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Original Article

Nambu-goldstone Modes in Two Segregated Bose-einstein Condensates Limited by Two Hard Walls

Hoang Van Quyet*

Hanoi Pedagogical University 2, 32 Nguyen Van Linh, Xuan Hoa, Phuc Yen, Vinh Phuc

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Abstract: The Nambu-Goldstone (NG) modes in the system of two segregated Bose-Einstein condensates (BECs) limited by two hard walls are studied by means of the Gross-Pitaevskii (GP) theory. Based on the double-parabola approximation (DPA) combining with the Bogoliubov-de Gennes (BdG) equations we found four NG modes that proves the failure of the Watanabe-Brauner counting rule and, furthermore, their dispersion relations depend explicitly on the geometrical structure.

Keywords: Bose-Einstein condensates, double-parabola approximation, interface, Nambu-Goldstone modes, Watanabe-Brauner counting rule.

1. Introduction

It is known that the Goldstone theorem [1, 2] has been a cornerstone for all physical systems with spontaneous breaking of symmetry. It states that any relativistic system for which a continuous, global symmetry is spontaneously broken must contains in its spectrum gapless modes called the Nambu-Goldstone (NG) modes and, moreover, the number of NG modes n_{NG} coincides with the number of broken symmetries $n_{BS}, n_{NG} = n_{BS}$.

In recent years there is an increasing interest on the investigation of NG modes in those relativistic systems which violate the Lorentz invariance as well as in non-relativistic systems [3-13] since in these the number of NG modes is usually greater than the number of broken symmetries. This trend is pioneered by the work of Nielsen and Chadha [3] where a new counting rule was formulated for relativistic systems violating the Lorentz invariance: the number of type I plus twice number of type II

* Corresponding author.
E-mail address: hoangvanquyet@hpu2.edu.vn

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of NG modes are equal or greater than the number of broken generators, here the type I and type II, respectively consists of NG modes with odd andeven power of momentum in the low momentum limit. This counting rule was extended for non-relativistic systems by Watanabe and Brauner [8] who conjectured a new counting rule which was later proved by Hidaka [10] assuming the translational invariance

$$n_{BS} - n_{NG} = \frac{1}{2}\text{rank}\langle [Q_a, Q_b]\rangle,$$

here Q_a, Q_b are the set of spontaneous broken conserved charges and

$$\langle [Q_a, Q_b]\rangle = \lim_{V\to\infty}\frac{1}{V}\langle 0 | [Q_a, Q_b] | 0\rangle,$$

with V being the volume of system under consideration.

For system of two segregated BECs with the spontaneous breaking of the translation along z-axis it was shown [12] that there exist only two NG modes: one phonon mode and one ripplon mode. Nevertheless, all foregoing studies have only dealt with infinite systems.

This work is devoted to investigating how the spatial limitation gives rise to the number as well as the dispersion relations of NG modes. To this end, we adopt the double-parabola approximation (DPA) developed in [14] to the Gross-Pitaevskii (GP) theory for two immiscible BECs. The power of DPA rests in the fact that it allows us to find systematically analytical formulae of many physical quantities with high reliability. To begin with, let us start from the GP Hamiltonian in the bulk

$$H_b = \sum_{j=1}^{2} \psi_j^* \Big(\frac{-\hbar^2}{2m_j}\nabla^2\Big)\psi_j + V(\psi_1, \psi_2), \tag{1a}$$

$$V(\psi_1, \psi_2) = \sum_{j=1}^{2}\Big[-\mu_j \psi_j^* \psi_j + \frac{g_{jj}}{2}|\psi_j|^4\Big] + g_{12}|\psi_1|^2|\psi_2|^2, \tag{1b}$$

here $\psi_j = \psi_j(\vec{x}), m_j, \mu_j (j = 1,2)$ are respectively the wave function, the atomic mass and the chemical potential of each species j, with $\vec{x} = (\vec{a}, z), \vec{a} = (x, y)$ to be denoted in what follows. The coupling constant is given as

$$g_{jj'} = 2\pi\hbar^2\Big(\frac{1}{m_j} + \frac{1}{m_{j'}}\Big)a_{jj'},$$

with $a_{jj'}$ being the $s-$wave scattering length. In the following we restrict ourselves to the case when two condensates are separated, that is when $g_{12}^2 - g_{11}g_{22} > 0.$ It is clear that $U(1)\times U(1)$ is the symmetry group of (1) and the presence of a hard wall breaks explicitly the translation. Hence, the number of broken symmetries $n_{BS} = 2.$

This article is organized as follows. The Section 2 is devoted to the boundary conditions. The DPA is briefly presented in Section 3. The application of DPA to finding the analytical expressions of NG modes is presented in Section 4. The conclusion and discussion are given in the last Section 5.

