\frac{1}{q}}}\sup_{t\geq t_0} l_t\|x(.)\|_{L_p[t_0,t]} \leq \frac{M_2\|x(t_0)\|}{(q\omega)^{\frac{1}{q}}}\sup_{t\geq t_0} l_t.$$

Therefore,

$$\boldsymbol{\|x(t)\| \leq e^{-\omega(t-t_0)}M\|x(t_0)\| + M\int_{t_0}^{t} e^{-\omega(t-\tau)}\left(\int_{t_0}^{\tau} k_{\tau,s}\|x(s)\|ds + l_\tau\|x(\tau)\|\right)d\tau}$$

$$\boldsymbol{\leq M_1\|x(t_0)\|}.$$

where $M_1 = M + MM_2(q\omega)^{\frac{-1}{q}}\left(\sup_{t\geq t_0}\left(\int_t^\infty k_{\tau,t}\, d\tau\right)^{\frac{1}{p}} + \sup_{t\geq t_0} l_t\right)$. We have the proof.

4. Bohl-Perron Theorem

This section continues to study the Bohl-Perron Theorem by considering the exponent stability to Eq. (2) via properties of mapping between weighted spaces $L^\gamma(t_0)$ and $C^\gamma(t_0)$ defined below. We construct an operator N and show that the exponential stability of Eq. (5) is equivalent the fact that the operator N is surjective.

Let $\gamma \geq 0$. Define two families of Banach spaces $L^\gamma(t_0)$ and $C^\gamma(t_0)$ as

$$L^\gamma(t_0) = \left\{ f\colon [t_0, \infty) \to X,\ \ f\ \ is\ measurable\ and\ \ \int_{t_0}^{\infty} e^{\gamma t}\|f(t)\|dt < \infty \right\},$$

$$C^\gamma(t_0) = \left\{ x\colon [t_0, \infty) \to X,\ \ x\ is\ continous\ x(t_0) = 0\ and\ \ \sup_{t\geq t_0} e^{\gamma t}\|x(t)\| < \infty \right\},$$

with the norms defined as follows

$$\|f\|_{L^\gamma(t_0)} = \int_{t_0}^{\infty} e^{\gamma t}\|f(t)\|dt \qquad \text{and} \qquad \|x\|_{C^\gamma(t_0)} = \sup_{t\geq t_0} e^{\gamma t}\|x(t)\|.$$

When$\gamma = 0$, the space $L^0(t_0)$ is $L_1([t_0, \infty), X)$ consisting all integrable functions and $C^0(t_0) = C_b([t_0, \infty), X)$ is the set of all bounded continuous.

To simplify notations, we write $L^\gamma(t_0)$, $C^\gamma(t_0)$ by $L^\gamma$and C^γ if there is no confusion. For any $f \in L^\gamma$ we consider equation

$$x'(t) = A(t)x(t) + \int_{t_0}^{t} H(t,s)x(s)\,d\mu(s) + f(t), \quad t \geq t_0, \tag{15}$$

with initial condition$x(t_0) = 0$. As is mentioned in the function $x(t), t \geq t_0$ is a solution of (15) if and only if

$$x(t) = \int_{t_0}^{t} \left(A(\tau)x(\tau) + \int_{t_0}^{\tau} H(\tau,s)x(s)d\mu(s) + f(\tau) \right) d\tau,\ \ t \geq t_0. \tag{16}$$

By using the constant variation formula (7), the solution x(t) of (15) is expressed as

$$x(t) = \int_{t_0}^{t} \Phi(t,s)f(s)\ ds,\ \ t \geq t_0. \tag{17}$$

Define $N_s f(t) = \int_{t_0}^{t} \Phi(t,s)f(s)\,ds, t \geq t_0,\ f \in L^\gamma(s)$. We write simply N for N_{t_0}.

