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Original Article

Strong Laws of Large Numbers for Weighted Sums of Hilbert-valued Coordinatewise PNQD Random Variables with Applications

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Abstract: The aim of this work is to investigate results on almost sure convergence of weighted sums of coordinatewise pairwise negatively quadrant dependent random variables taking values in Hilbert spaces. As an application, the almost sure convergence of degenerate von Mises-statistics is investigated.

Keywords: Negative quadrant dependence, Hilbert spaces, Weighted sums, Strong laws of large numbers.

1. Introduction

Let us consider a sequence $\{X_n, n \geq 1\}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. The concept of negatively quadrant dependent (NQD) random variables was introduced by Lehmann [1]. In particular, random variables X and Y are called NQD if

$$P(X \leq x, Y \leq y) \leq P(X \leq x)P(Y \leq y) \tag{1}$$

for all real numbers x, y. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be pairwise negatively quadrant dependent (PNQD) if every pair of random variables in the sequence satisfies (1). Obviously, a sequence of PNQD random variables contains a pairwise independent random variable sequence as special cases. It is important to note that (1) implies

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$$P(X > x, Y > y) \le P(X > x)P(Y > y) \tag{2}$$

for all real numbers x, y. Moreover, it follows that (2) implies (1), and hence, they are equivalent for PNQD. Ebrahimi and Ghosh [2] showed that (1) and (2) are not equivalent for $n \ge 3$.

We recall that the concept of coordinatewise PNQD random variables with values in Hilbert spaces. Let $\mathbb{H}$ be a real separable Hilbert space with the norm $\|\,.\,\|$ generated by an inner product $\langle \cdot, \cdot \rangle$ and let $\{e_j, j \in B\}$ be an orthonormal basis in $\mathbb{H}$. Let X be an $\mathbb{H}$-valued random variable, then $\langle X, e_j \rangle$ will be denoted by X^j.

Definition 1.1 *([3])* A sequence $\{X_n, n \ge 1\}$ of $\mathbb{H}$-valued random vectors is said to be coordinatewise PNQD if for each $j \in B$, the sequence of random variables $\{X_n^j, n \ge 1\}$ is PNQD.

There are no PNQD requirements between two different coordinates of each random variable in the notions of coordinatewise PNQD random variables taking value in Hilbert spaces (for more details see [4]). Obviously, if a sequence of $\mathbb{H}$-valued random variables is pairwise independent then it is coordinatewise PNQD. In [3], Example 2.3, the authors showed a sequence of $\mathbb{H}$-valued coordinatewise PNQD random variable which is not pairwise independent.

Recently, the results for weak laws of large numbers for weighted coordinatewise PNQD random vectors in Hilbert spaces in the case that the decay r-th order $0 < r < 2$ of tail probability was established by Dung and Son in [3]. In 2022, Son and Cuong [5] investigated the complete convergence and strong laws of large numbers for weighted sums of coordinatewise PNQD random variables taking values in Hilbert spaces. Hence, it is very significant to study limit properties of NQD random variables in probability theory. The main purpose of this paper is to establish the strong laws of large numbers for sequences of coordinatewise PNQD in Hilbert spaces with statistical applications.

The rest of this work is organized as follows. In Section 2, we provide some useful lemmas and several definitions that support our proofs. In Section 3, we discuss about the strong limit results for the weighted sums of coordinatewise PNQD sequence random variables. An application of the gereral Cramer-Von Mises statistics for coordinatewise PNQD random vectors in Hilbert spaces is given in Section 4.

Throughout this paper, by saying $\{X_n, n \ge 1\}$ is a sequence of $\mathbb{H}$-valued coordinatewise PNQD random variables, we mean that the $\mathbb{H}$-valued random variables are coordinatewise PNQD with respect to the orthonormal basis $\{e_j, j \in B\}$. The symbol C denotes a generic positive constant whose value may be different for each appearance.

Let $\{a_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$ be sequences of positive real numbers. We use notion $a_n \asymp b_n$ instead of $0 \le \liminf \frac{a_n}{b_n} \le \limsup \frac{a_n}{b_n} < \infty$; $a_n = o(b_n)$ means that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $a_n = O(b_n)$ means that $a_n \le Cb_n$ (for some $0 < C < \infty$) respectively. The indicator of A is denoted by $I(A)$.

To prove our main results, we need the following lemmas.

Lemma 1.1 ([1]) Let X and Y be ¡ -valued NQD random variables. If f and g are Borel fuctions, both of which are monotone increasing (or both are monotone decreasing), then $f(X)$ and $g(X)$ are NQD.