2. Boundary Conditions

For the system of BECs separated by an interface S and limited by two hard walls its total

$$H = \int_V H_b dV + \int_S H_S dS + \int_{W_1} H_{W_1} dS + \int_{W_2} H_{W_2} dS, \tag{2}$$

where H_S, H_{W_1} and H_{W_2} are respectively the Hamiltonians of interface, hard wall 1 and hard wall 2. Their forms determine the boundary conditions imposed on the condensates at each surface. Usually, the surface Hamiltonian is chosen in the phenomenological forms

$$H_\alpha = \sum_{j=1}^{2} c_j^\alpha \psi_j^{\alpha *} \psi_j^\alpha , \tag{3}$$

in which ψ_j^α is the surface field induced by the bulk field ψ_j on the surface $\alpha = S, W_1, W_2$ and c_j^α is called the surface chemical potential. The quantity $\Lambda_j = 1/c_j^\alpha$ with dimension of a length is called the extrapolation length which was introduced by de Gennes [15] and is determined from the system Hamiltonian [16]. The fact that the equilibrium values of the fields ψ_j minimize the total Hamiltonian given in (1), (2) and (3) lead to the time-independent GP equations in the bulk together with the corresponding boundary conditions at the surfaces

$$\Big[-\frac{\hbar^2}{2m_1}\Delta - \mu_1 + g_{11}|\psi_1|^2 + g_{12}|\psi_2|^2\Big]\psi_1 = 0, \tag{4a}$$

$$\Big[-\frac{\hbar^2}{2m_2}\Delta - \mu_2 + g_{22}|\psi_2|^2 + g_{12}|\psi_1|^2\Big]\psi_2 = 0, \tag{4b}$$

where $\psi_j (j = 1, 2)$ fulfills the boundary conditions at interface

$$\vec{n}\nabla \psi_j^\alpha = c_j^\alpha \psi_j^\alpha , \tag{5a}$$

and

$$\psi_j(\vec{a}, \tilde{z}) = \psi_j^S \text{ for } (\vec{a}, \tilde{z}) \in S. \tag{5b}$$

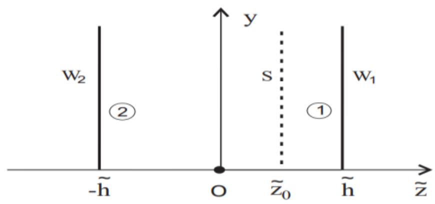


Figure 1. Two hard walls W_1, W_2 are located at $\tilde{z} = \pm\tilde{h}$ and the interface S at $\tilde{z} = \tilde{z}_0$.

In (5a), $\vec{n}$ is the unit vector normal to the interface and pointing inside the system. For a planar interface locating at $\tilde{z} = \tilde{z}_0$ and two hard walls at $\tilde{z} = \pm\tilde{h}$ as indicated in Fig. 1, Eqs. (5) turn out to be Robin boundary conditions

$$\Big(\frac{\partial \psi_j^S(\vec{a}, \tilde{z})}{\partial \tilde{z}}\Big)_{\tilde{z} = \tilde{z}_0 - 0} = c_j^S \psi_j^S(\vec{a}, \tilde{z} = \tilde{z}_0) = \Big(\frac{\partial \psi_j^S(\vec{a}, \tilde{z})}{\partial \tilde{z}}\Big)_{\tilde{z} = \tilde{z}_0 + 0} \tag{6a}$$

the interface, and at

$$\Big(\frac{\partial \psi_j^{W_1}(\vec{a}, \tilde{z})}{\partial \tilde{z}}\Big)_{\tilde{z} = \tilde{h}} = c_j^{W_1} \psi_j^{W_1}(\vec{a}, \tilde{z} = \tilde{h}), \tag{6b}$$

$$\Big(\frac{\partial \psi_j^{W_2}(\vec{a}, \tilde{z})}{\partial \tilde{z}}\Big)_{\tilde{z} = -\tilde{h}} = c_j^{W_2} \psi_j^{W_2}(\vec{a}, \tilde{z} = -\tilde{h}) \tag{6c}$$

at two hard walls.

The Robin conditions (6) expresses the continuity at the interface of the first derivative of condensate profiles with respect to $\tilde{z}$. In the case when the surface fields at two hard walls vanish, the boundary conditions at two hard walls are the Dirichlet ones, instead of the foregoing Robin conditions

$$\left[\psi_j^{w_j}(\vec{a},\tilde{z})\right]_{\tilde{z}=\pm\tilde{h}}=0. \tag{7}$$

It is necessary to emphasize that the boundary conditions at the interface (5) are the direct consequences of the interface Hamiltonian and, moreover, they are taken at equilibrium state of the system limited by two hard walls.