Theorem 4.1 For any $\gamma \geq 0$, if N maps $L^\gamma$to C^γ, then there exists a positive constant K such that for all $s \geq t_0$,

$$\|N_s\| \leq K.$$

Proof. Consider the case $s = t_0$. For every $t > t_0$, we define an operator $F_t\colon L^\gamma \to X$ by

$$F_t(f(.)) = e^{\gamma t}\int_{t_0}^{t} \Phi(t,s)f(s)\,ds = e^{\gamma t}Nf(t).$$

By the assumption of Theorem, the operator N maps L^γ to C^γ. Therefore,

$$\sup_{t\geq s}\|F_t(f)\| = \sup_{t\geq s} e^{\gamma t}\|Lf(t)\| < \infty,\quad f \in L^\gamma.$$

By using the Uniform Boundedness principle, we have $\sup_{t\geq t_0}\|F_t\| = K < \infty$. It is known that,

$$\|N\| = \sup_{f\in L^\gamma}\frac{\|Nf\|_{C^\gamma}}{\|f\|} = \sup_{f\in L^\gamma}\frac{\sup_{t\geq 0}\|F_t f\|}{\|f\|} = \sup_{t\geq t_0}\|F_t\| = K.$$

This means that we have the proof with $s = t_0$.

We pass to the case with arbitrary $s > t_0$. Let f(t) be a function in $L^\gamma(s)$. We extend the function f to a function $\bar{f}$ defined on $[t_0, \infty)$ as follows:

$$\bar{f} = \begin{cases} 0, & t_0 \le t \le s \\ f(t), & t > s. \end{cases}$$

It is seen that

$$N\bar{f}(t) = \int_0^t \Phi(t,\tau)\bar{f}(\tau)\,d\tau = N_s f(t).$$

Therefore, from (4.6) we get

$$\|N_s f\|_{C^\gamma} = \sup_{t \ge s} e^{\gamma t}\|N_s f(t)\| = \sup_{t \ge t_0} e^{\gamma t}\left\|L\bar{f}(t)\right\| = \left\|L\bar{f}(t)\right\|_{C^\gamma} \le K\left\|\bar{f}\right\|_{L^\gamma} = K\|f\|_{L^\gamma(s)}.$$

The proof is complete.

Theorem 4.2 Let $\gamma > 0$ be a positive number. The operator N maps L^γ to C^γ if and only if Eq. (5) is γ -exponentially stable.

Proof. First, we prove the necessary condition. Suppose that N maps L^γ to C^γ. We show that then (5) is γ -exponentially stable.

From Theorem 4.1, we see that N is a bounded operator from L^γ to C^γ with $\|N\| = K$. This means that if $f \in L^\gamma(s)$ and $0 \le s \le t$, then

$$e^{\gamma t}\left\|\int_s^t \Phi(t,u)f(u)\ du\right\| = \|Nf\|_{C^\gamma(s)} \le K\|f\|_{L^\gamma(s)}. \quad (18)$$

Let $\gamma > 0$ and $x \in X$, define the function

$$f_\sigma(u) = \begin{cases} \dfrac{1}{\rho} e^{-\frac{u-s}{\sigma} - \gamma u}\, v, \ u \ge s, \\ f(t), \ \ t_0 \le t \le s. \end{cases}$$

By a simple calculation we have

$$\int_{t_0}^{\infty} e^{\gamma u}\|f_\sigma(u)\|\ du = \frac{1}{\sigma}\int_{t_0}^{\infty} e^{-\frac{u-s}{\sigma}}\|v\|\ du = \|v\|.$$

i.e., $f_\sigma \in L^\gamma$ and $\|f_\sigma\|_{L^\gamma} = \|v\|$. Moreover,

$$\lim_{\sigma\to 0}\int_s^t \Phi(t,u)\, f_\sigma(u)du = \lim_{\sigma\to 0}\int_s^t \Phi(t,u)\frac{1}{\sigma}e^{-\frac{u-s}{\sigma} - \gamma u} v\, du$$
$$= \lim_{\sigma\to 0}\int_0^{\frac{t-s}{\sigma}} \Phi(t, s+\sigma\square)e^{-\square - \gamma(s+\sigma\square)} v\, d\square = e^{-\gamma s}\Phi(t,s)v.$$