Lemma 1.2 ([3, 4]) Let $\{X_n, n \ge 1\}$ be a sequence of $\mathbb{H}$-valued coordinatewise PNQD random variables with mean 0 and finite second moments. Then,

$$E\left\|\sum_{i=1}^{n} X_i\right\|^2 \le \sum_{i=1}^{n} E\|X_i\|^2,$$

and

$$E\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k} X_i\right\|^2\right) \le \log_2^2(2n)\sum_{i=1}^{n} E\|X_i\|^2.$$

Lemma 1.3 *([5])* Let $\{X_n, n \ge 1\}$ be a sequence of $\mathbb{H}$-valued coordinatewise PNQD random variables. Then, the series $\sum_{k=1}^{n} X_k$ converges a.s. if for some $c > 0$ the following three series are convergent:

i) $\sum_{n=1}^{\infty}\sum_{j\in B} P\left(\left|X_n^j\right| > c\right)$,

ii) $\sum_{n=1}^{\infty}\sum_{j\in B} EX_n^j(c)e_j$,

iii) $\sum_{n=1}^{\infty}\sum_{j\in B} VarX_n^j(c)\log_2^2 n$,

where $X_n^j = \langle X_n, e_j\rangle$, $X_n^j(c) = -cI(X_n^j < -c) + X_n^j I(|X_n^j| \le c) + cI(X_n^j > c)$ for $n \ge 1, j \in B$.

Definition 1.1 *([6])* A positive measurable function f on $[a,\infty)$ (for some $a \ge 0$) is is said to be regularly varying at infinity with index r $(r \in \mathbb{R})$, denoted by $f \in \mathcal{RV}_r$, if

$$\lim_{t\to\infty}\frac{f(tx)}{f(t)} = x^r \quad \text{for all } x > 0.$$

A regularly varying function with index zero ($r = 0$) is said to be vary slowly function. Without loss of generality, we may assume that $a = 0$. It is well known that a function f is regularly varying at infinity with index r if only if it can be written in the form $f(x) = x^r l(x)$, where $l(.)$ is slowly varying function.

It proves convenient to define $\log x = \max\{1, \log_2 x\}$ for $x > 0$.

Clearly, $x^r, x^r\log x, x^r\log\log x, x^r\dfrac{\log x}{\log\log x}$ are regularly varying functions at infinity with index r.

Lemma 1.4 *([5])* (Karamata's Theorem, see [7])

Let $f \in \mathcal{RV}_r$ be locally bounded on $[a,\infty)$ with $a > 0$. Then,

i) For $\sigma \ge -(r+1)$, $\dfrac{x^{\sigma+1}f(x)}{\int_a^x t^\sigma f(t)dt} \to \sigma + r + 1 \quad \text{as } x \to \infty,$

ii) For $\sigma < -(r+1)$, $\dfrac{x^{\sigma+1} f(x)}{\int_x^\infty t^\sigma f(t)dt} \to -(\sigma + r + 1)$ as $x \to \infty$.

2. The Main Results

In the next theorem, we study the strong laws of large numbers for a weighted sum of $\mathbb{H}$-valued coordinatewise PNQD random vectors with α-th order decay of tail probability.

Theorem 2.1 Let $\{X_n, n \ge 1\}$ be a sequence of $\mathbb{H}$-valued coordinatewise PNQD random vectors with zero mean, such that for each $n \ge 1$,

$$\sum_{j\in B} P\left(\left|X_n^j\right| > x\right) \asymp \mathrm{l}(x)x^{-\alpha} \quad \textit{for each } \alpha \in (1,2),$$

where l is a slowly varying function. Let $\{a_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$ be sequences of positive constants satisfying $0 < b_n \uparrow \infty$. Denote $c_n = \dfrac{b_n}{a_n \log n}$ for $n \ge 1$. Assume that

$$\sum_{n=1}^\infty \mathrm{l}(c_n)c_n^{-\alpha} < \infty. \tag{3}$$

Then
$$\frac{1}{b_n}\sum_{k=1}^n a_k X_k \to 0 \quad a.s. \ as \ n \to \infty.$$

Proof. For $n \ge 1, 1 \le k \le n, \ j \in B$, set

$$Y_k = \sum_{j\in B} Y_k^j e_j, \quad Y_k^j = -c_k I\left(X_k^j < -c_k\right) + X_k^j I\left(\left|X_k^j\right| \le c_k\right) + c_k I\left(X_k^j > c_k\right).$$