3. Ground States in Double-parabola Approximation

It is necessary to point out that the DPA is applied to the system of two immiscible BECs which are separated by interface. In this respect, let us remember that the equilibrium values of order parameters which minimize the Hamiltonian (2) are the solutions of the time-independent GP equations (4) together with the boundary condition at the interface.

Assume that the condensate 1 and condensate 2 occupies the region $\tilde{z} > z_0$ and $\tilde{z} < z_0$, respectively. Therefore, it is possible to restrict our consideration on the condensate profiles which depend only on $\tilde{z}$. For simplicity, they are also symbolized by $\Psi_j(\tilde{z})$ which satisfies the equations

$$-\frac{\hbar^2}{2m_1}\frac{d^2\Psi_1(\tilde{z})}{d\tilde{z}^2}-\mu_1\Psi_1(\tilde{z})+g_{11}\Psi_1^3(\tilde{z})+g_{12}\Psi_2^2(\tilde{z})\Psi_1(\tilde{z})=0, \tag{8a}$$

$$-\frac{\hbar^2}{2m_2}\frac{d^2\Psi_2(\tilde{z})}{d\tilde{z}^2}-\mu_2\Psi_2(\tilde{z})+g_{22}\Psi_2^3(\tilde{z})+g_{12}\Psi_1^2(\tilde{z})\Psi_2(\tilde{z})=0, \tag{8b}$$

the corresponding potential is

$$V(\Psi_1,\Psi_2)=\sum_{j=1}^{2}\left[-\mu_j|\Psi_j|^2+\frac{g_{jj}}{2}|\Psi_j|^4\right]+g_{12}|\Psi_1|^2|\Psi_2|^2, \tag{9}$$

where, without loss of generality, Ψ_j is assumed to be real.

Introducing the dimensionless quantities

$$z=\frac{\tilde{z}}{\xi_1},\phi_j=\frac{\Psi_j}{\sqrt{n_{j0}}},K=\frac{g_{12}}{\sqrt{g_{11}g_{22}}},h=\frac{\tilde{h}}{\xi_1},$$

here

$$\xi_j=\frac{\hbar}{\sqrt{2m_j\mu_j}},n_{j0}=\frac{\mu_j}{g_{jj}},$$

are the healing length and the bulk density of component *j*, respectively. For simplicity we consider the symmetric case $\xi_1=\xi_2=\xi$. Utilizing these parameters Eqs. (8) turn out to be

$$-\frac{d^2\phi_1}{dz^2}-\phi_1+\phi_1^3+K\phi_2^2\phi_1=0, \tag{10a}$$

$$-\frac{d^2\phi_2}{dz^2}-\phi_2+\phi_2^3+K\phi_1^2\phi_2=0, \tag{10b}$$

and the potential (9) is rewritten in the form

$$\tilde{V}(\phi_1,\phi_2)=\sum_{j=1}^{2}(-\phi_j^2+\frac{\phi_j^4}{2})+K\phi_1^2\phi_2^2. \tag{11}$$

In the context of surface and interfacial phenomena the DPA is established as follows. We assume that in the half-space $z < z\_0(z > z\_0)$ the condensate profiles behave like

$$\phi_j = 1+\varepsilon_j, \phi_{j'} = \delta_{j'}, with \begin{cases} z \ge z_0, (j, j') = (1,2), \\ z \le z_0, (j, j') = (2,1), \end{cases} \tag{12}$$

here the dimensionless real quantities ε_j and $\delta_{j'}$ are treated as small perturbations. Then expanding the potential (11) up to second order in ε_j and $\delta_{j'}$ we arrive at the potential in the DPA which consists of two quadratic potentials, each of them is to be used in the appropriate half-space

$$V_{DPA}(\phi_1,\phi_2) = \frac{1}{2}(2\phi_j - 1)(2\phi_j - 3) + (K-1)\phi_{j'}^2, \tag{13}$$

with

$$z \ge z_0, (j, j') = (1,2), z \le z_0, (j, j') = (2,1).$$

Next the DPA is used to calculate the ground states of the system. To do this, we employ the Dirichlet boundary conditions at the hard wall for both condensates

$$\phi_j(z = \pm h) = 0, \text{ with } j = (1,2). \tag{14}$$

Then we rewrite Eqs. (10) corresponding to DPA potential (13) on each side of the interface. In the right-hand side it reads