Hence,

$$e^{\gamma(t-s)}\|\Phi(t,s)v\| = e^{\gamma(t-s)}\lim_{\sigma\to 0}\left\|\int_s^t \Phi(t,u)\, f_\sigma(u)du\right\| \le Ke^{-\gamma s}\|f_\sigma\|_{L^\gamma(s)} = e^{-\gamma s}K \le K.$$

Thus,

$$\|\Phi(t,s)\| \le Ke^{-\gamma(t-s)}, \quad t \ge s \ge t_0.$$

This means that Eq. (5) is uniformly asymptotically stable. We will prove the inverse relation. For any $f \in L^\gamma$, by (18) it yields

$$e^{\gamma t}\|Nf(t)\| \le e^{\gamma t}\int_{t_0}^t \|\Phi(t,u)\|\|f(u)\|\,du \le Me^{\gamma t}\int_{t_0}^t e^{-\gamma(t-u)}\|f(u)\|\,du$$

$$\leq M \int_{t_0}^{t} e^{\gamma u} \|f(u)\| \, du \leq M \|f\|_{L^\gamma} < \infty.$$

Thus, $Lf \in C^\gamma$The proof is complete.

Remark 4.3 The argument dealt with in the proof of theorem 4.2 is still valid for $\gamma = 0$. Thus, if L maps L_1 to C_b then the solution of (5) with the initial condition $x(0) = 0$ is bounded.

Corollary 4.4 The equation (5) is γ -exponentially stable if and only if the solution of

$$y'(t) = A(t)y(t) + \gamma y(t) + \int_{t_0}^{t} H(t,\tau) e^{\gamma(t-\tau)} y(\tau) \, d\mu(\tau) + f(t), \quad t \geq t_0, \tag{19}$$

is bounded for all $f \in L^\gamma$.

Proof. Denote by $\Psi(t,s)$ the Cauchy operator of the homogeneous equation corresponding to (19), i.e., $\Psi(s,s) = I$ and

$$\Psi'(t,s) = A(t)\Psi(t,s) + \gamma\, \Psi(t,s) + \int_{t_0}^{t} H(t,\tau) e^{\gamma(t-\tau)} \Psi(\tau,s) \, d\mu(\tau).$$

From (6) we get

$$\frac{d}{dt}(e^{\gamma t}\Psi(t,s)) = A(t) e^{\gamma t}\Psi(t,s) + \gamma\, e^{\gamma t}\Psi(t,s) + \int_{t_0}^{t} H(t,\tau) e^{\gamma(t-\tau)} e^{\gamma\tau} \Psi(\tau,s) \, d\mu(\tau).$$

The uniqueness of solutions says that $\Psi(t,s) = e^{\gamma t}\Phi(t,s)$.

Hence, the γ -exponential stability of (5) implies that the solution of (19) is bounded. Let *y(t)* be the solution of (19) with the initial condition $y(0) = 0$. By (18), this solution can be expressed as

$$y(t) = \int_{0}^{t} \Psi(t,\tau) f(\tau) \, d\tau = e^{\gamma t} \int_{0}^{t} \Psi(t,\tau) f(\tau) \, d\tau = e^{\gamma t} N f(t).$$

The boundedness of *y(t)* says that N maps $L^\gamma$to C^γ. Therefore, by theorem 4.2, Eq. (5) is exponentially stable. The proof is complete.

Acknowledgments

The author would like to thank the referees for giving precious comments and suggestions. This work was done under the partial support of the Science and Technology Foundation of Electric Power University under grant number ĐTKHCN.01/2022.

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