By the Karamata theorem and the assumption (3), we obtain

$$\begin{aligned}
\sum_{k=1}^\infty \sum_{j\in B} \log^2 k Var\left(\frac{a_k Y_k^j}{b_k}\right) &\le \sum_{k=1}^\infty \sum_{j\in B} c_k^{-2} E(Y_k^j)^2 \\
&\le \sum_{k=1}^\infty \sum_{j\in B} P\left(\left|X_k^j\right| > c_k\right) + \sum_{k=1}^\infty \sum_{j\in B} c_k^{-2} E(X_k^j)^2 I(\left|X_k^j\right| \le c_k) \\
&= \sum_{k=1}^\infty \sum_{j\in B} P\left(\left|X_k^j\right| > c_k\right) + \sum_{k=1}^\infty c_k^{-2} \int_0^{c_k} t \sum_{j\in B} P\left(\left|X_k^j\right| > t\right) dt \\
&\asymp \sum_{k=1}^\infty \mathrm{l}(c_k)c_k^{-\alpha} + \sum_{k=1}^\infty c_k^{-2} \int_0^{c_k} \mathrm{l}(t)t^{1-\alpha} dt \\
&\le C\sum_{k=1}^\infty \mathrm{l}(c_k)c_k^{-\alpha} + C\sum_{k=1}^\infty c_k^{-2}\mathrm{l}(c_k)c_k^{2-\alpha} \\
&\le C\sum_{k=1}^\infty \mathrm{l}(c_k)c_k^{-\alpha} < \infty.
\end{aligned}$$

Using **Lemma 1.3** and the Kronecker lemma, we have $\sum_{k=1}^\infty \dfrac{a_k\left(Y_k - EY_k\right)}{b_k}$ converges, which implies that

$$\sum_{k=1}^{n} a_k \left(Y_k - EY_k\right) = o\left(b_n\right) \quad a.s. \tag{4}$$

Noting that $EX_n^j = 0$, we also have by the Karamata theorem and the assumption (3) again that

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{a_k \log k}{b_k} \|EY_k\| &= \sum_{k=1}^{\infty} c_k^{-1} \left\| \sum_{j\in B} E(Y_k^j e_j) \right\| \\
&\le \sum_{k=1}^{\infty} c_k^{-1} \left\| \sum_{j\in B} EX_k^j I\left(\left|X_k^j\right| \le c_k\right) e_j \right\| + \sum_{k=1}^{\infty} \sum_{j\in B} P\left(\left|X_k^j\right| > c_k\right) \\
&\le \sum_{k=1}^{\infty} c_k^{-1} \sum_{j\in B} E\left|X_k^j\right| I\left(\left|X_k^j\right| > c_k\right) + \sum_{k=1}^{\infty} \sum_{j\in B} P\left(\left|X_k^j\right| > c_k\right) \\
&\le \sum_{k=1}^{\infty} c_k^{-1} \int_{c_k}^{\infty} \sum_{j\in B} P\left(\left|X_k^j\right| > t\right) dt + \sum_{k=1}^{\infty} \sum_{j\in B} P\left(\left|X_k^j\right| > c_k\right) \\
&\asymp \sum_{k=1}^{\infty} c_k^{-1} \int_{c_k}^{\infty} \mathrm{l}(t) t^{-\alpha} dt + \sum_{k=1}^{\infty} \mathrm{l}(c_k) c_k^{-\alpha} \\
&\le C \sum_{k=1}^{\infty} c_k^{-1} \mathrm{l}\left(c_k\right) c_k^{-\alpha+1} + C \sum_{k=1}^{\infty} \mathrm{l}(c_k) c_k^{-\alpha} \\
&\le C \sum_{k=1}^{\infty} \mathrm{l}(c_k) c_k^{-\alpha} < \infty,
\end{aligned}$$

which implies that $\sum_{k=1}^{\infty} \frac{a_k}{b_k} EY_k$ converges a.s.

By the assumption of $\{b_n, n \ge 1\}$, applying the Kronecker lemma, it follows that

$$\sum_{k=1}^{n} a_k EY_k = o\left(b_n\right). \tag{5}$$

Combining (4) and (5), we obtain $\sum_{k=1}^{n} a_k Y_k = o(b_n)$ a.s. as $n \to \infty$.

To complete the proof of **Theorem 2.1**, we only need to show that $P\left(X_n \ne Y_n, i.o.\right) = 0$.