$$-\frac{1}{2}\frac{d^2\phi_1}{dz^2} + 2(\phi_1 - 1) = 0, \tag{15a}$$

$$-\frac{1}{2}\frac{d^2\phi_2}{dz^2} + (K-1)\phi_2 = 0. \tag{15b}$$

The solutions to Eqs. (15) satisfying (14) can be simply found

$$\phi_1 = Ae^{-\sqrt{2}z}(e^{2\sqrt{2}z} - e^{2\sqrt{2}h}) - e^{\sqrt{2}(h-z)} + 1, \tag{16a}$$

$$\phi_2 = Be^{-\sqrt{K-1}z}(e^{2\sqrt{K-1}z} - e^{2\sqrt{K-1}h}), \tag{16b}$$

with two integral constants *A*, *B*. In the left-hand side we have that

$$-\frac{1}{2}\frac{d^2\phi_1}{dz^2} + (K-1)\phi_1 = 0,$$

$$-\frac{1}{2}\frac{d^2\phi_2}{dz^2} + 2(\phi_2 - 1) = 0,$$

whose solutions satisfying (14) are easily derived

$$\phi_1 = Ce^{-\sqrt{2(K-1)}(2h+z)}[e^{2\sqrt{2(K-1)}(h+z)} - 1], \tag{17a}$$

$$\phi_2 = 1 - e^{-\sqrt{2}(h+z)} + 2De^{-2h}\sinh[2(h+z)], \tag{17b}$$

with two integral constants C, D.

To proceed further let us determine the constants A, B, C, D by means of the boundary conditions at the interface. The first condition is the continuity at interface of first derivative of condensate profiles with respect to z given in (6a)

$$\left(\frac{d\phi_j}{dz}\right)_{z=z_0-0} = \left(\frac{d\phi_j}{dz}\right)_{z=z_0+0}. \tag{18a}$$

The second condition is the continuity of both condensates at interface, $z = z_0$,

$$\phi_j(z_0 - 0) = \phi_j(z_0 + 0). \tag{18b}$$

Substituting (16) and (17) into (18) gives

$$A=\frac{1}{\mathcal{X}}\Big(e^{\sqrt{2}z_0}[\sqrt{\eta}-\sqrt{\eta}_{\_}e^{\sqrt{2}h_-}+e^{2\sqrt{\eta}h_+}(\sqrt{\eta}-\sqrt{\eta}_+e^{\sqrt{2}h_-})]\Big), \tag{19a}$$

$$B=\frac{\sqrt{2}}{\mathcal{X}}(e^{\sqrt{2}h}-e^{\sqrt{2}z_0})^2 e^{\sqrt{\eta}(2h+z_0)}, \tag{19b}$$

$$C=\frac{\sqrt{2}}{\mathcal{Y}}(e^{\frac{\sqrt{2}h_+}{\xi}}-1)^2 e^{\frac{\sqrt{\eta}z_0}{\xi}}, \tag{19c}$$

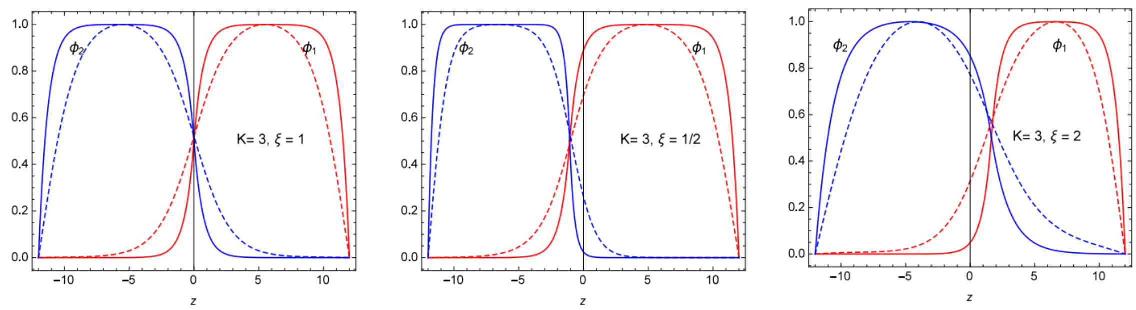
$$D=\frac{1}{\mathcal{Y}}\Big(e^{\frac{\sqrt{2}h}{\xi}}(\sqrt{\eta}_{\_}e^{\frac{2\sqrt{\eta}h}{\xi}}+\sqrt{\eta}_+e^{\frac{2\sqrt{\eta}z_0}{\xi}})-e^{\frac{\sqrt{2}h}{\xi}}\sqrt{\eta}(e^{\frac{2\sqrt{\eta}h+\sqrt{2}h_+}{\xi}}+e^{\frac{\sqrt{2}h_++2\sqrt{\eta}z_0}{\xi}})\Big). \tag{19d}$$

With

$$\eta=K-1, \sqrt{\eta}_\pm=\sqrt{\eta}\pm\sqrt{2}, h_\pm=h\pm z_0,$$

$$\mathcal{X}=\sqrt{\eta}_{\_}e^{2\sqrt{2}h}-\sqrt{\eta}_+e^{2\sqrt{2}z_0}+2e^{(2\sqrt{\eta}+\sqrt{2})h_+}(\sqrt{\eta}\sinh[\sqrt{2}h_{\_}]+\sqrt{2}\cosh[\sqrt{2}h_{\_}]),$$

$$\mathcal{Y}=e^{\frac{2\sqrt{\eta}z_0}{\xi}}[\sqrt{\eta}_+-\sqrt{\eta}_{\_}e^{\frac{2\sqrt{2}h_+}{\xi}}]-e^{\frac{2\sqrt{\eta}h}{\xi}}[-\sqrt{\eta}_{\_}+\sqrt{\eta}_+e^{\frac{2\sqrt{2}h_+}{\xi}}].$$



(a) (b) (c)

Figure 2. The condensate profiles calculated in DPA (solid lines) and in GP equations (dashed lines). The red and blue lines correspond to first and second components.

The condensate profiles in DPA and in GP theory are plotted in Fig. 2 for the symmetric case $m_1=m_2=m,\ g=g_{11}=g_{22}$. Fig. 2 tells that:

- For symmetric condensates, Fig. 2a, the interface is located at $z=0$, where two condensate profiles intersect. For asymmetric cases, Figs. 2b and 2c, the locations of interface slightly deviate from $z=0$.

- In all cases there is a qualitative accordance between the behaviors of both types of solutions. This fact proves that the DPA is a reliable approach.

Taking into account the condensate profiles found above we can easily calculate the surface chemical potential from (6).

$$c_j^S=\frac{1}{\xi_1}\frac{(\frac{d\phi_j}{dz})_{z=z_0}}{\phi_j(z_0)}=\begin{cases}\frac{1}{\xi_1}\sqrt{K-1}\coth\Big[\sqrt{K-1}\frac{h+z_0}{\xi_1}\Big] \text{ for } j=1,\\ -\frac{1}{\xi_2}\sqrt{K-1}\coth\Big[\sqrt{K-1}\frac{h-z_0}{\xi_2}\Big] \text{ for } j=2,\end{cases}$$

which vanish when K tends to 1.

4. Nambu-Goldstone Modes

For convenience the symmetric case is investigated in the BdG theory where the wave functions are expanded as

$$\psi_j(\vec{x},t) = [\Psi_j(z) + \delta\psi_j(\vec{a},z,t)]e^{-i\mu_j t/\hbar}, \tag{20a}$$

with

$$\delta\psi_j = u_{jk}(z)e^{i(\vec{k}\vec{a}-\omega t)} + v^*_{jk}(z)e^{-i(\vec{k}\vec{a}-\omega t)}. \tag{20b}$$

Substituting (20) into the time-dependent GP equations

$$\hbar\frac{\partial\psi_j}{\partial t} = \Big(-\frac{\hbar^2}{2m_j}\nabla^2 + g_{jj}|\psi_j|^2 + g_{jj'}|\psi_{j'}|^2\Big)\psi_j,$$

and taking the first order of $\delta\psi_j$ we have that

$$i\hbar\frac{\partial\delta\psi_j}{\partial t} = \Big(-\frac{\hbar^2}{2m_j}\nabla^2 - \mu_j + 2g_{jj}|\Psi_j|^2 + g_{jj'}|\Psi_{j'}|^2\Big)\delta\psi_j + g_{jj}|\Psi_j|^2\,\delta\psi_j^* + g_{jj'}\Psi_j\Psi_{j'}(\delta\psi_{j'} + \delta\psi^*_{j'}). \tag{21}$$

Basing on (12) we can rewrite (21) as follows

$$i\hbar\frac{\partial\delta\psi_j}{\partial t} = -\frac{\hbar^2}{2m_j}\nabla^2\delta\psi_j - \mu_j\delta\psi_j + g_{jj'}n_{j0}\delta\psi_j, \tag{22a}$$

$$i\hbar\frac{\partial\delta\psi_{j'}}{\partial t} = -\frac{\hbar^2}{2m_{j'}}\nabla^2\delta\psi_{j'} - \mu_{j'}\delta\psi_{j'} + 2g_{j'j'}n_{j'0}\delta\psi_{j'} + g_{j'j'}n_{j'0}\delta\psi^*_{j'}. \tag{22b}$$