By the assumption (3) again

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left(X_n \ne Y_n\right) &\le \sum_{n=1}^{\infty} \sum_{j\in B} P\left(\left|X_n^j - Y_n^j\right|^2 > 0\right) = \sum_{n=1}^{\infty} \sum_{j\in B} P\left(\left|X_n^j\right| > c_n\right) \\
&\asymp \sum_{n=1}^{\infty} \mathrm{l}(c_n) c_n^{-\alpha} < \infty.
\end{aligned}$$

It follows from the Borel - Cantelli lemma that $P\left(X_n \ne Y_n,\ i.o.\right) = 0.$ The proof is complete.

One version of **Theorem 2.1** for a sequence of $\mathbb{H}$-valued identically distributed coordinatewise NSD random vectors is Theorem 3.4 in [10].

Example 2.1 Suppose that $\{X_n, n \geq 1\}$ is a sequence of random vectors $\mathbb{H}$-valued PNQD whose distributions are defined by $P\left(X_n^j = 0\right) = 1 - \frac{\alpha_j}{q_n^r}$ and the tail probability

$$P\left(X_n^j > x\right) = \frac{\alpha_j}{\left(x + q_n\right)^r}, \text{ for } x > 0, j \in B, \tag{6}$$

where $r > 1, \alpha_j \geq 0, \sum_{j \in B} \alpha_j < \infty$ and q_n is a sequence of positive numbers such that

$$M = \max\left\{\alpha_j\right\} \leq q_n^r, \quad \text{for all } n.$$

By (6), we obtain $EX_n^j = \int_0^\infty P\left(X_n^j > x\right) dx = \int_0^\infty \frac{\alpha_j}{\left(x + q_n\right)^r} dx = \frac{\alpha_j}{\left(r-1\right) q_n^{r-1}} \leq \frac{M^{1/r}}{r-1} < \infty,$

and
$$EX_n = E\left(\sum_{j \in B} X_n^j\right) = \frac{\alpha}{\left(r-1\right) q_n^{r-1}}, \quad \text{where} \quad \alpha = \sum_{j \in B} \alpha_j.$$

It is easy to see that $\{X_n - EX_n, n \geq 1\}$ is a sequence of $\mathbb{H}$-valued coordinatewise PNQD with mean 0 and for $n \geq 1$, $\sum_{j \in B} P\left(\left|X_n^j - EX_n^j\right| > x\right) \asymp \sum_{j \in B} P\left(\left|X_n^j\right| > x\right) \asymp x^{-r}$.

Then
$$\sum_{n=1}^\infty P\left(\left|X_n - EX_n\right| > x\right) \leq \sum_{n=1}^\infty \sum_{j \in B} P\left(\left|X_n^j - EX_n^j\right| > x\right) \asymp \sum_{n=1}^\infty \sum_{j \in B} P\left(\left|X_n^j\right| > x\right).$$

From **Theorem 2.1**, for $n \geq 1$, let $a_n = 1$, $b_n = n^{1/r} \log^\alpha n$ and $c_n = \frac{n^{1/r}}{\log^{1-\alpha} n}$ for all $\alpha > 1 + \frac{1}{r}$, $l(x) \equiv 1 \ \forall x$, we get

$$\sum_{n=1}^\infty P\left(\left|X_n - EX_n\right| > c_n\right) \asymp \sum_{n=1}^\infty (c_n)^{-r} = \sum_{n=1}^\infty \frac{1}{n \log^{r(\alpha-1)} n} < \infty.$$

Applying **Theorem 2.1** for a sequence random variable PNQD $\{X_n - EX_n, n \geq 1\}$ we have

$$\frac{1}{n^{1/r} \log^\alpha n} \sum_{k=1}^n \left(X_k - EX_k\right) \to 0 \quad \text{a.s} \quad \text{as } n \to \infty.$$

The following theorem is obtained by using an extension of Theorem 3.5 in [11] for a sequence of $\mathbb{H}$-valued coordinatewise PNQD random vectors.

Let $\{c_n, n \geq 1\}$ be a sequence of positive numbers. For each $n \geq 1$, define $N(n)$ by

$$N\left(n\right) = Card\left\{i : c_i \leq n\right\},$$

where $Card(A)$ denotes the number elements of a set A. Note that, $N(0) = 0$ and N is non-decreasing integer valued function with $\lim_{n \to \infty} N(n) = \infty$.