Substituting (20b) into (22) and turning into the dimensionless form we get BdG equations

$$-u''_{jk} + (\kappa^2 + K - 1)u_{jk} = \varepsilon u_{jk}, \tag{23a}$$

$$-v''_{jk} + (\kappa^2 + K - 1)v_{jk} = -\varepsilon v_{jk}, \tag{23b}$$

$$-u''_{j'k} + (\kappa^2 + 1)u_{j'k} + v_{j'k} = \varepsilon u_{j'k}, \tag{24c}$$

$$-v''_{j'k} + (\kappa^2 + 1)v_{j'k} + u_{j'k} = -\varepsilon v_{j'k}, \tag{24d}$$

here $\kappa = k\xi/\sqrt{2}$ and $\varepsilon = \hbar\omega/gn_0$, with $\xi = \xi_j, n_0 = n_{j0}$.

Solutions of (23) are found straightforwardly

$$u_{1k} = A_1e^{-\alpha_- z}(e^{2\alpha_- z} - e^{2\alpha_- h}),$$

$$v_{1k} = B_1e^{-z\alpha_+}(e^{2\alpha_+ z} - e^{2\alpha_+ h}),$$

$$u_{2k} = C_1(e^{-\beta_- z} - e^{\beta_-(z-2h)}) + D_1(e^{\beta_+ z} - e^{\beta_+(2h-z)}),$$

$$v_{2k} = \frac{1}{\sqrt{\varepsilon^2+1}}e^{\beta_-(-2h-z)-\beta_+ z}(C_1\varepsilon_- e^{\beta_+ z}(e^{2\beta_- h} - e^{2\beta_- z}) + D_1\varepsilon_+ e^{\beta_-(2h+z)}(e^{2\beta_+ z} - e^{2\beta_+ h})),$$

for right-hand side of interface, and

$$u_{1k} = A_2(e^{-\beta_- z} - e^{\beta_-(z-2h)}) + B_2(e^{\beta_+ z} - e^{\beta_+(2h-z)}),$$

$$v_{1k} = \frac{1}{\sqrt{\varepsilon^2+1}}e^{\beta_-(-2h-z)-\beta_+ z}(A_2\varepsilon_- e^{\beta_+ z}(e^{2\beta_- h} - e^{2\beta_- z}) + B_2\varepsilon_+ e^{\beta_-(2h+z)}(e^{2\beta_+ z} - e^{2\beta_+ h})),$$

$$u_{2k} = C_2 e^{-\alpha_- z}(e^{2\alpha_- z} - e^{2\alpha_- h}),$$
$$v_{2k} = D_2 e^{-z\alpha_+}(e^{2\alpha_+ z} - e^{2\alpha_+ h}),$$

for left-hand side of interface.

Here A_j, B_j, C_j, D_j are integral constants, and

$$\alpha_\pm = \sqrt{\kappa^2 + K - 1 \pm \varepsilon}, \beta_\pm = \sqrt{\kappa^2 + 1 \pm \sqrt{\varepsilon^2 + 1}}, \varepsilon_\pm = \varepsilon(\sqrt{\varepsilon^2 + 1} \pm \varepsilon) \pm 1.$$

In order to determine the NG modes, let us introduce the functions

$$\Sigma_{jk} = u_{jk} + v_{jk}, \Delta_{jk} = u_{jk} - v_{jk}.$$

For sake the simplicity, we are now considering the case of strong segregation. For phonon mode corresponding to the component 2 in the right-hand side of interface, we impose on $\Sigma_{2k}(z)$ the Dirichlet boundary condition at interface

$$\Sigma_{2k}(z = z_0) = 0, \tag{24}$$

and on $\Delta_{2k}(z)$ the Robin boundary condition

$$\frac{\partial \Delta_{2k}}{\partial z}\Big|_{z=z_0} = 0, \tag{25}$$

which lead to the systems of two homogeneous, linear equations

$$M_{11}C_1 + M_{12}D_1 = 0, \tag{26a}$$

$$M_{21}C_1 + M_{22}D_1 = 0, \tag{26b}$$

here

$$M_{11} = 2(-\sqrt{\varepsilon^2 + 1} + \varepsilon + 1)e^{-\beta_- h}\sinh(\beta_-(h - z_0)), \tag{27a}$$

$$M_{12} = -2(\sqrt{\varepsilon^2 + 1} + \varepsilon + 1)e^{\beta_+ h}\sinh(\beta_+(h - z_0)), \tag{27b}$$

$$M_{21} = -2\beta_-(\sqrt{\varepsilon^2 + 1} - \varepsilon + 1)e^{-\beta_- h}\cosh(\beta_-(h - z_0)), \tag{27c}$$

$$M_{22} = -2\beta_+(\sqrt{\varepsilon^2 + 1} + \varepsilon - 1)e^{\beta_+ h}\cosh(\beta_+(h - z_0)). \tag{27d}$$

Taking into account (27) the existence of non-trivial solutions to Eqs. (26) requires that

$$M_{11}M_{22} - M_{21}M_{12} = 0,$$

yielding the first phonon mode

$$\varepsilon = 2\kappa + h\kappa^2 + \mathcal{O}[\kappa^3].$$

In the left-hand side of interface, the Robin boundary condition is used for both Σ_{1k} and Δ_{1k}, namely,

$$\frac{\partial \Sigma_{1k}}{\partial z}\Big|_{z=z_0} = -\sqrt{2}\Sigma_{1k}(z_0), \tag{28}$$

and

$$\frac{\partial \Delta_{1k}}{\partial z}\Big|_{z=z_0} = 0. \tag{29}$$

Eqs. (28) and (29) lead to a couple of homogeneous linear equations

$$M_{11}A + M_{12}B = 0, \tag{30a}$$

$$M_{21}A + M_{22}B = 0, \quad (30b)$$

here

$$M_{11} = (\sqrt{\varepsilon^2+1} - \varepsilon - 1)e^{-z_0\beta_-}(\beta_- + (\beta_- + \sqrt{2})e^{2(h+z_0)\beta_-} - \sqrt{2}), \quad (31a)$$

$$M_{12} = (\sqrt{\varepsilon^2+1} + \varepsilon + 1)e^{-(2h+z_0)\beta_+}(\beta_+ + (\beta_+ + \sqrt{2})e^{2(h+z_0)\beta_+} - \sqrt{2}), \quad (32b)$$

$$M_{21} = \beta_-(\sqrt{\varepsilon^2+1} - \varepsilon + 1)(-e^{-z_0\beta_-})(e^{2(h+z_0)\beta_-} + 1), \quad (33c)$$

$$M_{22} = \beta_+(\sqrt{\varepsilon^2+1} + \varepsilon - 1)(-e^{-(2h+z_0)\beta_+})(e^{2(h+z_0)\beta_+} + 1). \quad (34d)$$

Taking into account Eqs. (31) the second phonon mode is derived straightforwardly from the requirement that Eqs. (30) possesses non-trivial solutions

$$\varepsilon = 2\kappa - h\kappa^2 + \mathcal{O}[\kappa^3].$$

Finally, weak segregation of condensates is concerned. From continuous conditions at interface

$$u_{jk}(z_0 - 0) = u_{jk}(z_0 + 0),$$

$$v_{jk}(z_0 - 0) = v_{jk}(z_0 + 0),$$

with $j = (1, 2)$, we have

$$A_1 = \frac{e^{\alpha_- z_0}(B_2(e^{\beta_+ z_0} - e^{\beta_+(-(2h+z_0))}) - 2A_2 e^{\beta_- h}\sinh(\beta_-(h+z_0)))}{e^{2\alpha_- z_0} - e^{2\alpha_- h}}, \quad (35a)$$

$$B_1 = -\frac{e^{\beta_-(-h-z_0)-\beta_+(2h+z_0)+\alpha_+ z_0}(A_2\varepsilon_- e^{\beta_- h+\beta_+(2h+z_0)}(e^{2\beta_-(h+z_0)} - 1) + B_{11})}{\sqrt{\varepsilon^2+1}(e^{2\alpha_+ z_0} - e^{2\alpha_+ h})}, \quad (32b)$$

$$B_{11} = B_2\varepsilon_+(e^{\beta_-(h+z_0)} - e^{(\beta_- + 2\beta_+)(h+z_0)}),$$

$$C_{21} = C_1 + D_1(e^{(\beta_- + \beta_+)z_0} - e^{2\beta_+ h + \beta_- z_0 - \beta_+ z_0}), \quad (33c)$$

$$D_2 = \frac{e^{(\beta_- - \alpha_+)(-2h-z_0)-\beta_+ z_0}(\coth(\alpha_+(h+z_0)) - 1)D_{21}}{2\sqrt{\varepsilon^2+1}}, \quad (34d)$$