Theorem 2.2 Let $\{X_n, n \geq 1\}$ be a sequence of $\mathbb{H}$-valued coordinatewise PNQD random vectors with identical distributions and $\{a_n, n \geq 1\}$ be a sequence of positive numbers with $0 < A_n = \sum_{j=1}^{n} a_j \uparrow \infty$. Assume that

$$EX_1 = 0, \quad \sum_{j \in B} E\left|X_1^j\right|^r < \infty \quad \text{for } 1 < r < 2. \tag{7}$$

Denote $c_n = \dfrac{A_n}{a_n \log n}$ for $n \geq 1$. Assume that

$$N(n) = O(n^r), \quad n \geq 1. \tag{8}$$

Then

$$\frac{1}{A_n}\sum_{i=1}^{n} a_i X_i \to 0 \quad a.s. \quad as \quad n \to \infty. \tag{9}$$

Proof. It is important to note that the definition of $N(n)$ implies $c_n \to \infty$ as $n \to \infty$. Otherwise, there exist infinite subscripts i and some n_0 such that $c_i \leq n_0^r$, then $N(n_0) = \infty$, which is contrary to $N(n_0) \leq Cn_0^r$ from (8).

For each $n \geq 1, j \in B$, set

$$Y_n = \sum_{j \in B} Y_n^j e_j, \quad Y_n^j = -c_n I\left(X_n^j < -c_n\right) + X_n^j I\left(\left|X_n^j\right| \leq c_n\right) + c_n I\left(X_n^j > c_n\right).$$

Combining (7) and (8) yields

$$\begin{aligned}
\sum_{n=1}^{\infty} P(X_n \neq Y_n) &\leq \sum_{n=1}^{\infty}\sum_{j \in B} P\left(\left|X_n^j - Y_n^j\right|^2 > 0\right) = \sum_{n=1}^{\infty}\sum_{j \in B} P\left(\left|X_n^j\right| > c_n\right) \\
&= \sum_{n=1}^{\infty}\sum_{j \in B}\sum_{c_n \leq k < c_n + 1} P\left(\left|X_n^j\right| > c_n\right) \\
&\leq \sum_{k=1}^{\infty}\sum_{j \in B}\sum_{k-1 < c_n \leq k} P\left(\left|X_n^j\right| > k-1\right) \\
&= \sum_{j \in B}\sum_{k=1}^{\infty}\left(N(k) - N(k-1)\right) P\left(\left|X_1^j\right| > k-1\right) \\
&= \sum_{j \in B}\sum_{k=1}^{\infty}\left(N(k) - N(k-1)\right)\sum_{l=k}^{\infty} P\left(l-1 < \left|X_1^j\right| \leq l\right) \\
&= \sum_{j \in B}\sum_{l=1}^{\infty} P\left(l-1 < \left|X_1^j\right| \leq l\right)\sum_{k=1}^{l}\left(N(k) - N(k-1)\right) \\
&= \sum_{j \in B}\sum_{l=1}^{\infty} N(l) P\left(l-1 < \left|X_1^j\right| \leq l\right) \\
&\leq C\sum_{j \in B}\sum_{l=1}^{\infty} l^r P\left(l-1 < \left|X_1^j\right| \leq l\right) \\
&\leq C\sum_{j \in B} E\left|X_1^j\right|^r < \infty,
\end{aligned}$$

which implies that

$$\sum_{n=1}^{\infty} P\left(X_n \neq Y_n\right) \leq \sum_{n=1}^{\infty} \sum_{j\in B} P\left(\left|X_n^j\right| > c_n\right) < \infty. \tag{10}$$

It follows from the Borel-Cantelli lemma that

$$P\left(X_n \neq Y_n,\ i.o.\right) = 0. \tag{11}$$

Next, to complete the proof of **Theorem 2.2**, we only need to show that

$$\frac{1}{A_n}\sum_{i=1}^{n} a_i Y_i \to 0 \quad \text{a.s.} \quad \text{as } n \to \infty. \tag{12}$$

Combining (7), (8) and (10) we have

$$\begin{aligned}
\sum_{i=1}^{\infty}\sum_{j\in B} \log^2 i Var\left(\frac{a_i Y_i^j}{A_i}\right) &\leq \sum_{i=1}^{\infty}\sum_{j\in B} c_i^{-2} E(Y_i^j)^2 \\
&\leq \sum_{i=1}^{\infty}\sum_{j\in B} P\left(\left|X_i^j\right| > c_i\right) + \sum_{i=1}^{\infty}\sum_{j\in B} c_i^{-2} E(X_1^j)^2 I\left(\left|X_1^j\right| \leq c_i\right) \\
&\leq C + \sum_{j\in B}\sum_{i=1}^{\infty}\sum_{c_i \leq k < c_i+1} c_i^{-2} E(X_1^j)^2 I\left(\left|X_1^j\right| \leq c_i\right) \\
&\leq C + \sum_{j\in B}\sum_{i=1}^{\infty}\sum_{k-1 < c_i \leq k} c_i^{-2} E(X_1^j)^2 I\left(\left|X_1^j\right| \leq k\right) \\
&\leq C + \sum_{j\in B}\sum_{i=1}^{\infty} (N(k) - N(k-1))(k-1)^{-2} \sum_{l=1}^{k} E(X_1^j)^2 I(l-1 < |X_1^j| \leq l) \\
&\leq C + \sum_{j\in B}\sum_{l=2}^{\infty} E(X_1^j)^2 I(l-1 < |X_1^j| \leq l) \sum_{k=l}^{\infty} N(k)((k-1)^{-2} - k^{-2}) \\
&\leq C + \sum_{j\in B}\sum_{l=2}^{\infty} E(X_1^j)^2 I(l-1 < |X_1^j| \leq l) \sum_{k=l}^{\infty} k^{r-3} \\
&\leq C + \sum_{j\in B}\sum_{l=2}^{\infty} l^{r-2} I(l-1 < |X_1^j| \leq l) \\
&\leq\ C + C\sum_{j\in B} E\,|X_1^j|^r < \infty.
\end{aligned}$$

By **Lemma 1.3** and the Kronecker lemma, we obtain $A_n^{-1}\sum_{i=1}^{n} a_i\left(Y_i - EY_i\right) \to 0$ a.s. as $n \to \infty$.

In order to prove (12), it suffices to prove that $A_n^{-1}\sum_{i=1}^{n} a_i EY_i \to 0$ a.s as $n \to \infty$.

In fact, by (7), (8) and (10) again, it follows that

$$\sum_{i=1}^{\infty}\frac{a_i\log i}{A_i}\|EY_i\| = \sum_{i=1}^{\infty}c_i^{-1}\left\|\sum_{j\in B}EY_i^j e_j\right\|$$
$$\le \sum_{i=1}^{\infty}\sum_{j\in B}P\left(\left|X_i^j\right|>c_i\right)+\sum_{i=1}^{\infty}c_i^{-1}\left\|\sum_{j\in B}EX_i^j I\left(\left|X_i^j\right|\le c_i\right)e_j\right\|$$
$$\le \sum_{i=1}^{\infty}c_i^{-1}\sum_{j\in B}E\left|X_1^j\right|I\left(\left|X_1^j\right|>c_i\right)$$
$$\le C+\sum_{i=1}^{\infty}\sum_{j\in B}\sum_{c_i\le l<c_i+1}c_i^{-1}E\left|X_1^j\right|I\left(\left|X_1^j\right|>c_i\right)$$
$$\le C+\sum_{j\in B}\sum_{l=1}^{\infty}\sum_{l-1<c_i\le l}c_i^{-1}E\left|X_1^j\right|I\left(\left|X_1^j\right|>l-1\right)$$
$$\le C+\sum_{j\in B}\sum_{l=2}^{\infty}\left(N(l)-N(l-1)\right)(l-1)^{-1}\sum_{k=l-1}^{\infty}E\left|X_1^j\right|I\left(k<\left|X_1^j\right|\le k+1\right)$$
$$\le C+C\sum_{j\in B}\sum_{k=1}^{\infty}E\left|X_1^j\right|I\left(k<\left|X_1^j\right|\le k+1\right)\sum_{l=2}^{k+1}N(l)\left((l-1)^{-1}-l^{-1}\right)$$
$$\le C+C\sum_{j\in B}\sum_{k=1}^{\infty}E\left|X_1^j\right|I\left(k<\left|X_1^j\right|\le k+1\right)\sum_{l=2}^{k+1}l^{r-2}$$
$$\le C+C\sum_{j\in B}\sum_{k=1}^{\infty}k^{r-1}E\left|X_1^j\right|I\left(k<\left|X_1^j\right|\le k+1\right)$$
$$\le C+C\sum_{j\in B}\sum_{k=1}^{\infty}E\left|X_1^j\right|^r I\left(k<\left|X_1^j\right|\le k+1\right)$$
$$\le C\sum_{j\in B}E\left|X_1^j\right|^r<\infty,$$

which implies that

$$A_n^{-1}\sum_{i=1}^{n}a_iEY_i\to 0 \quad \text{as} \quad n\to\infty. \tag{13}$$

Hence, the desired result (9) follows from (11-13) immediately. The proof is completed.