Taking into account (34) the existence of non-trivial solutions to Eqs. (33) requires that

$$M_{11}M_{22} - M_{21}M_{12} = 0,$$

yielding the third phonon mode

$$\varepsilon = 2\kappa + \mathcal{O}[\kappa^2].$$

Applying conditions (28) and (29) for the component 2 in the left-hand side of interface, and using (32c), (32d) we have

$$M_{11}C_1 + M_{12}D_1 = 0, \quad (35a)$$

$$M_{21}C_1 + M_{22}D_1 = 0, \quad (35b)$$

here

$$M_{11} = \frac{1}{4\sqrt{\varepsilon^2+1}} e^{\beta_-(-(2h+z_0))}(e^{2\beta_- h} - e^{2\beta_- z_0})(\coth(\alpha_-(h+z_0)) - 1)(\coth(\alpha_+(h+z_0)) - 1)\Big(\alpha_-\sqrt{\varepsilon^2+1}(e^{2\alpha_-(h+z_0)} + 1)(e^{2\alpha_+(h+z_0)} - 1)$$
$$+4e^{(\alpha_- + \alpha_+)(h+z_0)}\sinh(\alpha_-(h+z_0))(\sqrt{2}(\sqrt{\varepsilon^2+1} + \varepsilon_-)\sinh(\alpha_+(h+z_0)) + \alpha_+\varepsilon_-\cosh(\alpha_+(h+z_0)))\Big),$$

$$M_{12} = -\frac{1}{4\sqrt{\varepsilon^2+1}}\Big(e^{\beta_+(-z_0)}(e^{2\beta_+ h} - e^{2\beta_+ z_0})(\coth(\alpha_-(h+z_0)) - 1)(\coth(\alpha_+(h+z_0)) - 1)\Big(\alpha_-\sqrt{\varepsilon^2+1}(e^{2\alpha_-(h+z_0)} + 1)(e^{2\alpha_+(h+z_0)} - 1)$$
$$+4e^{(\alpha_- + \alpha_+)(h+z_0)}\sinh(\alpha_-(h+z_0))(\sqrt{2}(\sqrt{\varepsilon^2+1} + \varepsilon_+)\sinh(\alpha_+(h+z_0)) + \alpha_+\varepsilon_+\cosh(\alpha_+(h+z_0)))\Big)\Big),$$

$$M_{21} = -\frac{1}{4\sqrt{\varepsilon^2+1}}\Big(e^{\beta_-(-(2h+z_0))}(e^{2\beta_- h} - e^{2\beta_- z_0})(\alpha_+\varepsilon_-(e^{2\alpha_-(h+z_0)}-1)(e^{2\alpha_+(h+z_0)}+1)$$
$$-\alpha_-\sqrt{\varepsilon^2+1}(e^{2\alpha_-(h+z_0)}+1)(e^{2\alpha_+(h+z_0)}-1))(\coth(\alpha_-(h+z_0))-1)(\coth(\alpha_+(h+z_0))-1)\Big),$$
$$M_{22} = \frac{1}{4\sqrt{\varepsilon^2+1}}\Big(e^{\beta_+(-z_0)}(e^{2\beta_+ h} - e^{2\beta_+ z_0})(\alpha_+\varepsilon_+(e^{2\alpha_-(h+z_0)}-1)(e^{2\alpha_+(h+z_0)}+1)$$
$$-\alpha_-\sqrt{\varepsilon^2+1}(e^{2\alpha_-(h+z_0)}+1)(e^{2\alpha_+(h+z_0)}-1))(\coth(\alpha_-(h+z_0))-1)(\coth(\alpha_+(h+z_0))-1)\Big).$$

Taking into account (36) the existence of non-trivial solutions to Eqs. (35) requires that

$$M_{11}M_{22}-M_{21}M_{12}=0,$$

yielding the fourth phonon mode

$$\varepsilon = 2\kappa + \mathcal{O}[\kappa^2].$$

5. Conclusion and Discussion

In the preceding sections we presented the main content of the work. It consists of the following items:

The solutions derived from DPA are in qualitative agreement with those obtained directly from GP theory.

Adopting the DPA we derived successfully four NG modes that indicates the failure of the Watanabe-Brauner counting rule and, moreover, their dispersion relations depend on the spatial limitation. This result is our major success.

The existence of the interface Hamiltonian H_S is very important because it guarantees that the boundary condition at the interface (18) is taken at equilibrium state. For simplicity we chose the Dirichlet condition as the boundary condition at hard wall assuming that the hard wall Hamiltonian vanishes. Consequently, the total Hamiltonian of our system (2) was established. However, it is worth mentioning that there has been a long-standing interest in the problem of phase transition in confined geometry where the hard wall Hamiltonian play's crucial role [17-23]. In this regard, taking into consideration this quantity is the subject of our next study.

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