From **Theorem 2.1**, we can get the following almost sure convergence for weighted sum of coordinatewise PNQD random vectors. We set $b_n = n^{\alpha}\mathrm{l}(n)\in\mathcal{RV}_{\alpha}$ with $\mathrm{l}(.)$ is a slowly varying function, and the sequence of positive constants $\{a_n, n\ge 1\}$ is changed into an array of positive numbers $\{a_{ni}, 1\le i\le n, n\ge 1\}$ satisfying some conditions. The idea is mainly inspired by Shen [12].

Theorem 2.3 Let $\{X_n, n\ge 1\}$ be a sequence of $\mathbb{H}$-valued coordinatewise PNQD random vectors with $EX_n = 0$ and $E\left(X_n^j\right)^2 = \left(\sigma_n^j\right)^2 < \infty$ for each $n\ge 1, j\in B$. Let $\{a_{ni}, 1\le i\le n, n\ge 1\}$ be an array of positive numbers such that $a_{ni}^2\le Ca_{ii}^2$ for each $n\ge 1, 1\le i\le n$. If for some $\alpha > 1/2$

$$\sum_{i=1}^{\infty}\sum_{j\in B}\frac{a_{ii}^{2}\left(\sigma_{i}^{j}\right)^{2}}{i^{2\alpha-1}\ell^{2}(i)}<\infty, \tag{14}$$

where $\ell(.)$ is a slowly varying function.

Then $$\frac{1}{n^{\alpha}\ell(n)}\sum_{i=1}^{n}a_{ni}X_{i}\to 0 \quad \text{completely} \quad \text{as } n\to\infty,$$

and in consequence $$\frac{1}{n^{\alpha}\ell(n)}\sum_{i=1}^{n}a_{ni}X_{i}\to 0 \quad \text{a.s.} \quad \text{as } n\to\infty.$$

Proof. By the assumption and Lemma 3 it follows that

$$E\left\|\sum_{i=1}^{n}a_{ni}X_{i}\right\|^{2}\le\sum_{i=1}^{n}a_{ni}^{2}E\|X_{i}\|^{2}=\sum_{i=1}^{n}\sum_{j\in B}a_{ni}^{2}\left(\sigma_{i}^{j}\right)^{2}\le C\sum_{i=1}^{n}\sum_{j\in B}a_{ii}^{2}\left(\sigma_{i}^{j}\right)^{2}.$$

By the Markov's inequality, the inequality above and the Karamata theorem, we have that for any $\varepsilon>0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty}P\left(\left\|\sum_{i=1}^{n}a_{ni}X_{i}\right\|>\varepsilon n^{\alpha}\ell(n)\right)\le\sum_{n=1}^{\infty}\frac{1}{\varepsilon^{2}n^{2\alpha}\ell^{2}(n)}E\left\|\sum_{i=1}^{n}a_{ni}X_{i}\right\|^{2}\\
&\le C\sum_{n=1}^{\infty}\frac{1}{n^{2\alpha}\ell^{2}(n)}\sum_{i=1}^{n}\sum_{j\in B}a_{ii}^{2}\left(\sigma_{i}^{j}\right)^{2}=C\sum_{i=1}^{\infty}\sum_{j\in B}a_{ii}^{2}\left(\sigma_{i}^{j}\right)^{2}\sum_{n=i}^{\infty}\frac{1}{n^{2\alpha}\ell^{2}(n)}\\
&=C\sum_{i=1}^{\infty}\sum_{j\in B}a_{ii}^{2}\left(\sigma_{i}^{j}\right)^{2}\int_{i}^{\infty}\frac{1}{x^{2\alpha}\ell^{2}(x)}dx\\
&\le C\sum_{i=1}^{\infty}\sum_{j\in B}a_{ii}^{2}\left(\sigma_{i}^{j}\right)^{2}\frac{1}{i^{2\alpha-1}\ell^{2}(i)}<\infty.
\end{aligned}$$

Thus, $\dfrac{1}{n^{\alpha}\ell(n)}\sum_{i=1}^{n}a_{ni}X_{i}\to 0$ completely as $n\to\infty$, which implies that $\dfrac{1}{n^{\alpha}\ell(n)}\sum_{i=1}^{n}a_{ni}X_{i}$ converges almost surely to zero as $n\to\infty$. The proof is completed.

Application to General Von Mises Statistics

Statistics of Cramer-von Mises type is an important tool for testing statistical hypotheses. Next, we will consider general bivariate and degenerate von Mises-statistics (V-statistics). Let $h:\mathbb{R}^{2}\to\mathbb{R}$ be a symmetric, measurable function. We call $V_{n}=\sum_{i,j=1}^{n}c_{ni}c_{nj}h(X_{i},X_{j})$ be V-statistic with kernel h.

A V-statistic and its kernel h is said to be called degenerate, if $E(h(x,X_{i}))=0$ for all $x\in\mathbb{R}$. Furthermore, we assume that h is Lipschitz-continuous and positive definite, i.e.

$$\sum_{i,j=1}^{n}c_{i}c_{j}h(x_{i},x_{j})\ge 0$$

for all $c_{1},\dots,c_{n},x_{1},\dots,x_{n}\in\mathbb{R}$. Son [5] and [13] gave the almost sure convergence of degenerate von Mises-statistics with independent real valued data.

In the section, we use methods for random variables taking values in Hilbert spaces to obtain the conditions for the almost sure convergence of degenerate von Mises-statistics with pairwise independent real valued data.

Theorem 2.4 Let $\{X_n, n\ge 1\}$ be a sequence of real-valued pairwise independent random variables with mean 0. Let $\{a_{ni}, 1\le i\le n, n\ge 1\}$ be an array of positive numbers such that $a_{ni}^2 \le Ca_{ii}^2$ for each $1\le i\le n, n\ge 1, j\in B$. Let h be a Lipschitz-continuous positive definite kernel function such that for any $\alpha > \frac{1}{2}$,

$$\sum_{i=1}^{\infty}\frac{a_{ii}^2}{i^{2\alpha-1}}E\left|h(X_i, X_i)\right| < \infty.$$

Then

$$V_n = \frac{1}{n^{2\alpha}}\sum_{i,j=1}^{n} a_{ni}a_{nj}h(X_i, X_j) \to 0 \quad \textit{completely} \ \text{ as } n\to\infty.$$

Proof. Using Sun's version of Mercers theorem in [8] (see also [9]), we have the representation of h function under these above conditions as follows

$$h(x, y) = \sum_{l=1}^{\infty}\lambda_l\phi_l(x)\phi_l(y)$$

for orthonormal eigenfunctions $(\phi_l)_{l\in ¥}$ with the following properties:

- $E\phi_l(X_n) = 0$ and $E\phi_l^2(X_n) = 1$ for all $l \in ¥$,
- $\lambda_l \ge 0$ for all $l\in ¥$ and $\sum_{l=1}^{\infty}\lambda_l < \infty.$

Let $\mathcal{H}$ be a Hilbert space which consists of real-valued sequences $y = (y_l)_{l\in ¥}$ equipped with the inner product $\langle y, z\rangle = \sum_{l=1}^{\infty}\lambda_l y_l z_l.$

We consider the $\mathcal{H}$-valued random variables $Y_n = (\phi_l(X_n))_{l\in ¥}$. Then $\{Y_n, n\ge 1\}$ is a sequence of $\mathcal{H}$-valued coordinatewise PNQD random variables with mean 0 and

$$V_n = \frac{1}{n^{2\alpha}}\sum_{i,j=1}^{n} a_{ni}a_{nj}h(X_i, X_i) = \sum_{l=1}^{\infty}\lambda_l\left(\frac{1}{n^{\alpha}}\sum_{k=1}^{n}a_{nk}\phi_l(X_k)\right)^2 = \left\|\frac{1}{n^{\alpha}}\sum_{k=1}^{n}a_{nk}Y_k\right\|^2.$$

With $l(n)\equiv 1\ \forall n$, we check the assumption (14) in **Theorem 2.3**, it follows

$$\sum_{i=1}^{\infty}\sum_{j\in B}\frac{a_{ii}^2}{i^{2\alpha-1}}(\sigma_i^j)^2 = \sum_{i=1}^{\infty}\frac{a_{ii}^2}{i^{2\alpha-1}}E\|Y_i\|^2 = \sum_{i=1}^{\infty}\frac{a_{ii}^2}{i^{2\alpha-1}}E\left|h(X_i, X_i)\right| < \infty.$$

Hence, applying **Theorem 2.3** with $l(n)\equiv 1\ \forall n$ yields

$$\frac{1}{n^{\alpha}}\sum_{k=1}^{n}a_{nk}Y_k \to 0 \quad \text{a.s.} \quad \text{as } n\to\infty.$$

This implies that

$$V_n = \left\|\frac{1}{n^{\alpha}}\sum_{k=1}^{n}a_{nk}Y_k\right\|^2 \to 0 \ \text{ a.s.} n\to\infty.$$

3. Conclusion

In summary, we stated and proved the strong limit results for the weighted sums of coordinatewise PNQD sequence random variables. Moreover, we showed an application of the general Cramer-Von Mises statistics for coordinatewise PNQD random vectors in Hilbert spaces.

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